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An Operational Approach to Fractional Scale-Invariant Linear Systems

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Abstract: The fractional scale-invariant systems are introduced and studied, using an operational formalism. It is shown that the impulse and step responses of such systems belong to the vector space generated by some special functions here introduced. For these functions, the fractional scale derivative is a decremental index operator, allowing the construction of an algebraic framework that enables to compute the impulse and step responses of such systems. The effectiveness and accuracy of the method are demonstrated through various numerical simulations.

Keywords: operational calculus; Mellin transform; fractional scale-invariant; fractional scale derivative; stretching derivative; hadamard derivative

MSC: 26A33; 44A40



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1. Introduction

The mathematical framework often used to define systems is based on shift-invariant derivatives [1,2], resulting from the works of Leibniz, Euler, Lagrange, and Liouville. However, a different concept was introduced by C. Braccini and G. Gambardella: the form-invariant linear filtering. It was a new kind of processing that was applied to several different fields such as optical pattern recognition, image restoration and reconstruction from projections [3]. This was the first step into the introduction of the scale-invariant linear systems, really done by B. Yazici and R. L. Kashyap for analysis and modelling 1/f phenomena and in general the self-similar processes, namely the scale stationary processes [4,5]. In parallel, physicists started studying the importance of scale in several physical systems [6–13]. Although the concept of scale is not very well defined, since it is a parameter expressing relative relations [6], the concept of scale-invariant system is well defined. While the shift-invariant systems are related and use in their definition the usual convolution [14] (D’Alembert’s) and corresponding derivatives, the scale-invariant systems are based on the Mellin’s convolution [15]. To fully define these systems the fractional scale-derivatives were introduced and studied [16], generalizing the classic Hadamard definitions [17]. These derivatives allow the formalization of linear scale-invariant systems of the autoregressive-moving average (ARMA) type [16,18]. With the use of the Mellin transform, the transfer functions of these systems assume a form identical to the one got from the shift-invariant systems through the use of the Laplace transform [19].

The objective of this paper is the study and search for the impulse and step responses of these systems. For this purpose, we first find a sequence of functions for which the fractional scale derivative behaves as a decremental index operator and then develop an algebraic framework similar to the one we used in the shift-invariant systems [20]. This approach allows us to closed forms for both the impulse and step responses of the systems.

The paper is organized as follows. Section 2 contains some preliminary results [16]. In Section 3 the sequence of functions $\{u_k(\tau)\}_k$ for which Hadamard right (left) derivative and fractional scale derivatives of type Grünwald-Letnikov are decremental index operators is founded. Section 3.2 contains the algebraic framework needed to solve fractional scale-invariant systems. In Section 4 we present an operational method based on algebraic framework introduced in Section 3. Numerical examples are solved in Section 5. Finally, Section 6 contains the main conclusions.

2. Scale-Invariant Systems and Derivatives

2.1. The Mellin Convolution

Definition 1. We call a linear system scale-invariant or dilation-invariant (DI) if its input–output relation is given by the Mellin convolution [16]

$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta) \frac{d\eta}{\eta},$$

where $\tau \in \mathbb{R}^+$, and $g(\tau)$ is the impulse response: the response to $x(\tau) = \delta(\tau - 1)$.

We demand that the impulse response, $g(t)$, be at least

- piecewise continuous,
- with bounded variation.

Similarly to the shift-invariant case, where the exponentials are the eigenfunctions, the powers $x(\tau) = \tau^v$, $\tau \in \mathbb{R}^+$, $v \in \mathbb{C}$, are the eigenfunctions of the dilation-invariant systems. In fact, if the input is $x(\tau) = \tau^v$, then the output is

$$y(\tau) = x(\tau) \star g(\tau) = G(v)\tau^v,$$

where $G(v)$ is the transfer function given by

$$G(v) = \int_0^{\infty} g(u)u^{-v-1}du, \quad (1)$$

which is a modified version of the Mellin transform (MT) of the impulse response. The Mellin transform in (1), denoted by $\mathcal{M}[g(\tau)](v)$, has a parameter sign change $-v \rightarrow v$ relatively to the usual Mellin transform [15,21,22].

Suppose that $\mathcal{M}[f(\tau)](v)$ and $\mathcal{M}[g(\tau)](v)$ exist in the regions of convergence (ROC) $a_1 < \text{Re}(v) < b_1$ and $a_2 < \text{Re}(v) < b_2$, respectively. An important property of Mellin convolution is

$$\mathcal{M}[f(\tau) \star g(\tau)](v) = F(v)G(v), \quad \max(a_1, a_2) < \text{Re}(v) < \min(b_1, b_2),$$

where $F(v)$ and $G(v)$ are the Mellin transforms of f and g , respectively. The inverse Mellin transform related to (1) is

$$x(\tau) = \mathcal{M}^{-1}[X(v)] = \frac{1}{2\pi i} \int_{\gamma} X(v)\tau^v dv, \quad \tau \in \mathbb{R}^+,$$

where γ is vertical straight line in the ROC of the transform.

2.2. Scale-Derivatives

Definition 2. Let $\alpha \in \mathbb{R}$. We define the α -order scale derivative (SD) as the operator \mathfrak{D}_s obeying the rule [16,23]

$$\mathfrak{D}_s^\alpha \tau^v = v^\alpha \tau^v, \quad \tau \in \mathbb{R}^+, v \in \mathbb{C},$$

for $Re(v) > 0$ (expansion or stretching case) or $Re(v) < 0$ (shrinking case).

If a function $x(\tau)$ has Mellin transform $X(v)$, then it has fractional scale-derivative that is given by

$$\mathfrak{M}[\mathfrak{D}_s^\alpha x(\tau)] = v^\alpha X(v), \tag{2}$$

for a suitable ROC. The way how we express v^α imposes a form for the derivative. To start, we notice that we can consider two situations corresponding to the sign of $Re(v)$. If it is positive, we obtain stretching derivatives, while if it is negative, we obtain the shrinking one. For both, we can express the inverse Mellin transform of $v^\alpha X(v)$ in two different forms: summation or integral.

2.2.1. Stretching Derivatives: $Re(v) > 0$ [16]

Let $\varepsilon(\tau)$ be Heaviside step function. We have two ways of expressing v^α :

$$v^\alpha = \begin{cases} \lim_{q \rightarrow 1^+} \left[\frac{(1 - q^{-v})}{\ln q} \right]^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{-nv} \\ \mathcal{M} \left[\frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} \varepsilon(\tau - 1) \right] \end{cases} \tag{3}$$

These expressions lead to the following scale derivative [16]:

$$\mathfrak{D}_{s^+}^\alpha x(\tau) = \begin{cases} \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(\tau q^{-n}) \\ \frac{1}{\Gamma(-\alpha)} \int_0^\tau \left[x(u) - \sum_{n=0}^{N-1} \frac{(-1)^n \mathfrak{D}_{s^+}^N x(\tau)}{n!} \ln^n(\tau/u) \right] \ln^{-\alpha-1}(\tau/u) \frac{du}{u} \end{cases} \tag{4}$$

(the last expression is valid for any real order, provided that we assume the summation to be null for $N \leq 0$). These relations can alternatively be expressed by the Hadamard derivatives:

1. Hadamard right derivative [17,24]

$$\mathfrak{D}_{s^+}^\alpha x(\tau) = \frac{1}{\Gamma(N - \alpha)} \mathfrak{D}_{s^+}^N \int_1^\infty x(\tau/\eta) \ln^{N-\alpha-1}(\eta) \frac{d\eta}{\eta}. \tag{5}$$

2. Hadamard–Liouville right derivative [16]

$$\mathfrak{D}_{s^+}^\alpha x(\tau) = \frac{1}{\Gamma(N - \alpha)} \int_1^\infty [\mathfrak{D}_{s^+}^N x(\tau)] \ln^{N-\alpha-1}(\tau/u) \frac{du}{u}.$$

These derivatives are equivalent from the Mellin transform point of view, although not from numerical aspects.

2.2.2. Shrinking Derivatives: $Re(v) < 0$ [16]

We have again two ways of expressing v^α :

$$v^\alpha = \begin{cases} \lim_{q \rightarrow 1^+} (-1)^\alpha \left[\frac{(1 - q^v)}{\ln q} \right]^\alpha = \sum_{n=0}^\infty \frac{(-\alpha)_n}{n!} q^{nv} \\ \mathcal{M} \left[\frac{\ln^{-\alpha-1}(1/\tau)}{\Gamma(-\alpha)} \varepsilon(1 - \tau) \right] \end{cases}$$

These expressions lead to the following scale derivative [16]:

$$\mathfrak{D}_{s-}^\alpha x(\tau) = \begin{cases} \lim_{q \rightarrow 1^+} (-1)^\alpha \ln^{-\alpha}(q) \sum_{n=0}^\infty \frac{(-\alpha)_n}{n!} x(\tau q^n) \\ \frac{1}{\Gamma(-\alpha)} \int_\tau^\infty \left[x(u) - \sum_{n=0}^{N-1} \frac{\mathfrak{D}_{s-}^N x(\tau)}{n!} \ln^n(u/\tau) \right] \ln^{-\alpha-1}(u/\tau) \frac{du}{u} \end{cases} \tag{6}$$

that can alternatively be expressed by the Hadamard derivatives

1. Hadamard left derivative [17,24]

$$\mathfrak{D}_{s-}^\alpha x(\tau) = \frac{1}{\Gamma(N - \alpha)} \mathfrak{D}_{s-}^N \int_0^1 x(\tau/\eta) \ln^{N-\alpha-1}(1/\eta) \frac{d\eta}{\eta}.$$

2. Hadamard–Liouville left derivative [16]

$$\mathfrak{D}_{s-}^\alpha x(\tau) = \frac{1}{\Gamma(N - \alpha)} \int_0^1 \left[\mathfrak{D}_{s-}^N x(\tau) \right] \ln^{N-\alpha-1}(u/\tau) \frac{du}{u}.$$

2.3. ARMA Type Systems

Definition 3. We define the dilation (scale)-invariant fractional autoregressive-moving average (DI-FARMA) system through

$$\sum_{k=0}^{N_0} a_k \mathfrak{D}_{s\pm}^{\alpha_k} y(\tau) = \sum_{k=0}^{M_0} b_k \mathfrak{D}_{s\pm}^{\beta_k} x(\tau), \quad \tau \in \mathbb{R}^+, \tag{7}$$

where $\mathfrak{D}_{s\pm}^{\alpha_k, (\beta_k)}$, $k = 0, 1, 2, \dots$, mean the fractional $\alpha_k (\beta_k)$ -order scale-derivatives and N_0, M_0 are the system orders. The parameters a_k, b_k , $k = 0, 1, \dots$, are considered real numbers. Without losing generality, we set $a_{N_0} = 1$.

The results in [25] can be easily adapted. As the power τ^v is the eigenfunction of (7), we obtain easily the transfer function

$$G(v) = \frac{\sum_{k=0}^{M_0} b_k v^{\beta_k}}{\sum_{k=0}^{N_0} a_k v^{\alpha_k}},$$

which is a bit difficult to manipulate [26]. In the following, we shall be considering the so-called “commensurate” systems described by differential equations with the format

$$\sum_{k=0}^{N_0} a_k \mathfrak{D}_{s\pm}^{k\alpha} y(\tau) = \sum_{k=0}^{M_0} b_k \mathfrak{D}_{s\pm}^{k\alpha} x(\tau), \quad \tau \in \mathbb{R}^+.$$

The corresponding transfer function is

$$G(v) = \frac{\sum_{k=0}^{M_0} b_k v^{k\alpha}}{\sum_{k=0}^{N_0} a_k v^{k\alpha}},$$

where we assume that $M_0 < N_0$, for simplicity. The objective of our work is to find the impulse and step responses of systems. As expected, we have two cases in agreement with the assumed ROC.

3. The Algebraic Framework for Solving DI-FARMA Systems

3.1. Sequence of Basic Functions for Fractional Scale Derivatives

Definition 4. Let S be an operator and $\{u_k(\tau)\}_{k \in \mathbb{Z}}$ a sequence of functions, verifying

$$S u_k(\tau) = u_{k-1}(\tau), \quad k \in \mathbb{Z}. \tag{8}$$

We say that S is decremental index operator of the sequence of functions $\{u_k(\tau)\}_{k \in \mathbb{Z}}$.

Remark 1. A well known example involves the generalized shift-invariant derivatives, D , and the power functions, stating

$$D \frac{t^n}{n!} \varepsilon(t) = \frac{t^{n-1}}{(n-1)!} \varepsilon(t), \quad n \in \mathbb{Z}$$

It has been extended and studied by several researchers [27–29].

The main goal in this section is to find sequence of functions $\{u_k(\tau)\}_{k \in \mathbb{Z}}$ verifying (8) for the scale derivatives, mainly the Hadamard’s.

Hadamard Right (Left) Derivative

In the following we shall be addressing the stretching derivative case ($Re(v) > 0$); the case of the shrinking derivative ($Re(v) < 0$) is analogous. We remember the result shown in Section 2.2.1

$$\mathcal{M}^{-1}[v^{-\alpha}] (\tau) = \frac{\ln^{\alpha-1}(\tau)}{\Gamma(\alpha)} \varepsilon(\tau - 1), \quad Re(v) > 0, \tag{9}$$

which will be used in this subsection.

Theorem 1. Let $u_k(\tau)$ be given by

$$u_k(\tau) = \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau - 1), \quad k \in \mathbb{N}. \tag{10}$$

Then,

$$\mathfrak{D}_{s+}^{\alpha} u_k(\tau) = u_{k-1}(\tau), \quad k \in \mathbb{N}. \tag{11}$$

Proof. Let $\alpha > 0, 1 \leq \tau$ and $k \in \mathbb{N}$. Using (9) with order $k\alpha$,

$$\mathcal{M} \left[\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau - 1) \right] = v^{-k\alpha}, \quad \alpha > 0, \quad k \geq 1. \tag{12}$$

Following (2), the derivative $\mathfrak{D}_{s+}^\alpha \left(\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) \right)$ is given by

$$\begin{aligned} \mathfrak{D}_{s+}^\alpha \left(\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) \right) &= \mathcal{M}^{-1} \left[v^\alpha \mathcal{M} \left[\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) \right] \right] (\tau) \\ &= \mathcal{M}^{-1} \left[v^{-(k-1)\alpha} \right] (\tau) \end{aligned}$$

Finally, from (12) we obtain that

$$\mathfrak{D}_{s+}^\alpha \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) = \frac{\ln^{(k-1)\alpha-1}(\tau)}{\Gamma((k-1)\alpha)} \varepsilon(\tau-1).$$

for $\alpha > 0$ and $k \geq 1$. \square

Remark 2. For the next result, we need to observe that [30]

$$u_0(\tau) = \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) = \delta(\ln(\tau)) \varepsilon(\tau-1) = \delta(\tau-1).$$

Lemma 1. Let $\alpha > 0$ and $1 \leq \tau$. For $u_0(\tau) = \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1)$,

$$\mathfrak{D}_{s+}^\alpha u_0(\tau) = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} \varepsilon(\tau-1).$$

Proof. From Remark 2 and (5) we obtain

$$\mathfrak{D}_{s+}^\alpha u_0(\tau) = \frac{1}{\Gamma(N-\alpha)} \mathfrak{D}_{s+}^N \int_1^\infty \delta\left(\frac{\tau}{\eta} - 1\right) \ln^{N-\alpha-1}(\eta) \frac{d\eta}{\eta}$$

Performing the variable change $\omega = \frac{\tau}{\eta}$ we have that

$$\int_1^\infty \delta\left(\frac{\tau}{\eta} - 1\right) \ln^{N-\alpha-1}(\eta) \frac{d\eta}{\eta} = \int_0^\tau \delta(\omega - 1) \ln^{N-\alpha-1}\left(\frac{\tau}{\omega}\right) \frac{d\omega}{\omega}.$$

It follows that

$$\frac{1}{\Gamma(N-\alpha)} \mathfrak{D}_{s+}^N \int_0^\tau \delta(\omega - 1) \ln^{N-\alpha-1}\left(\frac{\tau}{\omega}\right) \frac{d\omega}{\omega} = 0, \quad \tau < 1$$

and

$$\frac{1}{\Gamma(N-\alpha)} \mathfrak{D}_{s+}^N \int_0^\tau \delta(\omega - 1) \ln^{N-\alpha-1}\left(\frac{\tau}{\omega}\right) \frac{d\omega}{\omega} = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)}, \quad \tau \geq 1.$$

Therefore

$$\mathfrak{D}_{s+}^\alpha u_0(\tau) = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} \varepsilon(\tau-1).$$

\square

The previous Lemma tells us that $\mathfrak{D}_{s+}^\alpha u_0(\tau) = u_{-1}(\tau)$. So, we wonder what about $\mathfrak{D}_{s+}^\alpha u_{-1}(\tau)$. For this, from the additivity of operator \mathfrak{D}_{s+}^α we have that

$$\begin{aligned} \mathfrak{D}_{s+}^\alpha u_{-1}(\tau) &= \mathfrak{D}_{s+}^\alpha (\mathfrak{D}_{s+}^\alpha u_0(\tau)) \\ &= \mathfrak{D}_{s+}^{2\alpha} u_0(\tau) \\ &= \frac{\ln^{-2\alpha-1}(\tau)}{\Gamma(-2\alpha)} \varepsilon(\tau - 1) \\ &= u_{-2}(\tau) \end{aligned} \tag{13}$$

The penultimate equality follows from Lemma 1 with the order of the derivative equal to 2α . Following this reasoning we obtain that $\mathfrak{D}_{s+}^\alpha u_{-k}(\tau) = \mathfrak{D}_{s+}^\alpha u_{-k-1}(\tau), k \in \mathbb{N}$. Finally, we conclude that

Corollary 1.

$$\mathfrak{D}_{s+}^\alpha u_k(\tau) = \mathfrak{D}_{s+}^\alpha u_{k-1}(\tau), \text{ for any } k \in \mathbb{Z}. \tag{14}$$

Therefore Hadamard right (left) derivative is a decremental index operator on the sequence of functions (10).

Remark 3. The condition $\text{Re}(v) < 0$ leads to the $0 < \tau \leq 1$ case. The construction of the $u_k(\tau)$'s is similar to case $1 \leq \tau$. Here we get that

$$u_k(\tau) = \frac{\ln^{k\alpha-1}\left(\frac{1}{\tau}\right)}{\Gamma(k\alpha)} \varepsilon(1 - \tau).$$

It is not difficult to verify that

$$\mathfrak{D}_{s+}^\alpha u_k(\tau) = u_{k-1}(\tau), \text{ for any } k \in \mathbb{Z}.$$

Remark 4. As we previously mentioned the Hadamard right (left) derivative and scale derivative of type Grünwald-Letnikov are equivalent. Despite this, in Appendix A we verify that the scale derivative of type Grünwald-Letnikov is a decremental index operators on the same sequence of functions (10).

In some situations, it is more convenient to use the step response instead of the impulse response because this one has a singularity at $\tau = 1$. Therefore, another definition of $u_k(\tau)$ for which both Hadamard right (left) derivative and fractional scale derivative of type Grünwald-Letnikov are decremental index operators is possible and desired. For $1 \leq \tau$

$$u_k(\tau) = \frac{\ln^{k\alpha}(\tau)}{\Gamma(k\alpha + 1)} \varepsilon(\tau - 1), \quad k \in \mathbb{Z},$$

and when $0 < \tau \leq 1$

$$u_k(\tau) = \frac{\ln^{k\alpha}\left(\frac{1}{\tau}\right)}{\Gamma(k\alpha + 1)} \varepsilon(1 - \tau), \quad k \in \mathbb{Z}.$$

The justification can be seen in Appendix B.

3.2. The Framework

Let $u_k(\tau), k \in \mathbb{Z}$, having Mellin transform $U_k(v)$. Then, for any $k, m \in \mathbb{N}$

$$\mathcal{M}[u_k(\tau) \star u_m(\tau)](v) = U_k(v)U_m(v).$$

As $U_k(v) = v^{k\alpha}$ and $U_m(v) = v^{m\alpha}$, hence

$$U_k(v)U_m(v) = v^{(k+m)\alpha}, \quad \text{Re}(v) > 0,$$

and

$$u_k(\tau) \star u_m(\tau) = \mathcal{M}^{-1} \left[v^{(k+m)\alpha} \right] = u_{k+m}(\tau).$$

Therefore, we extend this product to any $k, m \in \mathbb{Z}$ as follows

$$u_k(\tau) \star u_m(\tau) = u_{k+m}(\tau). \tag{15}$$

Remark 5. Observe that

$$\mathfrak{D}_{s+}^{n\alpha} u_k(\tau) = u_{-n}(\tau) \star u_k(\tau) = u_{k-n}(\tau), \quad k \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

Consider $\{u_k(\tau)\}_{k \in \mathbb{Z}}$ as a sequence of basic functions. Let \mathcal{F} be the set of all the formal Laurent series

$$\mathcal{F} = \left\{ \sum_{k=-n}^{\infty} c_k u_k(\tau) : n \in \mathbb{N}_0, c_k \in \mathbb{C} \right\}.$$

Several properties of \mathcal{F} and the product (15) can be founded in [20,31]. Observe that

- $u_0(\tau)$ is the neutral element of the Mellin convolution, $u_0(\tau) \star u_k(\tau) = u_k(\tau)$;
- the inverse element of $u_k(\tau)$ is $u_{-k}(\tau)$, $u_k(\tau) \star u_{-k}(\tau) = u_0(\tau)$.

By means of Cauchy series product, we can extend the Mellin product, \star , to any elements of \mathcal{F} as follows. Let a_j, b_j , two sequences of real numbers and define

$$a = \sum_{j=-k_a}^{\infty} a_j u_j(\tau),$$

and

$$b = \sum_{j=-k_b}^{\infty} b_j u_j(\tau),$$

as elements of \mathcal{F} . Then

$$a \star b = f = \sum_{n=-k_a-k_b}^{\infty} f_n u_n(\tau),$$

where

$$f_n = \sum_{-k_a \leq k \leq n+k_b} a_k b_{n-k}.$$

This product is associative and commutative. Under this multiplication \mathcal{F} is a field.

Definition 5. Let us define the function

$$\mathcal{E}_{\gamma,n}(\tau) = \sum_{k=n+1}^{\infty} \binom{k-1}{n} \gamma^{k-n-1} u_k(\tau), \quad \gamma \in \mathbb{C}, n \in \mathbb{N}_0.$$

that we will call α -log-exponential function.

We could define an analogue to the Mittag-Leffler function [16], but this one is more useful.

Let us introduce also the sequential convolution

$$(u_{-1}(\tau) - \gamma u_0(\tau))_{\star}^{m+1} = \underbrace{(u_{-1}(\tau) - \gamma u_0(\tau)) \star \cdots \star (u_{-1}(\tau) - \gamma u_0(\tau))}_{m+1\text{-times}}.$$

It is not difficult to verify that

$$(u_{-1}(\tau) - \gamma u_0(\tau)) \star \mathcal{E}_{\gamma,0}(\tau) = u_0(\tau),$$

and in general,

$$(u_{-1}(\tau) - \gamma u_0(\tau))_{\star}^{n+1} \star \mathcal{E}_{\gamma,n}(\tau) = u_0(\tau).$$

The α -log-exponential function has some useful properties [31]:

1. Derivative on a parameter

$$\frac{D_{\gamma}^n}{n!} \mathcal{E}_{\gamma,0}(\tau) = \mathcal{E}_{\gamma,n}(\tau),$$

where D_{γ} means usual derivative with respect to γ .

2. Convolution of two different α -log-exponential functions, but with the same parameter γ

$$\mathcal{E}_{\gamma,m}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \mathcal{E}_{\gamma,m+n+1}(\tau).$$

3. Convolution of two different γ parameters α -log-exponential functions

For $\gamma_1 \neq \gamma_2$,

$$\mathcal{E}_{\gamma_1,0}(\tau) \star \mathcal{E}_{\gamma_2,0}(\tau) = \frac{\mathcal{E}_{\gamma_1,0}(\tau) - \mathcal{E}_{\gamma_2,0}(\tau)}{\gamma_1 - \gamma_2},$$

$$\mathcal{E}_{\gamma_1,m}(\tau) \star \mathcal{E}_{\gamma_2,n}(\tau) = \sum_{l=0}^m \frac{\binom{n+l}{l}(-1)^l}{(\gamma_1 - \gamma_2)^{1+n+l}} \mathcal{E}_{\gamma_1,m-l}(\tau) + \sum_{k=0}^n \frac{\binom{m+k}{k}(-1)^k}{(\gamma_2 - \gamma_1)^{1+m+k}} \mathcal{E}_{\gamma_2,n-k}(\tau).$$

Next, we will introduce and prove some other properties which will allow us to deduce that functions $\mathcal{E}_{\gamma,m}(\tau)$ are the generating elements of solution space of fractional scale-invariant systems.

Theorem 2. Let $n \geq 1$. Then

$$u_{-1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \mathcal{E}_{\gamma,n-1}(\tau) + \gamma \mathcal{E}_{\gamma,n}(\tau).$$

Proof. Observe that

$$\begin{aligned} u_{-1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= u_{-1}(\tau) \star \sum_{k=n+1}^{\infty} \binom{k-1}{n} \gamma^{k-n-1} u_k(\tau) \\ &= \sum_{k=n+1}^{\infty} \binom{k-1}{n} \gamma^{k-n-1} u_{k-1}(\tau). \end{aligned}$$

Using the basic recurrence $\binom{k-1}{n} = \binom{k-2}{n-1} + \binom{k-2}{n}$, we obtain

$$\begin{aligned} u_{-1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= \sum_{k=n+1}^{\infty} \left[\binom{k-2}{n-1} + \binom{k-2}{n} \right] \gamma^{k-n-1} u_{k-1}(\tau) \\ &= \sum_{k=n+1}^{\infty} \binom{k-2}{n-1} \gamma^{k-n-1} u_{k-1}(\tau) + \sum_{k=n+2}^{\infty} \binom{k-2}{n} \gamma^{k-n-1} u_{k-1}(\tau) \\ &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} \gamma^{k-n} u_k(\tau) + \gamma \sum_{k=n+1}^{\infty} \binom{k-1}{n} \gamma^{k-n-1} u_k(\tau) \\ &= \mathcal{E}_{\gamma,n-1}(\tau) + \gamma \mathcal{E}_{\gamma,n}(\tau). \end{aligned}$$

□

Remark 6. Observe that letting $n = 0$ in Theorem 2 we get

$$u_{-1}(\tau) \star \mathcal{E}_{\gamma,0}(\tau) = u_0(\tau) + \gamma \mathcal{E}_{\gamma,0}(\tau) \tag{16}$$

Theorem 3. Let N and n be positive integers such that $N \leq n$. Then we have

$$u_{-N}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \sum_{k=0}^N \binom{N}{k} \gamma^{N-k} \mathcal{E}_{\gamma,n-k}(\tau).$$

Proof. The proof is done by induction with $N \leq n$. For the base step of the induction ($N = 1$), we appeal to Lemma 2. For $N = 2$, due to $u_{-2}(\tau) = u_{-1}(\tau) \star u_{-1}(\tau)$ we have that

$$\begin{aligned} u_{-2}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= u_{-1}(\tau) \star (u_{-1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau)) \\ &= u_{-1}(\tau) \star (\mathcal{E}_{\gamma,n-1}(\tau) + \gamma \mathcal{E}_{\gamma,n}(\tau)) \\ &= \gamma^2 \mathcal{E}_{\gamma,n}(\tau) + 2\gamma \mathcal{E}_{\gamma,n-1}(\tau) + \mathcal{E}_{\gamma,n-2}(\tau). \end{aligned}$$

Suppose that theorem is valid for $N - 1$. This is

$$u_{-N+1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \sum_{k=0}^{N-1} \binom{N-1}{k} \gamma^{N-k-1} \mathcal{E}_{\gamma,n-k}(\tau). \tag{17}$$

Now, we will prove that theorem is valid for N . Observe that

$$\begin{aligned} u_{-N}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= u_{-1}(\tau) \star (u_{-N+1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau)) \\ &= \sum_{k=0}^{N-1} \binom{N-1}{k} \gamma^{N-k-1} (u_{-1}(\tau) \star \mathcal{E}_{\gamma,n-k}(\tau)) \\ &= \sum_{k=0}^{N-1} \binom{N-1}{k} \gamma^{N-k-1} (\mathcal{E}_{\gamma,n-k-1}(\tau) + \gamma \mathcal{E}_{\gamma,n-k}(\tau)) \\ &= \sum_{k=0}^N \binom{N}{k} \gamma^{N-k} \mathcal{E}_{\gamma,n-k}(\tau). \end{aligned} \tag{18}$$

□

Theorem 4. Let $l \in \mathbb{N}$. Then

$$u_{-n-l}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \sum_{k=n+1}^{n+l} \binom{k-1}{n} \gamma^{k-n-1} u_{k-n-l}(\tau) + \sum_{k=0}^n \binom{n+l}{n-k} \gamma^{k+l} \mathcal{E}_{\gamma,k}(\tau). \tag{19}$$

Proof. The proof is by induction on $l \in \mathbb{N}$. For the base step of the induction ($l = 1$) we have that

$$\begin{aligned} u_{-n-1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= u_{-1}(\tau) \star (u_{-n}(\tau) \star \mathcal{E}_{\gamma,n}(\tau)) \\ &= u_{-1}(\tau) \star \left(\sum_{k=0}^n \binom{n}{k} \gamma^k \mathcal{E}_{\gamma,k}(\tau) \right) \\ &= u_0(\tau) + \gamma \mathcal{E}_{\gamma,0}(\tau) + \sum_{k=1}^n \binom{n}{k} \gamma^k (\mathcal{E}_{\gamma,k-1}(\tau) + \gamma \mathcal{E}_{\gamma,k}(\tau)) \\ &= u_0(\tau) + \sum_{k=0}^n \binom{n+1}{k+1} \gamma^{k+1} \mathcal{E}_{\gamma,k}(\tau). \end{aligned}$$

In the penultimate equality we apply the Remark 6 to the first term of the sum. Suppose that the theorem is valid for $l - 1$, this is

$$u_{-n-l+1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) = \sum_{k=n+1}^{n+l-1} \binom{k-1}{n} \gamma^{k-n-1} u_{k-n-l+1}(\tau) + \sum_{k=0}^n \binom{n+l-1}{n-k} \gamma^{k+l-1} \mathcal{E}_{\gamma,k}(\tau). \tag{20}$$

Now, we will prove that theorem is valid for l . Observe that

$$\begin{aligned} u_{-n-l}(\tau) \star \mathcal{E}_{\gamma,n}(\tau) &= u_{-1}(\tau) \star (u_{-n-l+1}(\tau) \star \mathcal{E}_{\gamma,n}(\tau)) \\ &= \sum_{k=n+1}^{n+l-1} \binom{k-1}{n} \gamma^{k-n-1} u_{k-n-l}(\tau) + \sum_{k=0}^n \binom{n+l-1}{n-k} \gamma^{k+l-1} (u_{-1}(\tau) \star \mathcal{E}_{\gamma,k}(\tau)) \\ &= \sum_{k=n+1}^{n+l} \binom{k-1}{n} \gamma^{k-n-1} u_{k-n-l}(\tau) + \sum_{k=0}^n \binom{n+l}{n-k} \gamma^{k+l} \mathcal{E}_{\gamma,k}(\tau). \end{aligned} \tag{21}$$

Again, Remark 6 is applied in the last equality. \square

Theorem 4 suggests that impulse (step) response to a fractional scale-invariant system is a linear combination of functions $\mathcal{E}_{\gamma,n}(t)$.

4. Impulse and Step Responses

4.1. The AR Case

Consider an AR system. Its impulse response is given by the solution of the equation

$$p(\mathfrak{D}_{s+}^{\alpha})y(\tau) = \delta(\tau - 1), \tag{22}$$

where $p(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, $a_n \in \mathbb{C}$, $a_m = 1$, and its roots $\gamma_1, \gamma_2, \dots, \gamma_m$ have multiplicity 1. Consider that $y(\tau) \in \mathcal{F}$. In terms of the convolution \star , the Equation (22) can be rewritten as

$$(a_m u_{-m}(\tau) + a_{m-1} u_{-m+1}(\tau) + \dots + a_1 u_{-1}(\tau) + a_0 u_0(\tau)) \star y(\tau) = u_0(\tau).$$

Assume that impulse response is given by a linear combination of α -log-exponential functions

$$r_{\delta}(\tau) = c_1 \mathcal{E}_{\gamma_1,0}(\tau) + \dots + c_m \mathcal{E}_{\gamma_m,0}(\tau).$$

From (19), it is not difficult to deduce that

$$u_{-m}(\tau) \star \mathcal{E}_{\gamma_i,0}(\tau) = \sum_{k=1}^m \gamma_1^{k-1} u_{-m+k}(\tau) + \gamma_i^m \mathcal{E}_{\gamma_i,0}(t), \quad \text{for any } m \in \mathbb{N}.$$

A simple computation leads to

$$(a_m u_{-m}(\tau) + a_{m-1} u_{-m+1}(\tau) + \dots + a_0 u_0(\tau)) \star r_{\delta}(\tau) = \sum_{k=1}^m c_k \sum_{j=0}^{m-1} \sum_{i=0}^{m-j-1} a_{i+1} \gamma_k^i u_{-j}(\tau). \tag{23}$$

Now, in order to find the impulse response we obtain a linear system with m equations which can be reduced recursively to

$$\begin{aligned} c_1 + c_2 + c_3 + \dots + c_m &= 0 \\ \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3 + \dots + \gamma_m c_m &= 0 \\ &\vdots \\ \gamma_1^{m-1} c_1 + \gamma_2^{m-1} c_2 + \gamma_3^{m-1} c_3 + \dots + \gamma_m^{m-1} c_m &= 1. \end{aligned}$$

The coefficient matrix is invertible, since it has Vandermonde format and the roots γ_j are distinct. Therefore, the system has a unique solution. Finally, using the c_i 's, we obtain $r_\delta(\tau)$, the solution to fractional system (22).

For the case of roots with multiplicity greater than one, we must propose an alternative solution. Consider that a given root, γ_k , has multiplicity $m_k > 1$. To obtain the solution, we need to add another linear combination of α -log-exponentials to the previous solution

$$c_{k,0}\mathcal{E}_{\gamma_k,0}(\tau) + c_{k,1}\mathcal{E}_{\gamma_k,1}(\tau) + \dots + c_{k,m_k}\mathcal{E}_{\gamma_k,m_k}(\tau)$$

With the obtained guess of the solution we are led to a linear system with m equations. The coefficient matrix is of generalized Vandermonde-type [32] having non null determinant. To fix ideas, consider the system (22) but with $\gamma_4, \gamma_5, \dots, \gamma_m$ simple roots of $p(x)$ and γ_1 a root with multiplicity 3 ($\gamma_1 = \gamma_2 = \gamma_3$). The proposed solution assumes the form

$$r_\delta(\tau) = c_1\mathcal{E}_{\gamma_1,0}(\tau) + c_2\mathcal{E}_{\gamma_2,1}(\tau) + c_3\mathcal{E}_{\gamma_3,2}(\tau) + c_4\mathcal{E}_{\gamma_4,0}(\tau) + \dots + c_m\mathcal{E}_{\gamma_m,0}(\tau)$$

As in the previous case, from

$$(a_m u_{-m}(\tau) + a_{m-1} u_{-m+1}(\tau) + \dots + a_1 u_{-1}(\tau) + a_0 u_0(\tau)) \star r_\delta(\tau) = u_0(\tau)$$

we obtain a system of m linear equations that can be recursively reduced to the following system

$$\begin{aligned} c_1 + c_4 + \dots + c_m &= 0 \\ \gamma_1 c_1 + c_2 + \gamma_4 c_4 + \dots + \gamma_m c_m &= 0 \\ \gamma_1^2 c_1 + 2\gamma_2 c_2 + \gamma_3 c_3 + \gamma_4^2 c_4 + \dots + \gamma_m^2 c_m &= 0 \\ &\vdots \\ &= \vdots \\ \gamma_1^{m-1} c_1 + (m-1)\gamma_2^{m-2} c_2 + \frac{(m-1)(m-2)}{2!} \gamma_3^{m-1} c_3 + \gamma_4^{m-1} c_4 + \dots + \gamma_m^{m-1} c_m &= 1. \end{aligned}$$

This system has a coefficient matrix of generalized Vandermonde-type whose determinant is equal to

$$\prod_{3 < i < j \leq m} (\gamma_i - \gamma_j)^{m_i m_j},$$

where m_i, m_j are the multiplicities of the roots γ_i, γ_j , respectively. It follows that the system has a unique solution.

We have proven that the solution to system (22) is an element of vector space generated by the set

$$\{\mathcal{E}_{\gamma,n}(\tau) : \gamma \in \mathbb{C}, n \in \mathbb{N}_0\}. \tag{24}$$

Furthermore, in (p. 338, [31]) it is proved that the set (24) is linearly independent.

4.2. The ARMA Case

By means of the method above described, we can solve the more general problem

$$p(\mathcal{D}_{s+}^\alpha)y(\tau) = q(\mathcal{D}_{s+}^\alpha)\delta(\tau - 1),$$

where $p(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$ and $q(x) = b_r x^r + b_{r-1} x^{r-1} + \dots + b_1 x + b_0$ are polynomials of degree m and r , respectively, with constant coefficients in \mathbb{C} . We proceed as follows. Firstly, we compute the impulsive response $r_\delta(\tau)$. In different words, we solve the fractional system

$$p(\mathcal{D}_{s+}^\alpha)y(\tau) = \delta(\tau - 1).$$

Later, the solution is given by the convolution

$$y(\tau) = (b_r u_{-r}(\tau) + b_{r-1} u_{-r+1}(\tau) + \dots + b_1 u_{-1}(\tau) + b_0 u_0(\tau)) \star r_\delta(\tau).$$

This is known as cascade connection of two systems

$$\begin{aligned} \text{System } S_1 : & \quad p(\mathfrak{D}_{s+}^\alpha)r_\delta(\tau) = \delta(\tau - 1) \\ \text{System } S_2 : & \quad y(\tau) = q(\mathfrak{D}_{s+}^\alpha)r_\delta(\tau) \end{aligned} \tag{25}$$

5. Examples

In all the following examples, we will compute both impulse and step response, by means of operational method. In order to compare ours with the classic transform method, we include the computation of impulse response in Example 1 using the Mellin transform.

Example 1. Consider the fractional scale-invariant linear system

$$p(\mathfrak{D}_{s+}^\alpha)y(\tau) = \delta(\tau - 1). \tag{26}$$

with $p(x) = x^2 + 2$. The roots of $p(x)$ are $\sqrt{2}i$ and $-\sqrt{2}i$ of multiplicity 1.

- Operational method:
By means of our operational method, the system can be rewritten as

$$(u_{-2}(\tau) + 2u_0(\tau)) \star y(\tau) = u_0(\tau).$$

We propose the solution

$$y(\tau) = c_1\mathcal{E}_{\sqrt{2}i,0}(\tau) + c_2\mathcal{E}_{-\sqrt{2}i,0}(\tau).$$

Following the presented in Section 3.2 we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ \sqrt{2}ic_1 - \sqrt{2}ic_2 &= 1 \end{aligned} .$$

The solution to system of equations is $c_1 = \frac{1}{2\sqrt{2}i}$ and $c_2 = -\frac{1}{2\sqrt{2}i}$. Therefore, for $\tau \geq 1$ the impulse response is

$$\begin{aligned} r_\delta(\tau) &= \frac{1}{2\sqrt{2}i} \left(\sum_{k=1}^{\infty} \left((\sqrt{2}i)^{k-1} - (-\sqrt{2}i)^{k-1} \right) \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau - 1) \right) \\ &= \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha-1}(\tau)}{\Gamma(2k\alpha)} \varepsilon(\tau - 1), \end{aligned}$$

and step response

$$r_u(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha}(\tau)}{\Gamma(2k\alpha + 1)} \varepsilon(\tau - 1).$$

Remark 7. When $\alpha = 1$, $r_\delta(\tau) = \frac{1}{\sqrt{2}} \sin(\sqrt{2} \ln(\tau))$ and $r_u(\tau) = \frac{1 - \cos(\sqrt{2} \ln(\tau))}{2}$.

The Figures 1 and 2 show the graphical representation of the solutions with $\tau \geq 1$ and several values of α .

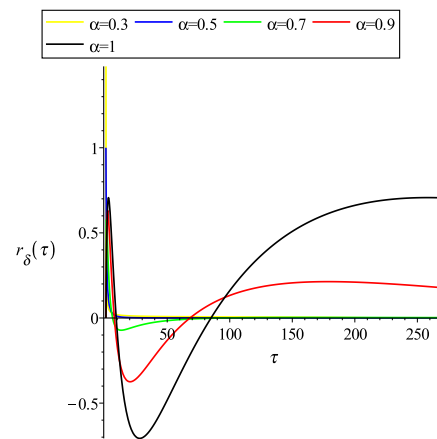


Figure 1. Impulse response of Example 1 with $1 \leq \tau$ and several values of α .

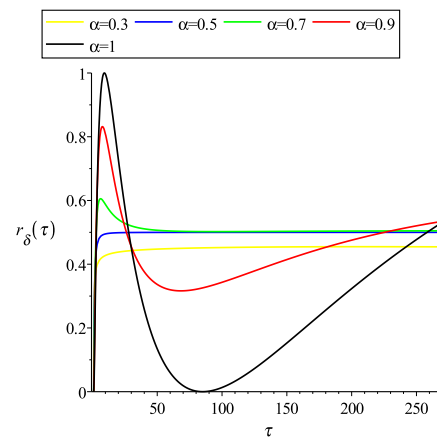


Figure 2. Step response of Example 1 with $1 \leq \tau$ and several values of α .

For $0 < \tau \leq 1$, the impulse response is

$$r_{\delta}(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha-1}\left(\frac{1}{\tau}\right)}{\Gamma(2k\alpha)} \varepsilon(1-\tau),$$

and step response

$$r_u(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha}\left(\frac{1}{\tau}\right)}{\Gamma(2k\alpha+1)} \varepsilon(1-\tau).$$

Remark 8. When $\alpha = 1$,

$$r_{\delta}(\tau) = \frac{1}{\sqrt{2}} \sin\left(\sqrt{2} \ln\left(\frac{1}{\tau}\right)\right) \varepsilon(1-\tau),$$

and

$$r_u(\tau) = \frac{1 - \cos\left(\sqrt{2} \ln\left(\frac{1}{\tau}\right)\right)}{2} \varepsilon(1-\tau).$$

The Figures 3 and 4 show the graphical representation of the solutions with $0 < \tau \leq 1$ and several values of α .

- Mellin transform:**
 We only consider the case $\tau \geq 1$, the case $0 < \tau \leq 1$ is analog. By means of Mellin transform, the system (26) can be transformed to equation

$$\begin{aligned}
 Y(v) &= \frac{1}{v^{2\alpha} + 2} \\
 &= \frac{i}{2\sqrt{2}} \frac{1}{v^\alpha + \sqrt{2}i} - \frac{i}{2\sqrt{2}} \frac{1}{v^\alpha - \sqrt{2}i} \\
 &= v^{-\alpha} \frac{\frac{i}{2\sqrt{2}}}{1 + \frac{\sqrt{2}i}{v^\alpha}} - v^{-\alpha} \frac{\frac{i}{2\sqrt{2}}}{1 - \frac{\sqrt{2}i}{v^\alpha}}
 \end{aligned}$$

Using the geometric series, with $\sqrt{2} < v^\alpha$, we obtain that

$$\begin{aligned}
 Y(v) &= \frac{i}{4} v^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \left(\frac{2i}{v^\alpha}\right)^k - \frac{i}{4} v^{-\alpha} \sum_{k=0}^{\infty} \left(\frac{2i}{v^\alpha}\right)^k \\
 &= \sum_{k=1}^{\infty} (-2)^{k-1} v^{-2k\alpha}.
 \end{aligned}$$

Computing the inverse Mellin transform we obtain the impulsive response

$$\begin{aligned}
 r_\delta(\tau) &= \mathcal{M}^{-1} \left[\sum_{k=1}^{\infty} (-2)^{k-1} v^{-2k\alpha} \right] \\
 &= \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha-1}(\tau)}{\Gamma(2k\alpha)} \varepsilon(\tau - 1).
 \end{aligned}$$

For the step response we apply (A1) and obtain

$$r_u(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha}(\tau)}{\Gamma(2k\alpha + 1)} \varepsilon(\tau - 1).$$

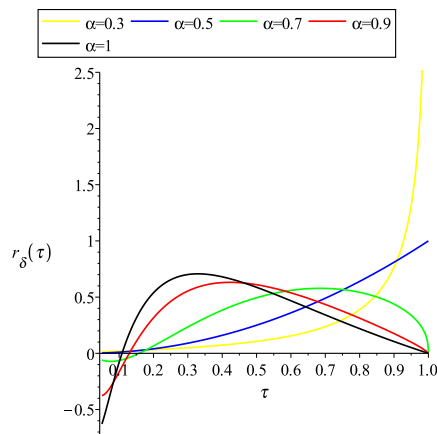


Figure 3. Impulse response of Example 1 with $0 < \tau \leq 1$ and several values of α .

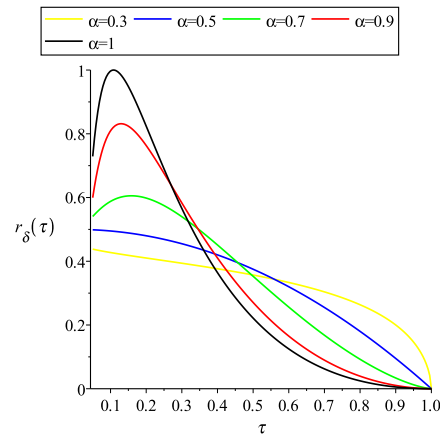


Figure 4. Step response of Example 1 with $0 < \tau \leq 1$ and several values of α .

Example 2. Consider the fractional scale-invariant linear system

$$p(\mathfrak{D}_{s+}^\alpha)y(\tau) = q(\mathfrak{D}_{s+}^\alpha)\delta(\tau - 1),$$

with $p(x) = x^2 + 2$ and $q(x) = x - \sqrt{2}$. Observe that system is related with the Example 1. Following (25) we only need to compute the convolution

$$y(\tau) = \left(u_{-1}(\tau) - \sqrt{2}u_0(\tau)\right) \star \left(\frac{1}{2\sqrt{2}i}\mathcal{E}_{\sqrt{2}i,0}(\tau) - \frac{1}{2\sqrt{2}i}\mathcal{E}_{-\sqrt{2}i,0}(\tau)\right).$$

Simplifying we obtain that

$$\begin{aligned} y(\tau) &= \left(u_{-1}(\tau) - \sqrt{2}u_0(\tau)\right) \star \left(\sum_{k=1}^{\infty} (-2)^{k-1} u_{2k}(\tau)\right) \\ &= \sum_{k=1}^{\infty} (-2)^{k-1} u_{2k-1}(\tau) - \sqrt{2} \sum_{k=1}^{\infty} (-2)^{k-1} u_{2k}(\tau). \end{aligned}$$

For $\tau \geq 1$, the impulse response is

$$r_\delta(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{(2k-1)\alpha-1}(\tau)}{\Gamma((2k-1)\alpha)} \varepsilon(\tau - 1) - \sqrt{2} \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha-1}(\tau)}{\Gamma(2k\alpha)} \varepsilon(\tau - 1),$$

and step response

$$r_u(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{(2k-1)\alpha}(\tau)}{\Gamma((2k-1)\alpha + 1)} \varepsilon(\tau - 1) - \sqrt{2} \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha}(\tau)}{\Gamma(2k\alpha + 1)} \varepsilon(\tau - 1).$$

The Figures 5 and 6 show the graphical representation of the solutions with $\tau \geq 1$ and several values of α .

For $0 < \tau \leq 1$, the impulse response is

$$r_\delta(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{(2k-1)\alpha-1}\left(\frac{1}{\tau}\right)}{\Gamma((2k-1)\alpha)} \varepsilon(1 - \tau) - \sqrt{2} \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha-1}\left(\frac{1}{\tau}\right)}{\Gamma(2k\alpha)} \varepsilon(1 - \tau),$$

and step response

$$r_u(\tau) = \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{(2k-1)\alpha}\left(\frac{1}{\tau}\right)}{\Gamma((2k-1)\alpha + 1)} \varepsilon(1 - \tau) - \sqrt{2} \sum_{k=1}^{\infty} (-2)^{k-1} \frac{\ln^{2k\alpha}\left(\frac{1}{\tau}\right)}{\Gamma(2k\alpha + 1)} \varepsilon(1 - \tau).$$

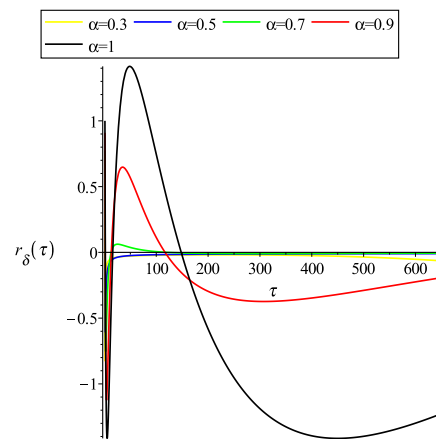


Figure 5. Impulse response of Example 2 with $1 \leq \tau$ and several values of α .

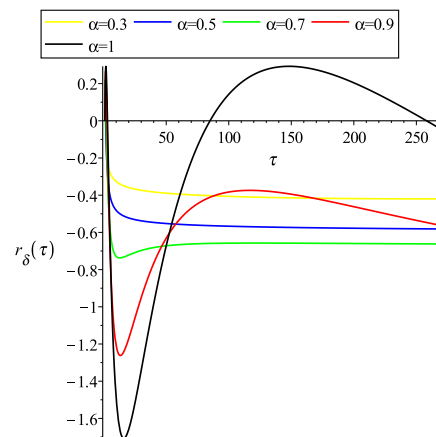


Figure 6. Step response of Example 2 with $1 \leq \tau$ and several values of α .

The Figures 7 and 8 show the graphical representation of the solutions with $0 < \tau \leq 1$ and several values of α .

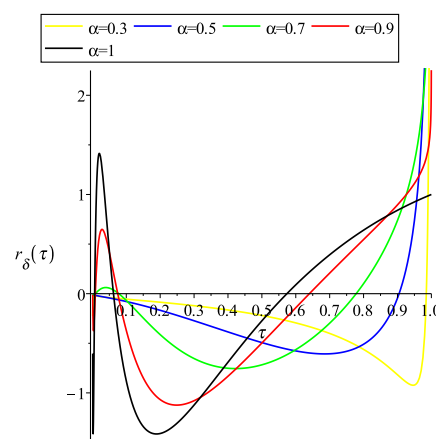


Figure 7. Impulse response of Example 2 with $0 < \tau \leq 1$ and several values of α .

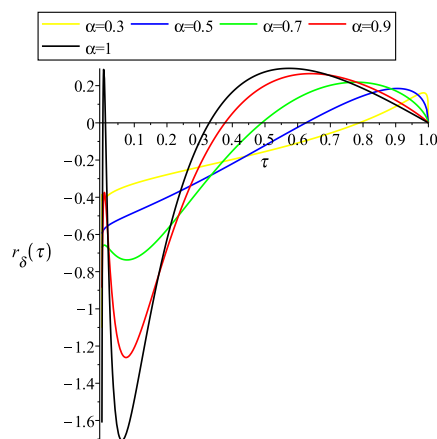


Figure 8. Step response of Example 2 with $0 < \tau \leq 1$ and several values of α .

6. Conclusions

We have proved that both Hadamard right (left) derivative and fractional scale derivative of type Grünwald-Letnikov are decremental index operators on the same sequence of functions $\{u_k\}_{k \in \mathbb{Z}}$. The Mellin convolution suggested to us how to define an algebraic product, which allowed us to construct a simple mathematical method to resolve fractional scale-invariant systems. The method relies on the roots of characteristic polynomial and the resolution of a linear system of equations. As expected, in our simulations, there is no convergence issue.

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Appendix A

The fractional scale derivatives of type Grünwald-Letnikov were defined in (4) and (6).

Definition A1. Let $q > 1$. The following expressions,

$$\mathfrak{D}_{s+}^\alpha x(\tau) = \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^\infty \frac{(-\alpha)_n}{n!} x(\tau q^{-n}),$$

and

$$\mathfrak{D}_{s-}^\alpha x(\tau) = \lim_{q \rightarrow 1^+} (-1)^\alpha \ln^{-\alpha}(q) \sum_{n=0}^\infty \frac{(-\alpha)_n}{n!} x(\tau q^n),$$

represent scale-derivatives that we can call stretching and shrinking Grünwald-Letnikov type derivatives, respectively.

Suppose that $\tau > 1$ and $k \geq 1$. Hence

$$\mathfrak{D}_{s+}^\alpha \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) = \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^\infty \frac{(-\alpha)_n}{n!} \frac{\ln^{k\alpha-1}(\tau q^n)}{\Gamma(k\alpha)}.$$

Using the Mellin transform and its inverse the previous relation can be rewritten as

$$\begin{aligned} \mathfrak{D}_{s+}^{\alpha} \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) &= \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} \frac{1}{2\pi i} \int_{\gamma} \mathcal{M} \left[\frac{\ln^{k\alpha-1}(\tau q^{-n})}{\Gamma(k\alpha)} \right] \tau^v dv \\ &= \frac{1}{2\pi i} \int_{\gamma} \lim_{q \rightarrow 1^+} \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{-nv}}{\ln^{\alpha}(q)} \mathcal{M} \left[\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \right] \tau^v dv. \end{aligned}$$

By binomial theorem and properties of limits it is not difficult to verify that

$$v^{\alpha} = \left[\frac{(1 - q^{-v})}{\ln(q)} \right]^{\alpha} = \lim_{q \rightarrow 1^+} \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{-nv}}{\ln^{\alpha}(q)}, \quad \text{Re}(v) > 0.$$

It follows that

$$\begin{aligned} \mathfrak{D}_{s+}^{\alpha} \frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \varepsilon(\tau-1) &= \frac{1}{2\pi i} \int_{\gamma} v^{\alpha} \mathcal{M} \left[\frac{\ln^{k\alpha-1}(\tau)}{\Gamma(k\alpha)} \right] \tau^v dv \\ &= \frac{1}{2\pi i} \int_{\gamma} v^{-(k-1)\alpha} \tau^v dv \\ &= \frac{\ln^{(k-1)\alpha-1}(\tau)}{\Gamma((k-1)\alpha)} \varepsilon(\tau-1). \end{aligned}$$

The last equal follows from (3).

Theorem A1. Let $\alpha < 0$ and $\tau > 1$. Then

$$\mathfrak{D}_{s+}^{\alpha} \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} \varepsilon(\tau-1).$$

Proof. Observe that

$$\frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) = \delta(\tau-1).$$

Hence

$$\begin{aligned} \mathfrak{D}_{s+}^{\alpha} \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) &= \mathfrak{D}_{s+}^{\alpha} \delta(\tau-1) \\ &= \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} \delta(\tau q^{-n} - 1). \end{aligned}$$

By properties of delta function, $\delta(\tau q^{-n} - 1) = q^n \delta(\tau - q^n)$. It follows that

$$\mathfrak{D}_{s+}^{\alpha} \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) = \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^n \delta(\tau - q^n).$$

By Mellin transform we get that $\delta(\tau - q^n) = \frac{1}{2\pi i} \int_{\gamma} q^{n(v-1)} \tau^v dv$. So

$$\begin{aligned} \mathfrak{D}_{s+}^{\alpha} \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau-1) &= \lim_{q \rightarrow 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^n \frac{1}{2\pi i} \int_{\gamma} q^{n(v-1)} \tau^v dv \\ &= \frac{1}{2\pi i} \int_{\gamma} \lim_{q \rightarrow 1^+} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} \ln^{-\alpha}(q) q^{nv} \tau^v dv. \end{aligned}$$

By binomial Theorem and properties of limits we have that

$$\lim_{q \rightarrow 1^+} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} \ln^{-\alpha}(q) q^{nv} = \lim_{q \rightarrow 1^+} \left(\frac{q^v - 1}{\ln q} \right)^\alpha = v^\alpha.$$

Therefore

$$\begin{aligned} \mathfrak{D}_{s^+}^\alpha \frac{\ln^{-1}(\tau)}{\Gamma(0)} \varepsilon(\tau - 1) &= \frac{1}{2\pi i} \int_{\gamma} v^\alpha \tau^v dv \\ &= \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} \varepsilon(\tau - 1). \end{aligned}$$

□

Finally, by means of reasoning similar to (14) we obtain that

$$\mathfrak{D}_{s^+}^\alpha u_k(\tau) = \mathfrak{D}_{s^+}^\alpha u_{k-1}(\tau), \text{ for any } k \in \mathbb{Z}.$$

Remark A1. The case $0 < \tau \leq 1$ is similar to case $1 \leq \tau$.

Appendix B

Observe that (3) is valid for all $\alpha > 0$. If $\alpha > 0$, then $\alpha + 1 > 0$. So

$$\mathcal{M}^{-1} \left[v^{-(\alpha+1)} \right] (\tau) = \frac{\ln^\alpha(\tau)}{\Gamma(\alpha + 1)} \varepsilon(\tau - 1), \quad \text{Re}(v) > 0. \tag{A1}$$

The case $\text{Re}(v) < 0$ is analogous. With this inverse Mellin transform we can prove that

$$\mathfrak{D}_{s^+}^\alpha \frac{\ln^{k\alpha}(\tau)}{\Gamma(k\alpha + 1)} \varepsilon(\tau - 1) = \frac{\ln^{(k-1)\alpha}(\tau)}{\Gamma((k-1)\alpha + 1)} \varepsilon(\tau - 1), \quad k \in \mathbb{N}.$$

In the case when $k = 0$, it is true that

$$\mathfrak{D}_{s^+}^\alpha \varepsilon(\tau - 1) = \frac{\ln^{-\alpha}(\tau)}{\Gamma(-\alpha + 1)} \varepsilon(\tau - 1).$$

The proof runs as Lemma 1. The same argument as in (13) applies to show

$$\mathfrak{D}_{s^+}^\alpha \frac{\ln^{-k\alpha}(\tau)}{\Gamma(-k\alpha + 1)} \varepsilon(\tau - 1) = \frac{\ln^{(-k-1)\alpha}(\tau)}{\Gamma((-k-1)\alpha + 1)} \varepsilon(\tau - 1), \quad k \in \mathbb{N}.$$

Therefore the new definition is given by

$$u_k(\tau) = \frac{\ln^{k\alpha}(\tau)}{\Gamma(k\alpha + 1)} \varepsilon(\tau - 1).$$

References

1. Roberts, M.J. *Signals and Systems: Analysis Using Transform Methods and Matlab*; McGraw-Hill: New York, NY, USA, 2003.
2. Shmaliy, Y. *Continuous-Time Systems*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2007.
3. Braccini, C.; Gambardella, G. Form-invariant linear filtering: Theory and applications. *IEEE Trans. Acoust. Speech Signal Process.* **1986**, *34*, 1612–1628. [[CrossRef](#)]
4. Yazici, B.; Kashyap, R. A class of second-order stationary self-similar processes for 1/f phenomena. *IEEE Trans. Signal Process.* **1997**, *45*, 396–410. [[CrossRef](#)]
5. Ortigueira, M.D. On the fractional linear scale invariant systems. *IEEE Trans. Signal Process.* **2010**, *58*, 6406–6410. [[CrossRef](#)]
6. Nottale, L. The theory of scale relativity. *Int. J. Mod. Phys. A* **1992**, *7*, 4899–4936. [[CrossRef](#)]
7. Narasimha, R.; Rao, R. Modeling variable-bit-rate video traffic using linear scale-invariant systems. In *Wavelet and Independent Component Analysis Applications IX*; SPIE: Bellingham, WA, USA, 2002; Volume 4738, pp. 79–89.

8. Cresson, J. Scale relativity theory for one-dimensional non-differentiable manifolds. *Chaos Solitons Fractals* **2002**, *14*, 553–562. [[CrossRef](#)]
9. Cresson, J. Scale calculus and the Schrödinger equation. *J. Math. Phys.* **2003**, *44*, 4907–4938. [[CrossRef](#)]
10. Khaluf, Y.; Ferrante, E.; Simoens, P.; Huepe, C. Scale invariance in natural and artificial collective systems: A review. *J. R. Soc. Interface* **2017**, *14*, 20170662. [[CrossRef](#)] [[PubMed](#)]
11. Rao, R.; Zhao, W. Image modeling with linear scale-invariant systems. In *Wavelet Applications VI*; SPIE: Bellingham, WA, USA, 1999; Volume 3723, pp. 407–418.
12. Gorenflo, R.; Luchko, Y.; Mainardi, F. Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Comput. Appl. Math.* **2000**, *118*, 175–191. [[CrossRef](#)]
13. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific: Singapore, 2014.
14. Hirschman, I.I.; Widder, D.V. *The Convolution Transform*; Courier Corporation: Honolulu, HI, USA, 2012.
15. Poularikas, A. *The Transforms and Applications Handbook*; CRC Press LLC: Boca Raton, FL, USA, 2000.
16. Ortigueira, M.; Bohannan, G. Fractional Scale Calculus: Hadamard vs. Liouville. *Fractal Fract.* **2023**, *7*, 296. [[CrossRef](#)]
17. Samko, S.; Kilbas, A.; Marichev, O. *Fractional Integrals and Derivatives*; Gordon and Breach: New York, NY, USA, 1993.
18. Ortigueira, M.; Valério, D. *Fractional Signals and Systems*; De Gruyter: Berlin, Germany, 2020.
19. Ortigueira, M.D.; Machado, J.T. The 21st century systems: An updated vision of continuous-time fractional models. *IEEE Circuits Syst. Mag.* **2022**, *22*, 36–56. [[CrossRef](#)]
20. Bengochea, G.; Ortigueira, M. An operational approach to solve fractional continuous-time linear systems. *Int. J. Dyn. Control* **2017**, *5*, 61–71. [[CrossRef](#)]
21. Butzer, P.; Jansche, S. A direct approach to the Mellin transform. *J. Fourier Anal. Appl.* **1997**, *3*, 325–376. [[CrossRef](#)]
22. Luchko, Y.; Kiryakova, V. The Mellin integral transform in fractional calculus. *Fract. Calc. Appl. Anal.* **2013**, *16*, 405–430. [[CrossRef](#)]
23. Hadamard, J. Essai sur l'étude des fonctions données par leur développement de Taylor. *J. Math. Pures Appl.* **1892**, *8*, 101–186.
24. Kilbas, A.; Srivastava, H.; Trujillo, J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
25. Ortigueira, M.; Bengochea, G.; Machado, J. Substantial, tempered, and shifted fractional derivatives: Three faces of a tetrahedron. *Math. Methods Appl. Sci.* **2021**, *44*, 9191–9209. [[CrossRef](#)]
26. Ortigueira, M.; Bengochea, G. Non-commensurate fractional linear systems: New results. *J. Adv. Res.* **2020**, *25*, 11–17. [[CrossRef](#)] [[PubMed](#)]
27. Appell, P. Sur une classe de polynômes. *Ann. Sci. de l'École Norm. Supérieure.* **1880**, *9*, 119–144. [[CrossRef](#)]
28. Rota, G.; Taylor, B. The classical umbral calculus. *SIAM J. Math. Anal.* **1994**, *25*, 694–711. [[CrossRef](#)]
29. Roman, S. *The Umbral Calculus*; Springer: Berlin/Heidelberg, Germany, 2005.
30. Gel'fand, I.; Shilov, G. *Generalized Functions*; Academic Press: Cambridge, MA, USA, 1964.
31. Bengochea, G.; Verde-Star, L. Linear algebraic foundations of the operational calculi. *Adv. Appl. Math.* **2011**, *47*, 330–351. [[CrossRef](#)]
32. Verde-Star, L. Inverses of generalized Vandermonde matrices. *J. Math. Anal. Appl.* **1988**, *131*, 341–353. [[CrossRef](#)]

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