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On the interpolation constants for variable Lebesgue spaces

Oleksiy Karlovych[1] | Eugene Shargrodsky[2,3]

[1] Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal
[2] Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom
[3] Fakultät Mathematik, Technische Universität Dresden, 01062 Dresden, Germany

Correspondence
*Oleksiy Karlovych Email: oyk@fct.unl.pt

Summary

For $\theta \in (0,1)$ and variable exponents $p_0(\cdot), q_0(\cdot)$ and $p_1(\cdot), q_1(\cdot)$ with values in $[1,\infty]$, let the variable exponents $p_\theta(\cdot), q_\theta(\cdot)$ be defined by

$$1/p_\theta(\cdot) := (1-\theta)/p_0(\cdot) + \theta/p_1(\cdot), \quad 1/q_\theta(\cdot) := (1-\theta)/q_0(\cdot) + \theta/q_1(\cdot).$$

The Riesz-Thorin type interpolation theorem for variable Lebesgue spaces says that if a linear operator T acts boundedly from the variable Lebesgue space $L^{p_j(\cdot)}$ to the variable Lebesgue space $L^{q_j(\cdot)}$ for $j = 0, 1$, then

$$\|T\|_{L^{p_\theta(\cdot)} \to L^{q_\theta(\cdot)}} \le C \|T\|^{1-\theta}_{L^{p_0(\cdot)} \to L^{q_0(\cdot)}} \|T\|^{\theta}_{L^{p_1(\cdot)} \to L^{q_1(\cdot)}},$$

where C is an interpolation constant independent of T. We consider two different modulars $\varrho^{\max}(\cdot)$ and $\varrho^{\rm sum}(\cdot)$ generating variable Lebesgue spaces and give upper estimates for the corresponding interpolation constants $C_{\max}$ and $C_{\rm sum}$, which imply that $C_{\max} \le 2$ and $C_{\rm sum} \le 4$, as well as, lead to sufficient conditions for $C_{\max} = 1$ and $C_{\rm sum} = 1$. We also construct an example showing that, in many cases, our upper estimates are sharp and the interpolation constant is greater than one, even if one requires that $p_j(\cdot) = q_j(\cdot)$, $j = 0, 1$ are Lipschitz continuous and bounded away from one and infinity (in this case $\varrho^{\max}(\cdot) = \varrho^{\rm sum}(\cdot)$).

KEYWORDS
Variable Lebesgue space, Riesz-Thorin interpolation theorem, interpolation constant, Calderón product, complex method of interpolation.

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1 | INTRODUCTION

Let (Ω, μ) be a complete σ-finite measure space. The space of all equivalence classes of complex-valued measurable almost everywhere finite functions will be denoted by $L^0(\Omega, \mu)$. By $\mathcal{P}(\Omega, \mu)$ denote the set of all measurable functions

$$p(\cdot) : \Omega \to [1, \infty],$$

which will be called variable exponents. For $p(\cdot) \in \mathcal{P}(\Omega, \mu)$, let

$$p_- := \operatorname*{ess\,inf}_{x \in \Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x),$$

and

$$\Omega_\infty^{p(\cdot)} := \{x \in \Omega : p(x) = \infty\}.$$

Since we allow variable exponents to take infinite values, some care is needed when we manipulate with them. We use the following conventions:

$$\infty \cdot 0 = 0, \quad t + \infty = \infty, \quad \frac{t}{\infty} = 0, \quad \frac{t}{0} = \infty, \quad \frac{\infty}{\infty} = 1, \quad t\frac{\infty}{\infty} = t,$$

where $t \in (0, \infty)$.

For $f \in L^0(\Omega, \mu)$ and $p(\cdot) \in \mathcal{P}(\Omega, \mu)$, consider

$$\varrho^{\max}_{p(\cdot)}(f) := \max\left\{ \int\limits_{\Omega\setminus\Omega^{p(\cdot)}_\infty} |f(x)|^{p(x)} d\mu(x), \|f\|_{L^\infty(\Omega^{p(\cdot)}_\infty)} \right\}, \quad \varrho^{\text{sum}}_{p(\cdot)}(f) := \int\limits_{\Omega\setminus\Omega^{p(\cdot)}_\infty} |f(x)|^{p(x)} d\mu(x) + \|f\|_{L^\infty(\Omega^{p(\cdot)}_\infty)}.$$

One can show that both $\varrho^{\max}_{p(\cdot)}(\cdot)$ and $\varrho^{\text{sum}}_{p(\cdot)}(\cdot)$ are convex modulars (in the sense of [8, Definition 2.1.1]).

The variable Lebesgue space $L^{p(\cdot)}_{\max}(\Omega, \mu)$ (resp. $L^{p(\cdot)}_{\text{sum}}(\Omega, \mu)$) is the set of all functions $f \in L^0(\Omega, \mu)$ such that $\varrho^{\max}_{p(\cdot)}(f/\lambda) < \infty$ (resp. $\varrho^{\text{sum}}_{p(\cdot)}(f/\lambda) < \infty$) for some $\lambda = \lambda(f) > 0$. It is well known that both $L^{p(\cdot)}_{\max}(\Omega, \mu)$ and $L^{p(\cdot)}_{\text{sum}}(\Omega, \mu)$ are Banach spaces with respect to the Luxemburg-Nakano norms

$$\|f\|^{\max}_{p(\cdot)} := \inf\{\lambda > 0 \ : \ \varrho^{\max}_{p(\cdot)}(f/\lambda) \le 1\}, \quad \|f\|^{\text{sum}}_{p(\cdot)} := \inf\{\lambda > 0 \ : \ \varrho^{\text{sum}}_{p(\cdot)}(f/\lambda) \le 1\},$$

respectively. It is not difficult to check that $L^{p(\cdot)}_{\max}(\Omega, \mu)$ and $L^{p(\cdot)}_{\text{sum}}(\Omega, \mu)$ coincide as sets and

$$\|f\|^{\max}_{p(\cdot)} \le \|f\|^{\text{sum}}_{p(\cdot)} \le 2\|f\|^{\max}_{p(\cdot)}, \quad f \in L^0(\Omega, \mu).$$

If $\mu\left(\Omega^{p(\cdot)}_\infty\right) = 0$, then $\varrho^{\max}_{p(\cdot)}(f) = \varrho^{\text{sum}}_{p(\cdot)}(f)$ for $f \in L^0(\Omega, \mu)$. Hence $\|f\|^{\max}_{p(\cdot)} = \|f\|^{\text{sum}}_{p(\cdot)}$ for $f \in L^0(\Omega, \mu)$. In this case and in the case $p(\cdot) \equiv \infty$, we omit max/sum and write simply $\varrho_{p(\cdot)}(f)$, $\|f\|_{p(\cdot)}$ and $L^{p(\cdot)}(\Omega, \mu)$.

We will frequently use the following consequence of [8, Lemma 2.1.14]:

$$\varrho^{\max}_{p(\cdot)}(f) \le 1 \iff \|f\|^{\max}_{p(\cdot)} \le 1, \qquad \varrho^{\text{sum}}_{p(\cdot)}(f) \le 1 \iff \|f\|^{\text{sum}}_{p(\cdot)} \le 1. \tag{1.1}$$

The modular $\varrho^{\text{sum}}_{p(\cdot)}(\cdot)$ and the corresponding norm $\|\cdot\|^{\text{sum}}_{p(\cdot)}$ were introduced by Kováčik and Rákosník [14] and used in the monograph [5], while the modular $\varrho^{\max}_{p(\cdot)}(\cdot)$ and the corresponding norm $\|\cdot\|^{\max}_{p(\cdot)}$ were considered by Edmunds and Rákosník [9]. We will show in Lemma 2.1 that the norm $\|\cdot\|^{\max}_{p(\cdot)}$ coincides with the norm $\|\cdot\|_{\overline{\varphi}_{p(\cdot)}}$ considered in the monograph [8, Section 3]. Note also that yet another equivalent norm $\|\cdot\|_{\widetilde{\varphi}_{p(\cdot)}}$ was studied systematically in [8, Section 3] (see also [5, Section 2.10.4]).

We believe that the norm $\|\cdot\|^{\max}_{p(\cdot)}$ is more natural than $\|\cdot\|^{\text{sum}}_{p(\cdot)}$. One of the reasons for that is the following result.

Theorem 1.1. Let (Ω, μ) be a complete finite measure space. If $p_n(\cdot) : \Omega \to [1, \infty)$ is a non-decreasing sequence of measurable (a.e. finite) functions converging to $p(\cdot) : \Omega \to [1, \infty]$ a.e. and $f \in L^{p(\cdot)}_{\max}(\Omega, \mu)$, then

$$\lim_{n\to\infty} \|f\|_{p_n(\cdot)} = \|f\|^{\max}_{p(\cdot)}. \tag{1.2}$$

Suppose that $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in \mathcal{P}(\Omega, \mu)$. For $\theta \in (0, 1)$, consider the variable exponents

$$\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}, \quad \frac{1}{q_\theta(x)} = \frac{1-\theta}{q_0(x)} + \frac{\theta}{q_1(x)}, \quad x \in \Omega. \tag{1.3}$$

A widely used version of the Riesz-Thorin type interpolation theorem for variable Lebesgue spaces is contained in [8, Corollary 7.1.4] (see also [19, 20, 21, 23] for related results). It says that if a linear operator T acts boundedly on the variable Lebesgue space $L^{p_j(\cdot)}(\Omega, \mu)$ equipped with the norm $\|\cdot\|_{\overline{\varphi}_{p_j(\cdot)}}$, then

$$\|T\|_{L^{p_\theta(\cdot)}\to L^{p_\theta(\cdot)}} \le 4\|T\|^{1-\theta}_{L^{p_0(\cdot)}\to L^{p_0(\cdot)}} \|T\|^{\theta}_{L^{p_1(\cdot)}\to L^{p_1(\cdot)}}. \tag{1.4}$$

Its proof is based on [8, Theorem 7.1.2], which says that the variable Lebesgue space $L^{p_\theta(\cdot)}(\Omega, \mu)$ equipped with the norm $\|\cdot\|_{\overline{\varphi}_{p_\theta(\cdot)}}$ is isomorphic to the space

$$X_{[\theta]}(\Omega, \mu) := \left[L^{p_0(\cdot)}(\Omega, \mu), L^{p_1(\cdot)}(\Omega, \mu)\right]_{[\theta]}$$

given by the lower (first) Calderón complex interpolation method (see [3] and [2, Chap. 4]). Unfortunately, one needs to place additional restrictions on $p_j(\cdot)$ for this isomorphism to hold (see, e.g., [7, Corollary A.2] where the condition $1 < (p_j)_- \le (p_j)_+ < \infty$, $j = 0, 1$ is used). This was mentioned by Karol Leśnik in his MathSciNet review MR3931352 in the more general setting of Musielak-Orlicz spaces. For the reader's convenience, we provide some details here in the case of variable Lebesgue spaces.

Lemma 1.2. Let $p_0(\cdot) : (0, 1) \to [1, \infty)$ be an unbounded measurable function and let $p_1(x) = \infty$ for all $x \in (0, 1)$. Then

$$X_{[1/2]}((0, 1), m) \ne L^{p_{1/2}(\cdot)}((0, 1), m). \tag{1.5}$$

Here $p_{1/2}(\cdot)$ is given by (1.3) with $\theta = 1/2$ and m is the standard Lebesgue measure on $(0,1)$.

Proof. It is clear that $L^{p_0(\cdot)}((0,1),m) \cap L^{p_1(\cdot)}((0,1),m) = L^\infty((0,1),m)$. Then, by [2, Theorem 4.2.2(a)], $L^\infty((0,1),m)$ is dense in $X_{[1/2]}((0,1),m)$. On the other hand, in view of [5, Theorem 2.75], since $p_{1/2}(\cdot) = 2p_0(\cdot)$ is unbounded, $L^\infty((0,1),m)$ is not dense in $L^{p_{1/2}(\cdot)}((0,1),m)$. Thus (1.5) holds. □

In spite of the above complication, [8, Corollary 7.1.4] holds in full generality. One can bypass using the lower (first) Calderón complex interpolation method by using the well-known fact that variable Lebesgue spaces are Banach lattices with the Fatou property and by employing the Riesz-Thorin type interpolation theorem for Calderón products of Banach lattices with the Fatou property (see [18, Theorem 3.11]). We identify the Calderón products of $L^{p_0(\cdot)}_{\max}(\Omega,\mu)$, $L^{p_1(\cdot)}_{\max}(\Omega,\mu)$ and $L^{p_0(\cdot)}_{\text{sum}}(\Omega,\mu)$, $L^{p_1(\cdot)}_{\text{sum}}(\Omega,\mu)$ with the variable Lebesgue spaces $L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$ and $L^{p_\theta(\cdot)}_{\text{sum}}(\Omega,\mu)$ for arbitrary variable exponents $p_0(\cdot), p_1(\cdot) \in \mathcal{P}(\Omega,\mu)$. Note that since $p_0(\cdot)$ and $p_1(\cdot)$ may have infinite values, our results do not follow from the results in [10, Theorem 6] and [17, Proposition 6.1] for Musielak-Orlicz spaces generated by finite-valued φ-functions with parameters. Moreover, we pay special attention to the values of interpolation constants, which are best possible in some cases. In particular, we prove that [8, Corollary 7.1.4] holds with the interpolation constant $C_{\max}(\theta, p_0, p_1) \le 2$ in place of 4.

Theorem 1.3 (Main result 1)**.** Suppose that (Ω,μ) is a complete σ-finite measure space. For $\theta \in (0,1)$ and $p_j(\cdot), q_j(\cdot) \in \mathcal{P}(\Omega,\mu)$, $j = 0,1$, let $p_\theta(\cdot)$ and $q_\theta(\cdot)$ be defined by (1.3).

(a) If a linear mapping
$$T : L^{p_0(\cdot)}_{\max}(\Omega,\mu) + L^{p_1(\cdot)}_{\max}(\Omega,\mu) \to L^{q_0(\cdot)}_{\max}(\Omega,\mu) + L^{q_1(\cdot)}_{\max}(\Omega,\mu)$$
is such that the restriction of T is bounded from $L^{p_j(\cdot)}_{\max}(\Omega,\mu)$ to $L^{q_j(\cdot)}_{\max}(\Omega,\mu)$ for $j = 0,1$, then the restriction of T is bounded from $L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$ to $L^{q_\theta(\cdot)}_{\max}(\Omega,\mu)$ and
$$\|T\|_{L^{p_\theta(\cdot)}_{\max} \to L^{q_\theta(\cdot)}_{\max}} \le C_{\max}(\theta, q_0, q_1) \|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\max} \to L^{q_0(\cdot)}_{\max}} \|T\|^{\theta}_{L^{p_1(\cdot)}_{\max} \to L^{q_1(\cdot)}_{\max}},$$
where
$$C_{\max}(\theta, q_0, q_1) := \left((1-\theta) \left\| \frac{q_\theta}{q_0} \right\|_\infty + \theta \left\| \frac{q_\theta}{q_1} \right\|_\infty \right)^{1/(q_\theta)_-}. \tag{1.6}$$

(b) We have
$$1 \le C_{\max}(\theta, q_0, q_1) \le 2^{1/(q_\theta)_-} \le 2. \tag{1.7}$$
Moreover, $C_{\max}(\theta, q_0, q_1) = 1$ if and only if $q_1(\cdot)/q_0(\cdot)$ is constant a.e.

(c) If a linear mapping
$$T : L^{p_0(\cdot)}_{\text{sum}}(\Omega,\mu) + L^{p_1(\cdot)}_{\text{sum}}(\Omega,\mu) \to L^{q_0(\cdot)}_{\text{sum}}(\Omega,\mu) + L^{q_1(\cdot)}_{\text{sum}}(\Omega,\mu)$$
is such that the restriction of T is bounded from $L^{p_j(\cdot)}_{\text{sum}}(\Omega,\mu)$ to $L^{q_j(\cdot)}_{\text{sum}}(\Omega,\mu)$ for $j = 0,1$, then the restriction of T is bounded from $L^{p_\theta(\cdot)}_{\text{sum}}(\Omega,\mu)$ to $L^{q_\theta(\cdot)}_{\text{sum}}(\Omega,\mu)$ and
$$\|T\|_{L^{p_\theta(\cdot)}_{\text{sum}} \to L^{q_\theta(\cdot)}_{\text{sum}}} \le C_{\text{sum}}(\theta, p_0, p_1, q_0, q_1) \|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\text{sum}} \to L^{q_0(\cdot)}_{\text{sum}}} \|T\|^{\theta}_{L^{p_1(\cdot)}_{\text{sum}} \to L^{q_1(\cdot)}_{\text{sum}}},$$
where
$$C_{\text{sum}}(\theta, p_0, p_1, q_0, q_1) := C^{(1)}_{\text{sum}}(\theta, p_0, p_1) C^{(2)}_{\text{sum}}(\theta, q_0, q_1) \tag{1.8}$$
and
$$C^{(1)}_{\text{sum}}(\theta, p_0, p_1) := \left(1 + \left\| \chi_{\Omega^{p_0(\cdot)}_\infty \setminus \Omega^{p_1(\cdot)}_\infty} \right\|_\infty \right)^{1-\theta} \left(1 + \left\| \chi_{\Omega^{p_1(\cdot)}_\infty \setminus \Omega^{p_0(\cdot)}_\infty} \right\|_\infty \right)^{\theta}, \tag{1.9}$$
$$C^{(2)}_{\text{sum}}(\theta, q_0, q_1) := (1-\theta) \left\| \frac{q_\theta}{q_0} \right\|_\infty + \theta \left\| \frac{q_\theta}{q_1} \right\|_\infty. \tag{1.10}$$

(d) We have
$$1 \le C_{\text{sum}}(\theta, p_0, p_1, q_0, q_1) \le \max\{2C^{(1)}_{\text{sum}}(\theta, p_0, p_1), 2C^{(2)}_{\text{sum}}(\theta, q_0, q_1)\} \le 4.$$
Moreover, $C^{(2)}_{\text{sum}}(\theta, q_0, q_1) = 1$ if and only if $q_1(\cdot)/q_0(\cdot)$ is constant a.e., and $C^{(1)}_{\text{sum}}(\theta, p_0, p_1) = 1$ if and only if
$$\mu\left(\Omega^{p_0(\cdot)}_\infty \setminus \Omega^{p_1(\cdot)}_\infty\right) + \mu\left(\Omega^{p_1(\cdot)}_\infty \setminus \Omega^{p_0(\cdot)}_\infty\right) = 0.$$

Inequality (1.4) is stated in several works (see [12, Theorem 2.1], [11, formula (3.11)], [25, Proposition 2.1]) with the interpolation constant 1 in place of 4 and with references to [7] and [22, Theorem 14.16], although (1.4) is proved there with the interpolation constant 4. Our next result shows that, in general, the interpolation constant is greater than 1, even if one requires that $p_j(\cdot) = q_j(\cdot)$, $j = 0, 1$, are Lipschitz continuous and bounded away from one and infinity (these regularity assumptions are stronger than the log-Hölder continuity common in the study of variable Lebesgue spaces, see [5, Definition 2.2] and [8, Definition 4.1.1]). On the other hand, it follows from Theorem 1.3(a),(b) that [11, formula (3.11)] is correct as stated.

Theorem 1.4 (Main result 2)**.** Let the underlying measure space be the interval $(-1, 1)$ with the standard Lebesgue measure. For every $\theta \in (0, 1)$, $\varepsilon > 0$, and every constant

$$r \geq \max\left\{\frac{1}{1-\theta}, \frac{1}{\theta}\right\}, \tag{1.11}$$

there exist Lipschitz continuous variable exponents

$$p_j(\cdot) : (-1, 1) \to (1, \infty), \quad j = 0, 1,$$

and a rank-one linear operator B such that

$$1 < (p_j)_- \leq (p_j)_+ < \infty, \quad j = 0, 1, \tag{1.12}$$

$$\frac{1}{r} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}, \quad x \in (-1, 1), \tag{1.13}$$

and

$$\|B\|_{L^{p_j(\cdot)} \to L^{p_j(\cdot)}} \leq 1, \quad j = 0, 1, \quad \|B\|_{L^r \to L^r} > 2^{1/r} - \varepsilon. \tag{1.14}$$

It follows from (1.14) that constant (1.6) is optimal for the spaces in Theorem 1.4 (see (1.7)). Inequality (1.11) implies that $r \geq 2$, and hence $2^{1/r} \leq \sqrt{2}$. Unfortunately, we do not know whether $C_{\max}(\theta, q_0, q_1)$ remains optimal in the cases where it is greater than $\sqrt{2}$.

The paper is organized as follows. In Section 2, we prove that $\|f\|^{\max}_{p(\cdot)} = \|f\|_{\overline{\varphi}_{p(\cdot)}}$ for all $f \in L^0(\Omega, \mu)$, which allows us to compare our results with results in [8], where the norm $\|\cdot\|_{\overline{\varphi}_{p(\cdot)}}$ is studied in detail. Further, we prove Theorem 1.1. Section 3 is central in the paper. Here we show that the spaces $L^{p_\theta(\cdot)}_{\max}(\Omega, \mu)$ and $L^{p_\theta(\cdot)}_{\text{sum}}(\Omega, \mu)$ are isomorphic to the Caderón products of the spaces $L^{p_0(\cdot)}_{\max}(\Omega, \mu)$, $L^{p_1(\cdot)}_{\max}(\Omega, \mu)$ and $L^{p_0(\cdot)}_{\text{sum}}(\Omega, \mu)$, $L^{p_1(\cdot)}_{\text{sum}}(\Omega, \mu)$, respectively. We pay special attention to the embedding constants $C_{\max}(\theta, p_0, p_1)$ and $C^{(j)}_{\text{sum}}(\theta, p_0, p_1)$, $j = 1, 2$. Theorem 1.3 is then an easy consequence of the above results and the interpolation theorem for Calderón products [18, Theorem 3.11]. In Section 4, we show by constructing an example that the constant $C_{\max}(\theta, q_0, q_1)$ in Theorem 1.3(a) can be attained if $p_j(\cdot) = q_j(\cdot)$ for $j = 0, 1$. Further, we modify that example in order to prove Theorem 1.4. In Section 5, by using results of Section 3, we show that the set of variable exponents $1/p(\cdot)$, such that the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue space $L^{p(\cdot)}$ over a quasimetric measure space (Ω, d, μ), is convex. This results extends a result by Cruz-Uribe [4, Corollary 3] from the Euclidean setting to the setting of quasimetric measure spaces.

2 | ON THE NORM $\|\cdot\|^{\text{MAX}}_{P(\cdot)}$

2.1 | Another point of view on the norm $\|\cdot\|^{\max}_{p(\cdot)}$

Following [8, Definition 3.1.2], for $t \geq 0$ and $1 \leq p < \infty$, let $\overline{\varphi}_p(t) := t^p$, and

$$\overline{\varphi}_\infty(t) := \infty \chi_{(1,\infty)}(t) = \begin{cases} 0 & \text{if } t \in [0, 1], \\ \infty & \text{if } t \in (1, \infty). \end{cases}$$

Further, for $p(\cdot) \in \mathcal{P}(\Omega, \mu)$ and $f \in L^0(\Omega, \mu)$, consider the semimodular

$$\overline{\varrho}_{p(\cdot)}(f) := \int_\Omega \overline{\varphi}_{p(x)}(|f(x)|)\, d\mu(x)$$

(see [8, Definition 2.1.1]) and the corresponding norm

$$\|f\|_{\overline{\varphi}_{p(\cdot)}} := \inf\left\{\lambda > 0 \ : \ \int_\Omega \overline{\varphi}_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}$$

(see [8, Theorems 2.1.7 and 3.2.13]).

It seems that the following equality has not been noticed before.

Lemma 2.1. If $p(\cdot) \in \mathcal{P}(\Omega, \mu)$, then

$$\|f\|_{p(\cdot)}^{\max} = \|f\|_{\overline{\varphi}_{p(\cdot)}} \quad \text{for all} \quad f \in L^0(\Omega, \mu).$$

Proof. Suppose that $\|f\|_{p(\cdot)}^{\max} < \infty$. If $\lambda \ge \|f\|_{p(\cdot)}^{\max}$, then $\|f/\lambda\|_{p(\cdot)}^{\max} \le 1$. It follows from (1.1) that $\varrho_{p(\cdot)}^{\max}(f/\lambda) \le 1$. Then

$$\|f/\lambda\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \le \varrho_{p(\cdot)}^{\max}(f/\lambda) \le 1.$$

Hence, for $x \in \Omega_\infty^{p(\cdot)}$,

$$\overline{\varphi}_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) = \overline{\varphi}_\infty\left(\frac{|f(x)|}{\lambda}\right) = 0.$$

Therefore

$$\int_\Omega \overline{\varphi}_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) = \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \le \varrho_{p(\cdot)}^{\max}(f/\lambda) \le 1.$$

It follows from the above inequality and [8, Lemma 3.2.4] that $\|f/\lambda\|_{\overline{\varphi}_{p(\cdot)}} \le 1$. Hence $\|f\|_{\overline{\varphi}_{p(\cdot)}} \le \lambda$ for every $\lambda \ge \|f\|_{p(\cdot)}^{\max}$. In particular,

$$\|f\|_{\overline{\varphi}_{p(\cdot)}} \le \|f\|_{p(\cdot)}^{\max}. \tag{2.1}$$

If $0 < \lambda < \|f\|_{p(\cdot)}^{\max}$, then by [8, Corollary 2.1.15],

$$1 < \|f/\lambda\|_{p(\cdot)}^{\max} \le \varrho_{p(\cdot)}^{\max}(f/\lambda).$$

Hence, if $0 < \lambda < \|f\|_{p(\cdot)}^{\max}$, then

$$\int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) > 1$$

or

$$\mu\left(\Omega_\infty^{p(\cdot)}\right) > 0 \quad \text{and} \quad \|f/\lambda\|_{L^\infty(\Omega_\infty^{p(\cdot)})} > 1.$$

In the former case, we have

$$\overline{\varrho}_{p(\cdot)}(f/\lambda) = \int_\Omega \overline{\varphi}_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \ge \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) > 1,$$

while in the latter one, $|f(x)| > \lambda$ on a subset of $\Omega_\infty^{p(\cdot)}$ of positive measure. Then

$$\overline{\varrho}_{p(\cdot)}(f/\lambda) = \int_\Omega \overline{\varphi}_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \ge \int_{\Omega_\infty^{p(\cdot)}} \overline{\varphi}_\infty\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) = \infty.$$

In both cases, we get $\overline{\varrho}_{p(\cdot)}(f/\lambda) > 1$. Then, by [8, Lemma 3.2.4], $\|f/\lambda\|_{\overline{\varrho}_{p(\cdot)}} > 1$, whence $\|f\|_{\overline{\varphi}_{p(\cdot)}} > \lambda$. Since this inequality holds for every $0 < \lambda < \|f\|_{p(\cdot)}$, we see that

$$\|f\|_{\overline{\varphi}_{p(\cdot)}} \ge (1-\varepsilon)\|f\|_{p(\cdot)}^{\max}, \quad 0 < \varepsilon < 1.$$

Passing to the limit as $\varepsilon \to 0^+$, we conclude that

$$\|f\|_{\overline{\varphi}_{p(\cdot)}} \ge \|f\|_{p(\cdot)}^{\max}. \tag{2.2}$$

Combining (2.1) and (2.2), we see that

$$\|f\|_{\overline{\varphi}_{p(\cdot)}} = \|f\|_{p(\cdot)}^{\max}$$

for every $f \in L^0(\Omega, \mu)$ such that $\|f\|_{p(\cdot)}^{\max} < \infty$.

Suppose now $\|f\|_{p(\cdot)}^{\max} = \infty$. The same argument as the one used in the case $0 < \lambda < \|f\|_{p(\cdot)}^{\max}$ above shows that $\|f\|_{\overline{\varphi}_{p(\cdot)}} \ge \lambda$ for every $0 < \lambda < \infty$, that is, $\|f\|_{\overline{\varphi}_{p(\cdot)}} = \infty$. □

2.2 | Proof of Theorem 1.1

Equality (1.2) trivially holds if $f=0$. So we can assume that $f\neq 0$. Take an arbitrary $\eta>0$ and set

$$\Omega_\eta := \left\{x\in\Omega\setminus\Omega_\infty^{p(\cdot)} : |f(x)|\le\eta\right\},\quad \Omega^\eta := \left\{x\in\Omega\setminus\Omega_\infty^{p(\cdot)} : |f(x)|>\eta\right\}.$$

Since $\mu(\Omega_\eta)\le\mu(\Omega)<\infty$ and

$$\left|\frac{f(x)}{\eta}\right|^{p_n(x)}\le 1^{p_n(x)}=1,\quad x\in\Omega_\eta,$$

one gets using the dominated convergence theorem

$$\int\limits_{\Omega_\eta}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)\to\int\limits_{\Omega_\eta}\left|\frac{f(x)}{\eta}\right|^{p(x)}d\mu(x)\quad\text{as}\quad n\to\infty. \tag{2.3}$$

We have

$$\left|\frac{f(x)}{\eta}\right|^{p_n(x)}\nearrow\left|\frac{f(x)}{\eta}\right|^{p(x)}\quad\text{as}\quad n\to\infty,\quad x\in\Omega^\eta.$$

Hence, by the monotone convergence theorem,

$$\int\limits_{\Omega^\eta}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)\to\int\limits_{\Omega^\eta}\left|\frac{f(x)}{\eta}\right|^{p(x)}d\mu(x)\quad\text{as}\quad n\to\infty. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\lim_{n\to\infty}\int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)=\int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\eta}\right|^{p(x)}d\mu(x). \tag{2.5}$$

If $\eta>\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}$, then $|f(x)|/\eta<1$ for a.e. $x\in\Omega_\infty^{p(\cdot)}$. Since $p_n(x)\uparrow\infty$ as $x\in\Omega_\infty^{p(\cdot)}$ and $n\to\infty$, we have

$$\left|\frac{f(x)}{\eta}\right|^{p_n(x)}\downarrow 0\quad\text{as}\quad n\to\infty,\quad x\in\Omega_\infty^{p(\cdot)}.$$

Hence it follows from $\mu(\Omega)<\infty$ and the dominated convergence theorem that

$$\eta>\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}\quad\Rightarrow\quad\lim_{n\to\infty}\int\limits_{\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)=0. \tag{2.6}$$

If $\eta<\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}$, then there exist $\delta>0$ and a measurable set $E\subseteq\Omega_\infty^{p(\cdot)}$ such that $\mu(E)>0$ and $|f(x)|/\eta\ge 1+\delta$ for $x\in E$. Hence

$$\left|\frac{f(x)}{\eta}\right|^{p_n(x)}\ge(1+\delta)^{p_n(x)}\nearrow\infty\quad\text{as}\quad n\to\infty,\quad x\in E.$$

Then it follows from the monotone convergence theorem that

$$\lim_{n\to\infty}\int\limits_{\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)\ge\lim_{n\to\infty}\int\limits_E\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)=\infty.$$

Thus

$$\eta<\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}\quad\Rightarrow\quad\lim_{n\to\infty}\int\limits_{\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\eta}\right|^{p_n(x)}d\mu(x)=\infty. \tag{2.7}$$

Take an arbitrary $\varepsilon>0$ and set $\lambda:=\|f\|_{p(\cdot)}$, $\lambda_\varepsilon:=(1+\varepsilon)\lambda$. In view (1.1), we have

$$\left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p(\cdot)})}\le\varrho_{p(\cdot)}^{\max}(f/\lambda)\le 1.$$

Hence $\lambda_\varepsilon>\lambda\ge\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}$. Then it follows from (2.5) and (2.6) that

$$\lim_{n\to\infty}\varrho_{p_n(\cdot)}(f/\lambda_\varepsilon)=\lim_{n\to\infty}\left(\int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\lambda_\varepsilon}\right|^{p_n(x)}d\mu(x)+\int\limits_{\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\lambda_\varepsilon}\right|^{p_n(x)}d\mu(x)\right)=\int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}}\left|\frac{f(x)}{\lambda_\varepsilon}\right|^{p(x)}d\mu(x)+0$$

$$\leq \frac{1}{1+\varepsilon} \int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leq \frac{\varrho_{p(\cdot)}^{\max}(f/\lambda)}{1+\varepsilon} \leq \frac{1}{1+\varepsilon}.$$

So, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\varrho_{p_n(\cdot)}(f/\lambda_\varepsilon) \leq 1 \quad \text{for all} \quad n \geq n_\varepsilon.$$

Then, by (1.1),

$$\|f\|_{p_n(\cdot)} \leq \lambda_\varepsilon = (1+\varepsilon)\|f\|_{p(\cdot)}^{\max} \quad \text{for all} \quad n \geq n_\varepsilon.$$

Hence

$$\limsup_{n\to\infty} \|f\|_{p_n(\cdot)} \leq (1+\varepsilon)\|f\|_{p(\cdot)}^{\max} \quad \text{for all} \quad \varepsilon > 0,$$

i.e.

$$\limsup_{n\to\infty} \|f\|_{p_n(\cdot)} \leq \|f\|_{p(\cdot)}^{\max}. \tag{2.8}$$

Let

$$\eta_0 := \liminf_{n\to\infty} \|f\|_{p_n(\cdot)}$$

and $\eta_\varepsilon := \eta_0 + \varepsilon$. There exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that

$$\|f\|_{p_{n_k}(\cdot)} \leq \eta_\varepsilon \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, by (1.1),

$$\int\limits_\Omega \left|\frac{f(x)}{\eta_\varepsilon}\right|^{p_{n_k}(x)} d\mu(x) = \varrho_{p_{n_k}}(f/\eta_\varepsilon) \leq 1 \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.9}$$

Then it follows from (2.5) and (2.9) that

$$\int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\eta_\varepsilon}\right|^{p(x)} d\mu(x) = \lim_{k\to\infty} \int\limits_{\Omega\setminus\Omega_\infty^{p(\cdot)}} \left|\frac{f(x)}{\eta_\varepsilon}\right|^{p_{n_k}(x)} d\mu(x) \leq \limsup_{k\to\infty} \int\limits_\Omega \left|\frac{f(x)}{\eta_\varepsilon}\right|^{p_{n_k}(x)} d\mu(x) \leq 1. \tag{2.10}$$

If $\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})} > \eta_\varepsilon$, then (2.7) with $\eta = \eta_\varepsilon$ contradicts (2.9). So, $\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \leq \eta_\varepsilon$, i.e. $\|f/\eta_\varepsilon\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \leq 1$. This inequality and (2.10) imply that $\varrho_{p(\cdot)}^{\max}(f/\eta_\varepsilon) \leq 1$. Then, by (1.1),

$$\|f\|_{p(\cdot)}^{\max} \leq \eta_\varepsilon = \liminf_{n\to\infty} \|f\|_{p_n(\cdot)} + \varepsilon \quad \text{for all} \quad \varepsilon > 0,$$

i.e.

$$\|f\|_{p(\cdot)}^{\max} \leq \liminf_{n\to\infty} \|f\|_{p_n(\cdot)}.$$

This and (2.8) imply (1.2). □

Remark 2.2. It is well known that (1.2) might not hold even for the standard Lebesgue spaces if $\mu(\Omega) = \infty$ (see, e.g., [24, Sect. 1.1]). Indeed, if $p_n < \infty$, $p = \infty$, and $f \equiv 1$, then the right-hand side of (1.2) is equal to 1, while the left-hand side is infinite. It is clear that (1.2) remains true in the case $\mu(\Omega) = \infty$ if the support of f has finite measure. On the other hand, (1.2) may fail even for a function belonging to the closure in $L^{p(\cdot)}(\Omega, \mu)$ of the set of essentially bounded measurable functions with supports of finite measure. Indeed, let $\Omega = \mathbb{R}$, μ be the standard Lebesgue measure, $p_n < \infty$, $p = \infty$, and $f(x) = \frac{1}{1+\log(1+|x|)}$. Then again the right-hand side of (1.2) equals 1, while the left-hand side is infinite.

3 | CALDERÓN PRODUCTS OF VARIABLE LEBESGUE SPACES

3.1 | Definitions of Calderón products of variable Lebesgue spaces

Suppose that (Ω, μ) be a complete σ-finite measure space. Let $p_j(\cdot) \in \mathcal{P}(\Omega, \mu)$ for $j = 0, 1$. Fix $0 < \theta < 1$. The Calderón product (see, e.g., [3, Section 13.5])

$$X_{\max}^\theta(\Omega, \mu) := \left(L_{\max}^{p_0(\cdot)}(\Omega, \mu)\right)^{1-\theta} \left(L_{\max}^{p_1(\cdot)}(\Omega, \mu)\right)^\theta \tag{3.1}$$

of the variable Lebesgue spaces $L_{\max}^{p_0(\cdot)}(\Omega, \mu)$ and $L_{\max}^{p_1(\cdot)}(\Omega, \mu)$ is the set of all functions $f \in L^0(\Omega, \mu)$ such that

$$|f| \leq \lambda |f_0|^{1-\theta} |f_1|^\theta \tag{3.2}$$

for some $\lambda > 0$ and $f_j \in L^{p_j(\cdot)}_{\max}(\Omega,\mu)$ with $\|f_j\|^{\max}_{p_j(\cdot)} \le 1$, $j = 0, 1$. The norm in $X^{\theta}_{\max}(\Omega,\mu)$ is the infimum over all $\lambda > 0$ such that (3.2) holds.

The Calderón product

$$X^{\theta}_{\mathrm{sum}}(\Omega,\mu) := \left(L^{p_0(\cdot)}_{\mathrm{sum}}(\Omega,\mu)\right)^{1-\theta}\left(L^{p_1(\cdot)}_{\mathrm{sum}}(\Omega,\mu)\right)^{\theta} \tag{3.3}$$

of the variable Lebesgue space $L^{p_0(\cdot)}_{\mathrm{sum}}(\Omega,\mu)$ and $L^{p_1(\cdot)}_{\mathrm{sum}}(\Omega,\mu)$ is defined similarly, replacing the norms $\|\cdot\|^{\max}_{p_j(\cdot)}$, $j = 0, 1$, by $\|\cdot\|^{\mathrm{sum}}_{p_j(\cdot)}$, $j = 0, 1$, respectively. Then $X^{\theta}_{\max}(\Omega,\mu)$ and $X^{\theta}_{\mathrm{sum}}(\Omega,\mu)$ are Banach lattices.

Kopaliani and Chelidze [13, Proposition 3.1] computed the Calderón product of two variable Lebesgue spaces under the assumptions that both variable exponents $p_j(\cdot)$, $j = 0, 1$, are bounded. Below we remove this assumption and show that $X^{\theta}_{\max}(\Omega,\mu)$ coincides with $L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$ (up to the equivalence of the norms) and $X^{\theta}_{\mathrm{sum}}(\Omega,\mu)$ coincides with $L^{p_\theta(\cdot)}_{\mathrm{sum}}(\Omega,\mu)$ (up to the equivalence of the norms), where $p_\theta(\cdot)$ is given by (1.3). Our proofs are similar to that of [15, pp. 179–180, Example 3].

3.2 | An auxiliary result leading to the proof of Theorem 1.3(b),(d)

We start calculating the Calderón products $X^{\theta}_{\max}(\Omega,\mu)$ and $X^{\theta}_{\mathrm{sum}}(\Omega,\mu)$ by proving the following lemma, which plays an important role in the study of optimality of our results.

Lemma 3.1. Let (Ω,μ) be a complete σ-finite measure space. For $\theta \in (0,1)$ and $p_j(\cdot) \in \mathcal{P}(\Omega,\mu)$, $j = 0, 1$, let the variable exponent $p_\theta(\cdot) \in \mathcal{P}(\Omega,\mu)$ be defined by (1.3). Set

$$M_0 := (1-\theta)\left\|\frac{p_\theta}{p_0}\right\|_\infty, \quad M_1 := \theta\left\|\frac{p_\theta}{p_1}\right\|_\infty \tag{3.4}$$

and

$$\alpha := \operatorname*{ess\,inf}_{x\in\Omega}\frac{p_1(x)}{p_0(x)}, \quad \beta := \operatorname*{ess\,sup}_{x\in\Omega}\frac{p_1(x)}{p_0(x)}. \tag{3.5}$$

Then

(a) $1 \le M_0 + M_1 \le 2$.

(b) $M_0 = 1$ if and only if $\beta = \infty$;

(c) $M_1 = 1$ if and only if $\alpha = 0$;

(d) $M_0 + M_1 = 2$ if and only if $\alpha = 0$ and $\beta = \infty$;

(e) $M_0 + M_1 = 1$ if and only if $p_1(\cdot)/p_0(\cdot)$ is constant a.e.

Proof. (a) It follows immediately from the definition of $p_\theta(\cdot)$ given by (1.3) that

$$(1-\theta)\frac{p_\theta(x)}{p_0(x)} + \theta\,\frac{p_\theta(x)}{p_1(x)} = 1, \quad x \in \Omega. \tag{3.6}$$

Hence

$$(1-\theta)\frac{p_\theta(x)}{p_0(x)} \le 1, \quad \theta\,\frac{p_\theta(x)}{p_1(x)} \le 1, \quad x \in \Omega,$$

whence $M_0 \le 1$, $M_1 \le 1$, and $M_0 + M_1 \le 2$. On the other hand, (3.4) and (3.6) imply that $M_0 + M_1 \ge 1$, which completes the proof of part (a).

(b) It follows from (1.3) that

$$\frac{p_0(x)}{p_\theta(x)} = 1 - \theta + \theta\,\frac{p_0(x)}{p_1(x)}, \quad x \in \Omega.$$

Then

$$\begin{aligned} M_0 &= (1-\theta)\left(\operatorname*{ess\,inf}_{x\in\Omega}\frac{p_0(x)}{p_\theta(x)}\right)^{-1} = (1-\theta)\left(1-\theta+\theta\operatorname*{ess\,inf}_{x\in\Omega}\frac{p_0(x)}{p_1(x)}\right)^{-1} \\ &= (1-\theta)\left(1-\theta+\theta\left(\operatorname*{ess\,sup}_{x\in\Omega}\frac{p_0(x)}{p_1(x)}\right)^{-1}\right)^{-1} = (1-\theta)\left(1-\theta+\frac{\theta}{\beta}\right)^{-1}. \end{aligned} \tag{3.7}$$

This identity immediately implies the statement of part (b).

(c) Similarly, by (1.3),

$$\frac{p_1(x)}{p_\theta(x)} = (1-\theta)\frac{p_1(x)}{p_0(x)} + \theta, \quad x \in \Omega.$$

Therefore,

$$M_1 = \theta \left(\operatorname*{ess\,inf}_{x\in\Omega} \frac{p_1(x)}{p_\theta(x)} \right)^{-1} = \theta \left((1-\theta) \operatorname*{ess\,inf}_{x\in\Omega} \frac{p_1(x)}{p_0(x)} + \theta \right)^{-1} = \theta\big((1-\theta)\alpha + \theta\big)^{-1} = \frac{\theta}{\alpha}\left(1-\theta+\frac{\theta}{\alpha}\right)^{-1}. \tag{3.8}$$

This yields the statement of part (c).

Part (d) is an immediate consequence of parts (b) and (c).

(e) If $p_1(\cdot)/p_0(\cdot)$ is constant a.e., then $\beta = \alpha$, and it follows from (3.7)–(3.8) that

$$M_0 + M_1 = (1-\theta)\left(1-\theta+\frac{\theta}{\alpha}\right)^{-1} + \frac{\theta}{\alpha}\left(1-\theta+\frac{\theta}{\alpha}\right)^{-1} = 1.$$

If $p_1(\cdot)/p_0(\cdot)$ is not constant, then $\beta > \alpha$, and it follows from (3.7)–(3.8) that

$$M_0 + M_1 > (1-\theta)\left(1-\theta+\frac{\theta}{\alpha}\right)^{-1} + \frac{\theta}{\alpha}\left(1-\theta+\frac{\theta}{\alpha}\right)^{-1} = 1,$$

which completes the proof of part (e). □

Theorem 1.3(b),(d) is an easy consequence of parts (a) and (e) of the above result.

3.3 | Continuous embedding $L^{p_\theta(\cdot)}_{\max}(\Omega,\mu) \hookrightarrow X^\theta_{\max}(\Omega,\mu)$

Lemma 3.2. Let (Ω,μ) be a complete σ-finite measure space. For $\theta \in (0,1)$ and $p_j(\cdot) \in \mathcal{P}(\Omega,\mu)$, $j = 0,1$, let the variable exponent $p_\theta(\cdot) \in \mathcal{P}(\Omega,\mu)$ be defined by (1.3) and the Calderón product $X^\theta_{\max}(\Omega,\mu)$ be defined by (3.1). If $f \in L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$, then $f \in X^\theta_{\max}(\Omega,\mu)$ and

$$\|f\|_{X^\theta_{\max}} \le \|f\|^{\max}_{p_\theta(\cdot)}. \tag{3.9}$$

Proof. It is easy to see that

$$p_\theta(x) = \begin{cases} \left(\dfrac{1-\theta}{p_0(x)} + \dfrac{\theta}{p_1(x)}\right)^{-1}, & x \in \Omega \setminus \left(\Omega_\infty^{p_0(\cdot)} \cup \Omega_\infty^{p_1(\cdot)}\right), \\ \dfrac{p_1(x)}{\theta}, & x \in \Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}, \\ \dfrac{p_0(x)}{1-\theta}, & x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}, \\ \infty, & x \in \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}. \end{cases} \tag{3.10}$$

Suppose that $f \in L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$. Without loss of generality, we can assume that $f \neq 0$. Then, by (1.1),

$$\varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right) \le 1. \tag{3.11}$$

Define the functions

$$h_0(x) := \begin{cases} \left(\dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{p_\theta(x)}{p_0(x)}}, & x \in \Omega \setminus \left(\Omega_\infty^{p_0(\cdot)} \cup \Omega_\infty^{p_1(\cdot)}\right), \\ \left(\dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{1}{1-\theta}}, & x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}, \\ 1, & x \in \Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}, \\ \dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}, & x \in \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}, \end{cases} \qquad h_1(x) := \begin{cases} \left(\dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{p_\theta(x)}{p_1(x)}}, & x \in \Omega \setminus \left(\Omega_\infty^{p_0(\cdot)} \cup \Omega_\infty^{p_1(\cdot)}\right), \\ \left(\dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{1}{\theta}}, & x \in \Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}, \\ 1, & x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}, \\ \dfrac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}, & x \in \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}. \end{cases} \tag{3.12}$$

Then, for $x \in \Omega$,

$$|f(x)| = \|f\|^{\max}_{p_\theta(\cdot)} (h_0(x))^{1-\theta} (h_1(x))^\theta. \tag{3.13}$$

It follows from (3.10), (3.11) and the first inequality in (3.12) that

$$\int\limits_{\Omega\setminus\Omega_\infty^{p_0(\cdot)}} (h_0(x))^{p_0(x)} d\mu(x) = \int\limits_{\Omega\setminus\left(\Omega_\infty^{p_0(\cdot)}\cup\Omega_\infty^{p_1(\cdot)}\right)} \left[\left(\frac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{p_\theta(x)}{p_0(x)}}\right]^{p_0(x)} d\mu(x) + \int\limits_{\Omega_\infty^{p_1(\cdot)}\setminus\Omega_\infty^{p_0(\cdot)}} \left(\frac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{\frac{p_0(x)}{1-\theta}} d\mu(x)$$

$$= \int\limits_{\Omega\setminus\Omega_\infty^{p_0(\cdot)}} \left(\frac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{p_\theta(x)} d\mu(x) \le \int\limits_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} \left(\frac{|f(x)|}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)^{p_\theta(x)} d\mu(x) \le \varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right) \le 1 \quad (3.14)$$

and

$$\|h_0\|_{L^\infty(\Omega_\infty^{p_0(\cdot)})} \le \max\left\{\|1\|_{L^\infty(\Omega_\infty^{p_0(\cdot)}\setminus\Omega_\infty^{p_1(\cdot)})}, \left\|\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right\|_{L^\infty(\Omega_\infty^{p_0(\cdot)}\cap\Omega_\infty^{p_1(\cdot)})}\right\} \le \max\left\{1, \varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)\right\} \le 1. \quad (3.15)$$

Combining (3.14) and (3.15), we see that

$$\varrho^{\max}_{p_0(\cdot)}(h_0) \le 1. \quad (3.16)$$

Similarly, (3.10), (3.11) and the second inequality in (3.12) yield

$$\int\limits_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} (h_1(x))^{p_1(x)} d\mu(x) \le \varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right) \le 1 \quad (3.17)$$

and

$$\|h_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})} \le \max\left\{1, \varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right)\right\} \le 1. \quad (3.18)$$

Combining (3.17) and (3.18), we immediately get

$$\varrho^{\max}_{p_1(\cdot)}(h_1) \le 1. \quad (3.19)$$

It follows from (3.16), (3.19), and (1.1) that

$$\left\|h_j\right\|^{\max}_{p_j(\cdot)} \le 1, \quad j = 0, 1. \quad (3.20)$$

Equality (3.13), inequalities (3.20) and the definition of the space $X^\theta_{\max}(\Omega,\mu)$ imply that $f \in X^\theta_{\max}(\Omega,\mu)$ and (3.9) holds. □

3.4 | Continuous embedding $L^{p_\theta(\cdot)}_{\text{sum}}(\Omega,\mu) \hookrightarrow X^\theta_{\text{sum}}(\Omega,\mu)$

Lemma 3.3. Let (Ω,μ) be a complete σ-finite measure space. For $\theta \in (0,1)$ and $p_j(\cdot) \in \mathcal{P}(\Omega,\mu)$, $j = 0, 1$, let the variable exponent $p_\theta(\cdot) \in \mathcal{P}(\Omega,\mu)$ be defined by (1.3) and the Calderón product $X^\theta_{\text{sum}}(\Omega,\mu)$ be defined by (3.3). If $f \in L^{p_\theta(\cdot)}_{\text{sum}}(\Omega,\mu)$, then $f \in X^\theta_{\text{sum}}(\Omega,\mu)$ and

$$\|f\|_{X^\theta_{\text{sum}}} \le C^{(1)}_{\text{sum}}(\theta, p_0, p_1)\|f\|^{\text{sum}}_{p_\theta(\cdot)}, \quad (3.21)$$

where $C^{(1)}_{\text{sum}}(\theta, p_0, p_1)$ is defined by (1.9).

Proof. The proof is similar to that of Lemma 3.2. Suppose that $f \in L^{p_\theta(\cdot)}_{\text{sum}}(\Omega,\mu)$. Without loss of generality, we can assume that $f \ne 0$. Then, by (1.1),

$$\varrho^{\text{sum}}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right) \le 1. \quad (3.22)$$

Define the functions

$$h_0(x) := \begin{cases} \left(\frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right)^{\frac{p_\theta(x)}{p_0(x)}}, & x \in \Omega \setminus \left(\Omega_\infty^{p_0(\cdot)} \cup \Omega_\infty^{p_1(\cdot)}\right), \\ \left(\frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right)^{\frac{1}{1-\theta}}, & x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}, \\ 1, & x \in \Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}, \\ \frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}, & x \in \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}, \end{cases} \qquad h_1(x) := \begin{cases} \left(\frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right)^{\frac{p_\theta(x)}{p_1(x)}}, & x \in \Omega \setminus \left(\Omega_\infty^{p_0(\cdot)} \cup \Omega_\infty^{p_1(\cdot)}\right), \\ \left(\frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right)^{\frac{1}{\theta}}, & x \in \Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}, \\ 1, & x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}, \\ \frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}, & x \in \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}. \end{cases} \tag{3.23}$$

Then, for $x \in \Omega$,

$$|f(x)| = \|f\|^{\text{sum}}_{p_\theta(\cdot)} (h_0(x))^{1-\theta} (h_1(x))^{\theta}. \tag{3.24}$$

Similarly to (3.14) and (3.17), we obtain from (3.10) and (3.23) that for $j = 0, 1$,

$$\int\limits_{\Omega \setminus \Omega_\infty^{p_j(\cdot)}} (h_j(x))^{p_j(x)} d\mu(x) \le \int\limits_{\Omega \setminus \Omega_\infty^{p_\theta(\cdot)}} \left(\frac{|f(x)|}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}}\right)^{p_\theta(x)} d\mu(x). \tag{3.25}$$

Let

$$\Delta_{0,1} := \left\| \chi_{\Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)}} \right\|_\infty, \quad \Delta_{1,0} := \left\| \chi_{\Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)}} \right\|_\infty. \tag{3.26}$$

Then it follows from (3.10) and (3.23) that

$$\|h_0\|_{L^\infty(\Omega_\infty^{p_0(\cdot)})} \le \max\left\{ \|1\|_{L^\infty(\Omega_\infty^{p_0(\cdot)} \setminus \Omega_\infty^{p_1(\cdot)})}, \left\| \frac{f}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}} \right\|_{L^\infty(\Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)})} \right\} \le \Delta_{0,1} + \left\| \frac{f}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}} \right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \tag{3.27}$$

and

$$\|h_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})} \le \max\left\{ \|1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_0(\cdot)})}, \left\| \frac{f}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}} \right\|_{L^\infty(\Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)})} \right\} \le \Delta_{1,0} + \left\| \frac{f}{\|f\|^{\text{sum}}_{p_\theta(\cdot)}} \right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})}. \tag{3.28}$$

Combining (3.22), (3.25) and (3.27)–(3.28), we see that

$$\varrho^{\text{sum}}_{p_0(\cdot)}\left(\frac{h_0}{1+\Delta_{0,1}}\right) \le \frac{\varrho^{\text{sum}}_{p_0(\cdot)}(h_0)}{1+\Delta_{0,1}} \le \frac{1}{1+\Delta_{0,1}} \left(\varrho^{\text{sum}}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right) + \Delta_{0,1} \right) \le 1, \tag{3.29}$$

$$\varrho^{\text{sum}}_{p_1(\cdot)}\left(\frac{h_1}{1+\Delta_{1,0}}\right) \le \frac{\varrho^{\text{sum}}_{p_1(\cdot)}(h_1)}{1+\Delta_{1,0}} \le \frac{1}{1+\Delta_{1,0}} \left(\varrho^{\text{sum}}_{p_\theta(\cdot)}\left(\frac{f}{\|f\|^{\max}_{p_\theta(\cdot)}}\right) + \Delta_{1,0} \right) \le 1. \tag{3.30}$$

It follows from (1.1) and (3.29)–(3.30) that

$$\left\| \frac{h_0}{1+\Delta_{0,1}} \right\|^{\text{sum}}_{p_0(\cdot)} \le 1, \quad \left\| \frac{h_1}{1+\Delta_{1,0}} \right\|^{\text{sum}}_{p_1(\cdot)} \le 1. \tag{3.31}$$

On the other hand, taking into account (3.26), we see that (3.24) can be rewritten as

$$\begin{aligned} |f(x)| &= (1+\Delta_{0,1})^{1-\theta} (1+\Delta_{1,0})^{\theta} \|f\|^{\text{sum}}_{p_\theta(\cdot)} \left(\frac{h_0(x)}{1+\Delta_{0,1}}\right)^{1-\theta} \left(\frac{h_1(x)}{1+\Delta_{1,0}}\right)^{\theta} \\ &= C^{(1)}_{\text{sum}}(\theta, p_0, p_1) \|f\|^{\text{sum}}_{p_\theta(\cdot)} \left(\frac{h_0(x)}{1+\Delta_{0,1}}\right)^{1-\theta} \left(\frac{h_1(x)}{1+\Delta_{1,0}}\right)^{\theta}, \quad x \in \Omega, \end{aligned} \tag{3.32}$$

where $C^{(1)}_{\text{sum}}(\theta, p_0, p_1)$ is defined by (1.9). Inequalities (3.31), equality (3.32) and the definition of the norm in the space $X^\theta_{\text{sum}}(\Omega, \mu)$ imply that $f \in X^\theta_{\text{sum}}(\Omega, \mu)$ and (3.21) holds. □

3.5 | Continuous embedding $X^\theta_{\max}(\Omega,\mu) \hookrightarrow L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$

Lemma 3.4. Let (Ω,μ) be a complete σ-finite measure space. For $\theta \in (0,1)$ and $p_j(\cdot) \in \mathcal{P}(\Omega,\mu)$, $j = 0,1$, let the variable exponent $p_\theta(\cdot) \in \mathcal{P}(\Omega,\mu)$ be defined by (1.3) and the Calderón product $X^\theta_{\max}(\Omega,\mu)$ be defined by (3.1). If $f \in X^\theta_{\max}(\Omega,\mu)$, then $f \in L^{p_\theta(\cdot)}_{\max}(\Omega,\mu)$ and

$$\|f\|^{\max}_{p_\theta(\cdot)} \le C_{\max}(\theta,p_0,p_1)\|f\|_{X^\theta_{\max}}, \tag{3.33}$$

where $C_{\max}(\theta,p_0,p_1)$ is defined by (1.6).

Proof. Let the constants M_0 and M_1 be defined by (3.4). Suppose that $\lambda > 0$ is such that $\|f\|_{X^\theta_{\max}} < \lambda$. Then

$$|f| \le \lambda |f_0|^{1-\theta}|f_1|^\theta \tag{3.34}$$

for some $f_j \in L^{p_j(\cdot)}_{\max}(\Omega,\mu)$ satisfying $\|f_j\|^{\max}_{p_j(\cdot)} \le 1$ with $j = 0,1$. Combining these inequalities with (1.1), we see that for $j = 0,1$,

$$\max\left\{ \int_{\Omega\setminus\Omega^{p_j(\cdot)}_\infty} |f_j(x)|^{p_j(x)}d\mu(x), \|f_j\|_{L^\infty(\Omega^{p_j(\cdot)}_\infty)} \right\} = \varrho^{\max}_{p_j(\cdot)}(f_j) \le 1. \tag{3.35}$$

Let $x \in \Omega \setminus \left(\Omega^{p_0(\cdot)}_\infty \cup \Omega^{p_1(\cdot)}_\infty\right)$. It follows from (3.34), (3.4), (3.6) and Young's inequality

$$ab \le a^r/r + b^s/s \tag{3.36}$$

with

$$a = |f_0(x)|^{(1-\theta)p_\theta(x)}, \quad b = |f_1(x)|^{\theta p_\theta(x)}, \quad r = \frac{p_0(x)}{(1-\theta)p_\theta(x)}, \quad s = \frac{p_1(x)}{\theta p_\theta(x)}$$

that

$$\begin{aligned}\left(\frac{|f(x)|}{\lambda}\right)^{p_\theta(x)} &\le \left(|f_0(x)|^{1-\theta}|f_1(x)|^\theta\right)^{p_\theta(x)} \le (1-\theta)\frac{p_\theta(x)}{p_0(x)}|f_0(x)|^{p_0(x)} + \theta\frac{p_\theta(x)}{p_1(x)}|f_1(x)|^{p_1(x)} \\ &\le M_0|f_0(x)|^{p_0(x)} + M_1|f_1(x)|^{p_1(x)}.\end{aligned} \tag{3.37}$$

If $\Omega^{p_0(\cdot)}_\infty \setminus \Omega^{p_1(\cdot)}_\infty$ has positive measure, then α defined by (3.5) is equal to zero. Hence, by Lemma 3.1(c), $M_1 = 1$. So, taking into account (3.10) and (3.34)–(3.35), one has for $x \in \Omega^{p_0(\cdot)}_\infty \setminus \Omega^{p_1(\cdot)}_\infty$,

$$\begin{aligned}\left(\frac{|f(x)|}{\lambda}\right)^{p_\theta(x)} &\le \left(|f_0(x)|^{1-\theta}|f_1(x)|^\theta\right)^{p_\theta(x)} = \left(|f_0(x)|^{1-\theta}|f_1(x)|^\theta\right)^{\frac{p_1(x)}{\theta}} \\ &\le \left(\|f_0\|^{1-\theta}_{L^\infty(\Omega^{p_0(\cdot)}_\infty)}\right)^{\frac{p_1(x)}{\theta}} |f_1(x)|^{p_1(x)} \le |f_1(x)|^{p_1(x)} = M_1\,|f_1(x)|^{p_1(x)}.\end{aligned} \tag{3.38}$$

Similarly, if $\Omega^{p_1(\cdot)}_\infty \setminus \Omega^{p_0(\cdot)}_\infty$ has positive measure, then β defined by (3.5) is equal to infinity. Hence, by Lemma 3.1(b), $M_0 = 1$. So, taking into account (3.10) and (3.34)–(3.35), one has for $x \in \Omega^{p_1(\cdot)}_\infty \setminus \Omega^{p_0(\cdot)}_\infty$,

$$\begin{aligned}\left(\frac{|f(x)|}{\lambda}\right)^{p_\theta(x)} &\le \left(|f_0(x)|^{1-\theta}|f_1(x)|^\theta\right)^{p_\theta(x)} = \left(|f_0(x)|^{1-\theta}|f_1(x)|^\theta\right)^{\frac{p_0(x)}{1-\theta}} \\ &\le |f_0(x)|^{p_0(x)}\left(\|f_1\|^\theta_{L^\infty(\Omega^{p_1(\cdot)}_\infty)}\right)^{\frac{p_0(x)}{1-\theta}} \le |f_0(x)|^{p_0(x)} = M_0\,|f_0(x)|^{p_0(x)}.\end{aligned} \tag{3.39}$$

Combining (3.37)–(3.39), we see that

$$\begin{aligned}\int_{\Omega\setminus\Omega^{p_\theta(\cdot)}_\infty} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) &= \int_{\Omega\setminus\left(\Omega^{p_0(\cdot)}_\infty\cup\Omega^{p_1(\cdot)}_\infty\right)} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) + \int_{\Omega^{p_0(\cdot)}_\infty\setminus\Omega^{p_1(\cdot)}_\infty} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) + \int_{\Omega^{p_1(\cdot)}_\infty\setminus\Omega^{p_0(\cdot)}_\infty} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) \\ &\le \int_{\Omega\setminus\left(\Omega^{p_0(\cdot)}_\infty\cup\Omega^{p_1(\cdot)}_\infty\right)} \left(M_0|f_0(x)|^{p_0(x)} + M_1|f_1(x)|^{p_1(x)}\right) d\mu(x) \\ &\quad + \int_{\Omega^{p_0(\cdot)}_\infty\setminus\Omega^{p_1(\cdot)}_\infty} M_1|f_1(x)|^{p_1(x)}d\mu(x) + \int_{\Omega^{p_1(\cdot)}_\infty\setminus\Omega^{p_0(\cdot)}_\infty} M_0|f_0(x)|^{p_0(x)}d\mu(x)\end{aligned}$$

$$= M_0 \int_{\Omega \setminus \Omega_\infty^{p_0(\cdot)}} |f_0(x)|^{p_0(x)} d\mu(x) + M_1 \int_{\Omega \setminus \Omega_\infty^{p_1(\cdot)}} |f_1(x)|^{p_1(x)} d\mu(x). \tag{3.40}$$

Finally, by (3.10), $\Omega_\infty^{p_\theta(\cdot)} = \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}$. If this set has positive measure, then it follows from (3.34) and Young's inequality (3.36) with

$$a = \|f_0\|^{1-\theta}_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})}, \quad b = \|f_1\|^{\theta}_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})}, \quad r = 1/(1-\theta), \quad s = 1/\theta$$

that

$$\begin{aligned}\left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} &\le \|f_0\|^{1-\theta}_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \|f_1\|^{\theta}_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \le (1-\theta)\|f_0\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} + \theta\|f_1\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \\ &\le (1-\theta)\|f_0\|_{L^\infty(\Omega_\infty^{p_0(\cdot)})} + \theta\|f_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})}.\end{aligned} \tag{3.41}$$

Combining (3.35) with (3.40)–(3.41), we see that

$$\int_{\Omega \setminus \Omega_\infty^{p_\theta(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) \le M_0 + M_1, \quad \left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \le 1. \tag{3.42}$$

It follows from the definitions in (3.4) and (1.6) that

$$C_{\max} := C_{\max}(\theta, p_0, p_1) = (M_0 + M_1)^{1/(p_\theta)_-}. \tag{3.43}$$

Then (3.42) and (3.43) yield

$$\begin{aligned}\varrho^{\max}_{p_\theta(\cdot)}\left(\frac{f}{C_{\max}(\theta, p_0, p_1)\lambda}\right) &= \max\left\{\int_{\Omega \setminus \Omega_\infty^{p_\theta(\cdot)}} \left|\frac{f(x)}{C_{\max}\lambda}\right|^{p_\theta(x)} d\mu(x), \left\|\frac{f}{C_{\max}\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})}\right\} \\ &\le \max\left\{\frac{1}{M_0 + M_1}\int_{\Omega \setminus \Omega_\infty^{p_\theta(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x), \left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})}\right\} \le 1.\end{aligned}$$

This inequality shows that $f \in L^{p_\theta(\cdot)}_{\max}(\Omega, \mu)$ and for every $\lambda > \|f\|_{X^\theta_{\max}}$, we have $\|f\|^{\max}_{p_\theta(\cdot)} \le C_{\max}(\theta, p_0, p_1)\lambda$. Passing to the limit as $\lambda \to \|f\|_{X^\theta_{\max}} + 0$, we arrive at (3.33). □

3.6 | Continuous embedding $X^\theta_{\mathrm{sum}}(\Omega, \mu) \hookrightarrow L^{p_\theta(\cdot)}_{\mathrm{sum}}(\Omega, \mu)$

Lemma 3.5. Let (Ω, μ) be a complete σ-finite measure space. For $\theta \in (0, 1)$ and $p_j(\cdot) \in \mathcal{P}(\Omega, \mu)$, $j = 0, 1$, let the variable exponent $p_\theta(\cdot) \in \mathcal{P}(\Omega, \mu)$ be defined by (1.3) and the Calderón product $X^\theta_{\mathrm{sum}}(\Omega, \mu)$ be defined by (3.3). If $f \in X^\theta_{\mathrm{sum}}(\Omega, \mu)$, then $f \in L^{p_\theta(\cdot)}_{\mathrm{sum}}(\Omega, \mu)$ and

$$\|f\|^{\mathrm{sum}}_{p_\theta(\cdot)} \le C^{(2)}_{\mathrm{sum}}(\theta, p_0, p_1)\|f\|_{X^\theta_{\mathrm{sum}}}, \tag{3.44}$$

where $C^{(2)}_{\mathrm{sum}}(\theta, p_0, p_1)$ is defined by (1.10).

Proof. The proof is similar to that of Lemma 3.4. Let the constants M_0 and M_1 be defined by (3.4). Suppose $\lambda > 0$ is such that $\|f\|_{X^\theta_{\mathrm{sum}}} < \lambda$. Then (3.34) holds for some $f_j \in L^{p_j(\theta)}_{\mathrm{sum}}(\Omega, \mu)$ satisfying $\|f_j\|^{\mathrm{sum}}_{p_j(\cdot)} \le 1$ with $j = 0, 1$. Combining these inequalities with (1.1), we see that for $j = 0, 1$,

$$\int_{\Omega \setminus \Omega_\infty^{p_j(\cdot)}} |f(x)|^{p_j(x)} d\mu(x) + \|f_j\|_{L^\infty(\Omega_\infty^{p_j(\cdot)})} = \varrho^{\mathrm{sum}}_{p_j(\cdot)}(f) \le 1. \tag{3.45}$$

It follows from (3.10) that $\Omega_\infty^{p_\theta(\cdot)} = \Omega_\infty^{p_0(\cdot)} \cap \Omega_\infty^{p_1(\cdot)}$. If this set has positive measure, then $1 - \theta \le M_0$ and $\theta \le M_1$. In this case, (3.41) implies that

$$\left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \le M_0\|f_0\|_{L^\infty(\Omega_\infty^{p_0(\cdot)})} + M_1\|f_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})}. \tag{3.46}$$

It follows from (3.40), (3.45), and (3.46) that

$$\begin{aligned}\varrho^{\rm sum}_{p_\theta(\cdot)}\left(\frac{f}{\lambda}\right) &= \int\limits_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} \left|\frac{f(x)}{\lambda}\right|^{p_\theta(x)} d\mu(x) + \left\|\frac{f}{\lambda}\right\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \\ &\le M_0 \int\limits_{\Omega\setminus\Omega_\infty^{p_0(\cdot)}} |f_0(x)|^{p_0(x)} d\mu(x) + M_1 \int\limits_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} |f_1(x)|^{p_1(x)} d\mu(x) + M_0\|f_0\|_{L^\infty(\Omega_\infty^{p_0(\cdot)})} + M_1\|f_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})} \\ &= M_0\varrho^{\rm sum}_{p_0(\cdot)}(f_0) + M_1\varrho^{\rm sum}_{p_1(\cdot)}(f_1) \le M_0 + M_1. \end{aligned} \tag{3.47}$$

The definitions in (1.10) and (3.4) imply that

$$C^{(2)}_{\rm sum}(\theta, p_0, p_1) = M_0 + M_1. \tag{3.48}$$

Lemma 3.1(a) yields $C^{(2)}_{\rm sum}(\theta, p_0, p_1) \ge 1$. Hence, taking into account (3.47) and (3.48), we get

$$\varrho^{\rm sum}_{p_\theta(\cdot)}\left(\frac{f}{C^{(2)}_{\rm sum}(\theta, p_0, p_1)\lambda}\right) \le \frac{1}{M_0 + M_1}\varrho^{\rm sum}_{p_\theta(\cdot)}\left(\frac{f}{\lambda}\right) \le 1.$$

This inequality shows that $f \in L^{p_\theta(\cdot)}_{\rm sum}(\Omega,\mu)$ and for every $\lambda > \|f\|_{X^\theta_{\rm sum}}$, we have $\|f\|^{\rm sum}_{p_\theta(\cdot)} \le C^{(2)}_{\rm sum}(\theta, p_0, p_1)\lambda$. Passing to the limit as $\lambda \to \|f\|_{X^\theta_{\rm sum}} + 0$, we arrive at (3.44). □

3.7 | Proof of Theorem 1.3(a),(c)

Let the variable exponents $p_\theta(\cdot)$ and $q_\theta(\cdot)$ be defined by (1.3). Consider the Calderón products

$$X^\theta_{\max}(\Omega,\mu) := \left(L^{p_0(\cdot)}_{\max}(\Omega,\mu)\right)^{1-\theta}\left(L^{p_1(\cdot)}_{\max}(\Omega,\mu)\right)^{\theta}, \quad X^\theta_{\rm sum}(\Omega,\mu) := \left(L^{p_0(\cdot)}_{\rm sum}(\Omega,\mu)\right)^{1-\theta}\left(L^{p_1(\cdot)}_{\rm sum}(\Omega,\mu)\right)^{\theta},$$
$$Y^\theta_{\max}(\Omega,\mu) := \left(L^{q_0(\cdot)}_{\max}(\Omega,\mu)\right)^{1-\theta}\left(L^{q_1(\cdot)}_{\max}(\Omega,\mu)\right)^{\theta}, \quad Y^\theta_{\rm sum}(\Omega,\mu) := \left(L^{q_0(\cdot)}_{\rm sum}(\Omega,\mu)\right)^{1-\theta}\left(L^{q_1(\cdot)}_{\rm sum}(\Omega,\mu)\right)^{\theta}.$$

Let $f \in L^{p_\theta(\cdot)}_{\max}(\Omega,\mu) = L^{p_\theta(\cdot)}_{\rm sum}(\Omega,\mu)$ (we understand the latter equality as equality of sets). By Lemmas 3.2 and 3.3,

$$\|f\|_{X^\theta_{\max}} \le \|f\|^{\max}_{p_\theta(\cdot)}, \tag{3.49}$$
$$\|f\|_{X^\theta_{\rm sum}} \le C^{(1)}_{\rm sum}(\theta, p_0, p_1)\|f\|^{\rm sum}_{p_\theta(\cdot)}. \tag{3.50}$$

It is well known that $L^{p_j(\cdot)}_{\max}(\Omega,\mu)$, $L^{p_j(\cdot)}_{\rm sum}(\Omega,\mu)$, $L^{q_j(\cdot)}_{\max}(\Omega,\mu)$, and $L^{q_j(\cdot)}_{\rm sum}(\Omega,\mu)$ have the Fatou property for $j = 0, 1$. Then it follows from [18, Theorem 3.11] that

$$\|Tf\|_{Y^\theta_{\max}} \le \|T\|_{X^\theta_{\max}\to Y^\theta_{\max}}\|f\|_{X^\theta_{\max}} \le \|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\max}\to L^{q_0(\cdot)}_{\max}}\|T\|^{\theta}_{L^{p_1(\cdot)}_{\max}\to L^{q_1(\cdot)}_{\max}}\|f\|_{X^\theta_{\max}}, \tag{3.51}$$
$$\|Tf\|_{Y^\theta_{\rm sum}} \le \|T\|_{X^\theta_{\rm sum}\to Y^\theta_{\max}}\|f\|_{X^\theta_{\rm sum}} \le \|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\rm sum}\to L^{q_0(\cdot)}_{\max}}\|T\|^{\theta}_{L^{p_1(\cdot)}_{\rm sum}\to L^{q_1(\cdot)}_{\max}}\|f\|_{X^\theta_{\rm sum}}. \tag{3.52}$$

In view of Lemmas 3.4 and 3.5, we have

$$\|Tf\|^{\max}_{q_\theta(\cdot)} \le C_{\max}(\theta, q_0, q_1)\|Tf\|_{Y^\theta_{\max}}, \tag{3.53}$$
$$\|Tf\|^{\rm sum}_{q_\theta(\cdot)} \le C^{(2)}_{\rm sum}(\theta, q_0, q_1)\|Tf\|_{Y^\theta_{\rm sum}}. \tag{3.54}$$

Combining (3.49), (3.51), and (3.53), we get

$$\|Tf\|^{\max}_{q_\theta(\cdot)} \le C_{\max}(\theta, q_0, q_1)\|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\max}\to L^{q_0(\cdot)}_{\max}}\|T\|^{\theta}_{L^{p_1(\cdot)}_{\max}\to L^{q_1(\cdot)}_{\max}}\|f\|^{\max}_{p_\theta(\cdot)},$$

which completes the proof of part (a). Similarly, gathering (3.50), (3.52), and (3.54), we obtain

$$\|Tf\|^{\rm sum}_{q_\theta(\cdot)} \le C^{(1)}_{\rm sum}(\theta, p_0, p_1)C^{(2)}_{\rm sum}(\theta, q_0, q_1)\|T\|^{1-\theta}_{L^{p_0(\cdot)}_{\rm sum}\to L^{q_0(\cdot)}_{\rm sum}}\|T\|^{\theta}_{L^{p_1(\cdot)}_{\rm sum}\to L^{q_1(\cdot)}_{\rm sum}}\|f\|^{\rm sum}_{p_\theta(\cdot)},$$

which completes the proof of part (c). □

4 | THE INTERPOLATION CONSTANT IN THEOREM 1.3 IS OPTIMAL IN SOME CASES

4.1 | Sharpness of the constant $C_{\max}(\theta, q_0, q_1)$ in Theorem 1.3(a)

The following result implies that the constant $C_{\max}(\theta, q_0, q_1)$ in Theorem 1.3(a) can be attained even if $p_j(\cdot) = q_j(\cdot)$ for $j = 0, 1$.

Theorem 4.1. Let the underlying measure space be the interval $(-1, 1)$ with the standard Lebesgue measure. For every $\theta \in (0, 1)$ and every constant

$$p \geq \max\left\{\frac{1}{1-\theta}, \frac{1}{\theta}\right\},$$

there exist variable exponents

$$p_j(\cdot) : (-1, 1) \to [1, \infty], \quad j = 0, 1,$$

such that

$$\frac{1}{p} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}, \quad x \in (-1, 1), \tag{4.1}$$

and the rank-one linear operator

$$(Kf)(x) := \int_0^1 f(t)\, dt, \quad x \in (-1, 1)$$

satisfies the following conditions

$$\|K\|_{L^{p_j(\cdot)}_{\max} \to L^{p_j(\cdot)}_{\max}} = 1, \quad j = 0, 1, \quad \|K\|_{L^p \to L^p} = 2^{1/p}. \tag{4.2}$$

Proof. Let

$$p_0(x) := \begin{cases} (1-\theta)p, & x \in (-1, 0], \\ \infty, & x \in (0, 1), \end{cases} \qquad p_1(x) := \begin{cases} \infty, & x \in (-1, 0], \\ \theta p, & x \in (0, 1). \end{cases}$$

Then (4.1) holds.

Suppose $f \in L^1(-1, 1)$ and

$$\lambda := \int_0^1 |f(t)|\, dt.$$

Then

$$\left|\frac{(Kf)(x)}{\lambda}\right| \leq 1, \quad x \in (-1, 1). \tag{4.3}$$

If $f \in L^{p_0(\cdot)}_{\max}(-1, 1)$, then

$$\begin{aligned} \|f\|^{\max}_{p_0(\cdot)} &= \inf\left\{\mu > 0 \ :\ \max\left\{\int_{-1}^{0} \left|\frac{f(x)}{\mu}\right|^{(1-\theta)p} dx, \left\|\frac{f}{\mu}\right\|_{L^\infty(0,1)}\right\} \leq 1\right\} \\ &\geq \inf\left\{\mu > 0 \ :\ \left\|\frac{f}{\mu}\right\|_{L^\infty(0,1)} \leq 1\right\} = \|f\|_{L^\infty(0,1)} \geq \|f\|_{L^1(0,1)} = \lambda. \end{aligned} \tag{4.4}$$

Taking into account (4.3), we get

$$\varrho^{\max}_{p_0(\cdot)}(Kf/\lambda) = \max\left\{\int_{-1}^{0} \left|\frac{(Kf)(x)}{\lambda}\right|^{(1-\theta)p} dx, \left\|\frac{Kf}{\lambda}\right\|_{L^\infty(0,1)}\right\} \leq \max\left\{\int_{-1}^{0} 1^{(1-\theta)p} dx, \|1\|_{L^\infty(0,1)}\right\} \leq 1. \tag{4.5}$$

It follows from (4.3)–(4.5) that for $f \in L^{p_0(\cdot)}_{\max}(-1, 1)$,

$$\|Kf\|^{\max}_{p_0(\cdot)} \leq \lambda \leq \|f\|^{\max}_{p_0(\cdot)}. \tag{4.6}$$

Similarly, if $f \in L^{p_1(\cdot)}_{\max}(-1, 1)$, then

$$\|f\|^{\max}_{p_1(\cdot)} = \inf\left\{\mu > 0 \ :\ \max\left\{\left\|\frac{f}{\mu}\right\|_{L^\infty(-1,0)}, \int_0^1 \left|\frac{f(x)}{\mu}\right|^{\theta p} dx\right\} \leq 1\right\}$$

$$\geq \inf\left\{\mu > 0 \;:\; \int_0^1 \left|\frac{f(x)}{\mu}\right|^{\theta p} dx \leq 1\right\} = \|f\|_{L^{\theta p}(0,1)} \geq \|f\|_{L^1(0,1)} = \lambda. \tag{4.7}$$

It follows from (4.3) that

$$\varrho^{\max}_{p_1(\cdot)}(Kf/\lambda) = \max\left\{\left\|\frac{Kf}{\lambda}\right\|_{L^\infty(-1,0)}, \int_0^1 \left|\frac{(Kf)(x)}{\lambda}\right|^{\theta p} dx\right\} \leq \max\left\{\|1\|_{L^\infty(-1,0)}, \int_0^1 1^{\theta p} dx\right\} = 1. \tag{4.8}$$

Combining (4.3) and (4.7)–(4.8), we get for $f \in L^{p_1(\cdot)}_{\max}(-1,1)$,

$$\|Kf\|^{\max}_{p_1(\cdot)} \leq \lambda \leq \|f\|^{\max}_{p_1(\cdot)}. \tag{4.9}$$

It follows from (4.6) and (4.9) that

$$\|K\|_{L^{p_j(\cdot)}_{\max} \to L^{p_j(\cdot)}_{\max}} \leq 1, \quad j = 0, 1. \tag{4.10}$$

On the other hand, for $f = \chi_{(0,1)}$, one has

$$\|f\|^{\max}_{p_0(\cdot)} = \inf\left\{\mu > 0 \;:\; \left\|\frac{1}{\mu}\right\|_{L^\infty(0,1)} \leq 1\right\} = 1, \tag{4.11}$$

$$\|f\|^{\max}_{p_1(\cdot)} = \inf\left\{\mu > 0 \;:\; \int_0^1 \left|\frac{1}{\mu}\right|^{\theta p} dx \leq 1\right\} = 1. \tag{4.12}$$

Further, $Kf = 1$ and for $\mu > 0$,

$$\varrho^{\max}_{p_0(\cdot)}(Kf/\mu) = \max\left\{\int_{-1}^0 \left|\frac{1}{\mu}\right|^{(1-\theta)p} dx, \left\|\frac{1}{\mu}\right\|_{L^\infty(0,1)}\right\} = \max\left\{\mu^{-(1-\theta)p}, \mu^{-1}\right\},$$

$$\varrho^{\max}_{p_1(\cdot)}(Kf/\mu) = \max\left\{\left\|\frac{1}{\mu}\right\|_{L^\infty(-1,0)}, \int_0^1 \left|\frac{1}{\mu}\right|^{\theta p} dx\right\} = \max\left\{\mu^{-1}, \mu^{-\theta p}\right\}.$$

Therefore

$$\|Kf\|^{\max}_{p_0(\cdot)} = \inf\{\mu > 0 \;:\; \varrho^{\max}_{p_0(\cdot)}(Kf/\mu) \leq 1\} = \inf\{\mu > 0 \;:\; \mu^{-(1-\theta)p} \leq 1\} = 1, \tag{4.13}$$

$$\|Kf\|^{\max}_{p_1(\cdot)} = \inf\{\mu > 0 \;:\; \varrho^{\max}_{p_1(\cdot)}(Kf/\mu) \leq 1\} = \inf\{\mu > 0 \;:\; \mu^{-\theta p} \leq 1\} = 1. \tag{4.14}$$

It follows from (4.11)–(4.14) that

$$\|K\|_{L^{p_j(\cdot)}_{\max} \to L^{p_j(\cdot)}_{\max}} \geq 1, \quad j = 0, 1. \tag{4.15}$$

Combining (4.10) and (4.15), we see that

$$\|K\|_{L^{p_j(\cdot)}_{\max} \to L^{p_j(\cdot)}_{\max}} = 1, \quad j = 0, 1. \tag{4.16}$$

It is easy to see that if $f \in L^p(-1,1)$, then

$$\|Kf\|_p = \left(\int_{-1}^1 \left|\int_0^1 f(t)\,dt\right|^p dx\right)^{1/p} = 2^{1/p}\left|\int_0^1 f(t)\,dt\right| \leq 2^{1/p}\left(\int_0^1 |f(t)|^p dt\right)^{1/p} \leq 2^{1/p}\left(\int_{-1}^1 |f(t)|^p dt\right)^{1/p} = 2^{1/p}\|f\|_p,$$

whence

$$\|K\|_{L^p \to L^p} \leq 2^{1/p}. \tag{4.17}$$

On other hand, we have for $f = \chi_{(0,1)}$,

$$\|f\|_p = 1, \quad (Kf)(x) = 1, \quad x \in (-1,1), \quad \|Kf\|_p = 2^{1/p}.$$

So,

$$\|K\|_{L^p \to L^p} \geq 2^{1/p}. \tag{4.18}$$

Thus

$$\|K\|_{L^p\to L^p} = 2^{1/p}, \tag{4.19}$$

which completes the proof of (4.2). □

It follows from Lemma 3.1 and Theorem 4.1 that for $p_j(\cdot)$, $j = 0, 1$, and p as in Theorem 4.1, one has $C(\theta, p_0, p_1) = 2^{1/p} > 1$ and

$$\|K\|_{L^p\to L^p} = C(\theta, p_0, p_1)\|K\|^{1-\theta}_{L^{p_0(\cdot)}_{\max}\to L^{p_0(\cdot)}_{\max}}\|K\|^{\theta}_{L^{p_1(\cdot)}_{\max}\to L^{p_1(\cdot)}_{\max}},$$

that is, the interpolation inequality in Theorem 1.3(a) becomes an equality.

Remark 4.2. The direct elementary proofs of (4.15) and (4.17) given above can be substituted with the following shorter but indirect argument. It follows from (4.18), Theorem 1.3(a),(b), and (4.10) that

$$2^{1/p} \le \|K\|_{L^p\to L^p} \le 2^{1/p}\|K\|^{1-\theta}_{L^{p_0(\cdot)}_{\max}\to L^{p_0(\cdot)}_{\max}}\|K\|^{\theta}_{L^{p_1(\cdot)}_{\max}\to L^{p_1(\cdot)}_{\max}} \le 2^{1/p}1^{1-\theta}1^{\theta} = 2^{1/p}.$$

Hence all the above non-strict inequalities are in fact equalities, and (4.16), (4.19) hold.

4.2 | Proof of Theorem 1.4

We can assume without loss of generality that $\varepsilon < 2$. Choose $\delta \in (0, 1)$ such that

$$\left(1 - \frac{\varepsilon}{3}\right)(1 + \delta) < 1, \quad \left(1 - \frac{\varepsilon}{3}\right)(1 - \delta)^{1-1/r} \ge 1 - \frac{\varepsilon}{2}, \tag{4.20}$$

and then choose

$$q > r \tag{4.21}$$

such that $q \ne 2r$ and

$$\left(1 - \frac{\varepsilon}{3}\right)(1 + \delta) + \left(1 - \frac{\varepsilon}{3}\right)^{q\min\{1-\theta,\theta\}} \le 1. \tag{4.22}$$

Let

$$p := \frac{qr}{q - r}.$$

Then

$$p \ne q, \quad p > r, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{4.23}$$

Define the variable exponents

$$p_0(x) := \begin{cases} (1-\theta)p, & x \in (-1, -\delta], \\ (1-\theta)\left(\dfrac{p-q}{2\delta pq}x + \dfrac{p+q}{2pq}\right)^{-1}, & x \in (-\delta, \delta), \\ (1-\theta)q, & x \in [\delta, 1), \end{cases} \qquad p_1(x) := \begin{cases} \theta q, & x \in (-1, -\delta], \\ \theta\left(\dfrac{q-p}{2\delta pq}x + \dfrac{p+q}{2pq}\right)^{-1}, & x \in (-\delta, \delta), \\ \theta p, & x \in [\delta, 1). \end{cases} \tag{4.24}$$

Taking into account the equality in (4.23), an easy calculation shows that the variable exponents defined by (4.24) satisfy (1.13).

It is easy to see that there exists $c \in (0, \infty)$ such that

$$\frac{dp_j(x)}{dx} \ne 0, \quad \left|\frac{dp_j(x)}{dx}\right| \le c \quad \text{for all} \quad x \in (-\delta, \delta), \quad j = 0, 1,$$

Hence $p_0(\cdot)$ and $p_1(\cdot)$ are Lipschitz continuous on $(-1, 1)$ and monotone on $[-\delta, \delta]$. Taking into account the latter observation, we easily get from (1.11), (4.21) and (4.23)–(4.24) that for all $x \in (-1, 1)$,

$$1 < (1-\theta)\min\{p, q\} \le p_0(x) \le (1-\theta)\max\{p, q\} < \infty, \quad 1 < \theta\min\{p, q\} \le p_1(x) \le \theta\max\{p, q\} < \infty.$$

These inequalities immediately imply (1.12).

Let A be the rank-one operator defined for $f \in L^1(-1, 1)$ by

$$(Af)(x) := \int_\delta^1 f(t)\,dt, \quad x \in (-1, 1),$$

and let

$$\lambda := \left(1-\frac{\varepsilon}{3}\right)^{-1}\int_{\delta}^{1}|f(t)|\,dt.$$

Then

$$\left|\frac{(Af)(x)}{\lambda}\right| \le 1-\frac{\varepsilon}{3}, \quad x\in(-1,1). \tag{4.25}$$

Taking into account (4.24), (4.25), (4.22) and (1.12), we get for all $f \in L^{p_0(\cdot)}(-1,1)$,

$$\begin{aligned}\varrho_{p_0(\cdot)}(Af/\lambda) &= \int_{-1}^{\delta}\left|\frac{(Af)(x)}{\lambda}\right|^{p_0(x)}dx + \int_{\delta}^{1}\left|\frac{(Af)(x)}{\lambda}\right|^{(1-\theta)q}dx \le \int_{-1}^{\delta}\left(1-\frac{\varepsilon}{3}\right)^{p_0(x)}dx + \int_{\delta}^{1}\left(1-\frac{\varepsilon}{3}\right)^{(1-\theta)q}dx\\ &\le \int_{-1}^{\delta}\left(1-\frac{\varepsilon}{3}\right)dx + \int_{\delta}^{1}\left(1-\frac{\varepsilon}{3}\right)^{q\min\{1-\theta,\theta\}}dx < \left(1-\frac{\varepsilon}{3}\right)(1+\delta) + \left(1-\frac{\varepsilon}{3}\right)^{q\min\{1-\theta,\theta\}} \le 1.\end{aligned}$$

Hence, for all $f \in L^{p_0(\cdot)}(-1,1)$,

$$\begin{aligned}\|Af\|_{p_0(\cdot)} &\le \lambda = \left(1-\frac{\varepsilon}{3}\right)^{-1}\int_{\delta}^{1}|f(x)|\,dx \le \left(1-\frac{\varepsilon}{3}\right)^{-1}\left(\int_{\delta}^{1}|f(x)|^{(1-\theta)q}dx\right)^{\frac{1}{(1-\theta)q}}\left(\int_{\delta}^{1}dx\right)^{1-\frac{1}{(1-\theta)q}}\\ &\le \left(1-\frac{\varepsilon}{3}\right)^{-1}\inf\left\{\tau>0\ :\ \int_{\delta}^{1}\left|\frac{f(x)}{\tau}\right|^{(1-\theta)q}dx \le 1\right\} \le \left(1-\frac{\varepsilon}{3}\right)^{-1}\inf\left\{\tau>0\ :\ \int_{-1}^{1}\left|\frac{f(x)}{\tau}\right|^{p_0(x)}dx \le 1\right\}\\ &= \left(1-\frac{\varepsilon}{3}\right)^{-1}\|f\|_{p_0(\cdot)}.\end{aligned} \tag{4.26}$$

Similarly, taking into account (4.24), (4.25), (4.22), and (1.12), we deduce for $f \in L^{p_1(\cdot)}(-1,1)$ that

$$\begin{aligned}\varrho_{p_1(\cdot)}(Af/\lambda) &= \int_{-1}^{-\delta}\left|\frac{(Af)(x)}{\lambda}\right|^{\theta q}dx + \int_{-\delta}^{1}\left|\frac{(Af)(x)}{\lambda}\right|^{p_1(x)}dx \le \int_{-1}^{-\delta}\left(1-\frac{\varepsilon}{3}\right)^{\theta q}dx + \int_{-\delta}^{1}\left(1-\frac{\varepsilon}{3}\right)^{p_1(x)}dx\\ &\le \int_{-1}^{-\delta}\left(1-\frac{\varepsilon}{3}\right)^{q\min\{1-\theta,\theta\}}dx + \int_{-\delta}^{1}\left(1-\frac{\varepsilon}{3}\right)dx < \left(1-\frac{\varepsilon}{3}\right)^{q\min\{1-\theta,\theta\}} + \left(1-\frac{\varepsilon}{3}\right)(1+\delta) \le 1.\end{aligned}$$

Therefore, for all $f \in L^{p_1(\cdot)}(-1,1)$,

$$\begin{aligned}\|Af\|_{p_1(\cdot)} &\le \lambda = \left(1-\frac{\varepsilon}{3}\right)^{-1}\int_{\delta}^{1}|f(x)|\,dx \le \left(1-\frac{\varepsilon}{3}\right)^{-1}\left(\int_{\delta}^{1}|f(x)|^{\theta p}dx\right)^{\frac{1}{\theta p}}\left(\int_{\delta}^{1}dx\right)^{1-\frac{1}{\theta p}}\\ &\le \left(1-\frac{\varepsilon}{3}\right)^{-1}\inf\left\{\tau>0\ :\ \int_{\delta}^{1}\left|\frac{f(x)}{\tau}\right|^{\theta p}dx \le 1\right\} \le \left(1-\frac{\varepsilon}{3}\right)^{-1}\inf\left\{\tau>0\ :\ \int_{-1}^{1}\left|\frac{f(x)}{\tau}\right|^{p_1(x)}dx \le 1\right\}\\ &\le \left(1-\frac{\varepsilon}{3}\right)^{-1}\|f\|_{p_1(\cdot)}.\end{aligned} \tag{4.27}$$

It follows (4.26)–(4.27) that

$$\|A\|_{L^{p_j(\cdot)}\to L^{p_j(\cdot)}} \le \left(1-\frac{\varepsilon}{3}\right)^{-1}, \quad j=0,1. \tag{4.28}$$

On the other hand, we have for $f = \chi_{[\delta,1]}$,

$$\|f\|_r = (1-\delta)^{1/r}, \quad (Af)(x) = 1-\delta, \quad x\in(-1,1), \quad \|Af\|_r = (1-\delta)2^{1/r}.$$

So,

$$\|A\|_{L^r\to L^r} \ge (1-\delta)^{1-1/r}2^{1/r}. \tag{4.29}$$

Finally, let

$$B := \left(1 - \frac{\varepsilon}{3}\right) A.$$

Then it follows from (4.28)–(4.29) and the second inequality in (4.20) that

$$\|B\|_{L^{p_j(\cdot)} \to L^{p_j(\cdot)}} \le 1, \quad j = 0, 1,$$

and

$$\|B\|_{L^r \to L^r} \ge \left(1 - \frac{\varepsilon}{3}\right)(1-\delta)^{1-1/r} 2^{1/r} \ge \left(1 - \frac{\varepsilon}{2}\right) 2^{1/r} > 2^{1/r} - \varepsilon,$$

which completes the proof of (1.14). □

5 | INTERPOLATION ESTIMATES FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON VARIABLE LEBESGUE SPACES

One says that that (Ω, d) is a quasimetric space if $d : \Omega \times \Omega \to [0, \infty)$ satisfies the following axioms:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;

(3) $d(x, y) \le A(d(x, z) + d(y, z))$ for all $x, y, z \in \Omega$ and some constant $A \ge 1$.

For $x \in \Omega$ and $r > 0$, consider the ball $B(x, r) := \{y \in \Omega : d(x, y) < r\}$. Let Ω be a quasimetric space with a distance function d and a measure μ such that $0 < \mu(B) < \infty$ for any ball $B \subset \Omega$. In this case, the triple (Ω, d, μ) will be called a quasimetric measure space.

For $f \in L^1_{\mathrm{loc}}(\Omega, \mu)$, the Hardy-Littlewood maximal operator is defined by

$$(Mf)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum is taken over all balls B containing x.

By $\mathcal{B}_M(\Omega)$ we denote the set of all variable exponents $p(\cdot) \in \mathcal{P}(\Omega, \mu)$ such that the Hardy-Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}(\Omega, \mu)$. In view of [1, Theorem 1.7], the set $\mathcal{B}_M(\Omega)$ is nontrivial, that is, it contains nonconstant elements. It was shown by Diening, Hästo, and Nekvinda [7, Corollary 2.5] that the set

$$\mathcal{R}_0(\mathbb{R}^d) := \{1/p(\cdot) \;:\; p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d),\ 1 < p_- \le p_+ < \infty\}$$

is convex. Further, Cruz-Uribe [4, Corollary 3] (see also [5, Theorems 3.36 and 3.38]) proved that if Ω is an open set in $\mathbb{R}^d$, then the set

$$\mathcal{R}(\Omega) := \{1/p(\cdot) \;:\; p(\cdot) \in \mathcal{B}_M(\Omega)\}$$

is convex. More precisely, his result reads as follows.

Theorem 5.1. Let Ω be an open set in $\mathbb{R}^d$. If $p_j(\cdot) \in \mathcal{B}_M(\Omega)$ for $j = 0, 1$, then for every $\theta \in (0, 1)$ the variable exponent $p_\theta(\cdot)$ defined by (1.3) belongs to $\mathcal{B}_M(\Omega)$ and

$$\|M\|_{L^{p_\theta(\cdot)}_{\mathrm{sum}} \to L^{p_\theta(\cdot)}_{\mathrm{sum}}} \le 96 \|M\|^{1-\theta}_{L^{p_0(\cdot)}_{\mathrm{sum}} \to L^{p_0(\cdot)}_{\mathrm{sum}}} \|M\|^{\theta}_{L^{p_1(\cdot)}_{\mathrm{sum}} \to L^{p_1(\cdot)}_{\mathrm{sum}}}. \tag{5.1}$$

Note that inequality (5.1) is stated in [4] with the constant 48, which seems to be a typo. This result was obtained as a consequence of the pointwise inequality

$$|T_f f| \le Mf \le 2T_f |f|, \tag{5.2}$$

where each T_f is a linear integral operator with a positive kernel (see [6]). On the other hand, it was shown in [4, Theorem 1] that if T is a linear integral operator with a positive kernel that satisfies $\|Tf\|^{\mathrm{sum}}_{p_j(\cdot)} \le B_j \|f\|^{\mathrm{sum}}_{p_j(\cdot)}$ for $j = 0, 1$ and all $f \in L^{p_j(\cdot)}_{\mathrm{sum}}(\Omega)$ with B_j independent of f, then

$$\|Tf\|^{\mathrm{sum}}_{p_\theta(\cdot)} \le 48 B_0^{1-\theta} B_1^{\theta} \|f\|^{\mathrm{sum}}_{p_\theta(\cdot)}.$$

It seems that it might not be entirely trivial to extend the proof of [4] from the Euclidean setting to the setting of quasimetric measure spaces as one would need an analogue of (5.2) for the latter. Nevertheless, the following extension and refinement of Theorem 5.1 is true.

Theorem 5.2. Let (Ω, d, μ) be a quasimetric measure space. If $p_j(\cdot) \in \mathcal{B}_M(\Omega)$ for $j = 0, 1$, then for every $\theta \in (0, 1)$ the variable exponent $p_\theta(\cdot)$ defined by (1.3) belongs to $\mathcal{B}_M(\Omega)$ and

$$\|M\|_{L^{p_\theta(\cdot)}_{\max} \to L^{p_\theta(\cdot)}_{\max}} \le C_{\max}(\theta, p_0, p_1) \|M\|^{1-\theta}_{L^{p_0(\cdot)}_{\max} \to L^{p_0(\cdot)}_{\max}} \|M\|^{\theta}_{L^{p_1(\cdot)}_{\max} \to L^{p_1(\cdot)}_{\max}}, \tag{5.3}$$

$$\|M\|_{L^{p_\theta(\cdot)}_{\mathrm{sum}} \to L^{p_\theta(\cdot)}_{\mathrm{sum}}} \le C_{\mathrm{sum}}(\theta, p_0, p_1, p_0, p_1) \|M\|^{1-\theta}_{L^{p_0(\cdot)}_{\mathrm{sum}} \to L^{p_0(\cdot)}_{\mathrm{sum}}} \|M\|^{\theta}_{L^{p_1(\cdot)}_{\mathrm{sum}} \to L^{p_1(\cdot)}_{\mathrm{sum}}}, \tag{5.4}$$

where the constants $C_{\max}(\theta, p_0, p_1)$ and $C_{\mathrm{sum}}(\theta, p_0, p_1, p_0, p_1)$ are defined by (1.6) and (1.8)–(1.10), respectively.

Proof. The idea of the proof is borrowed from [15, Theorem 15.13]. Consider the Calderón products

$$X^\theta_{\max}(\Omega, \mu) := \left(L^{p_0(\cdot)}_{\max}(\Omega, \mu)\right)^{1-\theta} \left(L^{p_1(\cdot)}_{\max}(\Omega, \mu)\right)^{\theta}, \quad X^\theta_{\mathrm{sum}}(\Omega, \mu) := \left(L^{p_0(\cdot)}_{\mathrm{sum}}(\Omega, \mu)\right)^{1-\theta} \left(L^{p_1(\cdot)}_{\mathrm{sum}}(\Omega, \mu)\right)^{\theta}.$$

Suppose that $f \in L^{p_\theta(\cdot)}_{\max}(\Omega, \mu) = L^{p_\theta(\cdot)}_{\mathrm{sum}}(\Omega, \mu)$ (we understand the latter equality as equality of sets). By Lemmas 3.2 and 3.3, $f \in X^\theta_{\max}(\Omega, \mu) = X^\theta_{\mathrm{sum}}(\Omega, \mu)$, and

$$\|f\|_{X^\theta_{\max}} \le \|f\|^{\max}_{p_\theta(\cdot)}, \tag{5.5}$$

$$\|f\|_{X^\theta_{\mathrm{sum}}} \le C^{(1)}_{\mathrm{sum}}(\theta, p_0, p_1) \|f\|^{\mathrm{sum}}_{p_\theta(\cdot)}. \tag{5.6}$$

Then (5.5) implies that

$$|f| \le \lambda |f_0|^{1-\theta} |f_1|^\theta \tag{5.7}$$

for some $\lambda > 0$ and $f_j \in L^{p_j(\cdot)}_{\max}(\Omega, \mu)$ with $\|f_j\|^{\max}_{p_j(\cdot)} \le 1$ and $j = 0, 1$. It follows from [16, Lemma 1 and Remark 2] with $\varphi(s, t) = s^{1-\theta} t^\theta$ and inequality (5.7) that

$$Mf \le \lambda M(|f_0|^{1-\theta} |f_1|^\theta) \le \lambda (Mf_0)^{1-\theta} (Mf_1)^\theta. \tag{5.8}$$

For $j = 0, 1$, we denote

$$L^{\max}_j := \|M\|_{L^{p_j(\cdot)}_{\max} \to L^{p_j(\cdot)}_{\max}}, \quad L^{\mathrm{sum}}_j := \|M\|_{L^{p_j(\cdot)}_{\mathrm{sum}} \to L^{p_j(\cdot)}_{\mathrm{sum}}}, \quad g_j := (L^{\max}_j)^{-1} M f_j.$$

Then (5.8) implies that

$$Mf \le \lambda (L^{\max}_0 g_0)^{1-\theta} (L^{\max}_1 g_1)^\theta = (L^{\max}_0)^{1-\theta} (L^{\max}_1)^\theta \lambda g_0^{1-\theta} g_1^\theta. \tag{5.9}$$

Since $\|g_j\|^{\max}_{p_j(\cdot)} \le \|f_j\|^{\max}_{p_j(\cdot)} \le 1$ for $j = 0, 1$, inequality (5.9) implies that $Mf \in X^\theta_{\max}(\Omega, \mu)$ and

$$\|Mf\|_{X^\theta_{\max}} \le (L^{\max}_0)^{1-\theta} (L^{\max}_1)^\theta \|f\|_{X^\theta_{\max}}. \tag{5.10}$$

Similarly to (5.10), we get

$$\|Mf\|_{X^\theta_{\mathrm{sum}}} \le (L^{\mathrm{sum}}_0)^{1-\theta} (L^{\mathrm{sum}}_1)^\theta \|f\|_{X^\theta_{\mathrm{sum}}}. \tag{5.11}$$

In view if Lemmas 3.4 and 3.5, we have

$$\|Mf\|^{\max}_{p_\theta(\cdot)} \le C_{\max}(\theta, p_0, p_1) \|Mf\|_{X^\theta_{\max}}, \tag{5.12}$$

$$\|Mf\|^{\mathrm{sum}}_{p_\theta(\cdot)} \le C^{(2)}_{\mathrm{sum}}(\theta, p_0, p_1) \|Mf\|_{X^\theta_{\mathrm{sum}}}. \tag{5.13}$$

Combining (5.5), (5.10), and (5.12), we obtain

$$\|Mf\|^{\max}_{p_\theta(\cdot)} \le C_{\max}(\theta, p_0, p_1) (L^{\max}_0)^{1-\theta} (L^{\max}_1)^\theta \|f\|^{\max}_{p_\theta(\cdot)},$$

which immediately implies (5.3). Similarly, gathering (5.6), (5.11), and (5.13), we get

$$\|Mf\|^{\mathrm{sum}}_{p_\theta(\cdot)} \le C^{(1)}_{\mathrm{sum}}(\theta, p_0, p_1) C^{(2)}_{\mathrm{sum}}(\theta, p_0, p_1) (L^{\mathrm{sum}}_0)^{1-\theta} (L^{\mathrm{sum}}_1)^\theta \|f\|^{\mathrm{sum}}_{p_\theta(\cdot)},$$

which yields (5.4). □

We emphasize that Theorems 5.2 and 1.3(d) imply that one can substitute the constant 96 in (5.1) by the constant $C_{\mathrm{sum}}(\theta, p_0, p_1, p_0, p_1)$ bounded by 4.

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