Integr. Equ. Oper. Theory (2023) 95:6
https://doi.org/10.1007/s00020-023-02725-8
Published online January 18, 2023

Integral Equations and Operator Theory

The Coburn Lemma and the Hartman–Wintner–Simonenko Theorem for Toeplitz Operators on Abstract Hardy Spaces

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Abstract. Let X be a Banach function space on the unit circle $\mathbb{T}$, let X' be its associate space, and let $H[X]$ and $H[X']$ be the abstract Hardy spaces built upon X and X', respectively. Suppose that the Riesz projection P is bounded on X and $a \in L^\infty \setminus \{0\}$. We show that P is bounded on X'. So, we can consider the Toeplitz operators $T(a)f = P(af)$ and $T(\overline{a})g = P(\overline{a}g)$ on $H[X]$ and $H[X']$, respectively. In our previous paper, we have shown that if X is not separable, then one cannot rephrase Coburn's lemma as in the case of classical Hardy spaces H^p, $1 < p < \infty$, and guarantee that $T(a)$ has a trivial kernel or a dense range on $H[X]$. The first main result of the present paper is the following extension of Coburn's lemma: the kernel of $T(a)$ or the kernel of $T(\overline{a})$ is trivial. The second main result is a generalisation of the Hartman–Wintner–Simonenko theorem saying that if $T(a)$ is normally solvable on the space $H[X]$, then $1/a \in L^\infty$.

Mathematics Subject Classification. Primary 47B35, 46E30.

Keywords. Banach function space, Toeplitz operator, Coburn's lemma, Normal solvability, Fredholmness, Invertibility.

1. Introduction

Let E be a Banach space and $\mathcal{B}(E)$ be the Banach algebra of all bounded linear operators on E. For $A \in \mathcal{B}(E)$, let

$$\operatorname{Ker} A := \{x \in E \ : \ Ax = 0\}, \quad \operatorname{Ran} A := \{Ax \ : \ x \in E\}.$$

Following [8, Section 4.1], an operator $A \in \mathcal{B}(E)$ is said to be normally solvable if $\operatorname{Ran} A$ is closed in E.

For a function $f \in L^1$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, let

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}$$

be the Fourier coefficients of f. Let X be a Banach space of measurable complex-valued functions on $\mathbb{T}$ continuously embedded into L^1. Let

$$H[X] := \{g \in X \ : \ \widehat{g}(n) = 0 \quad \text{for all} \quad n < 0\}$$

denote the abstract Hardy space built upon the X. In the case $X = L^p$, where $1 \le p \le \infty$, we will use the standard notation $H^p := H[L^p]$. We will also use the following notation:

$$\mathbf{e}_m(z) := z^m, \quad z \in \mathbb{C}, \quad m \in \mathbb{Z}.$$

Consider the operators $\mathcal{C}$ and P, defined for a function $f \in L^1$ and an a.e. point $t \in \mathbb{T}$ by

$$(\mathcal{C}f)(t) := \frac{1}{\pi i}\, \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad (Pf)(t) := \frac{1}{2}(f(t) + (\mathcal{C}f)(t)),$$

respectively, where the integral is understood in the Cauchy principal value sense. The operator $\mathcal{C}$ is called the Cauchy singular integral operator and the operator P is called the Riesz projection. Assume that the Riesz projection is bounded on X. For $a \in L^\infty$, the Toeplitz operator with symbol a is defined by

$$T(a)f = P(af), \quad f \in H[X].$$

It is clear that $T(a) \in \mathcal{B}(H[X])$ and $\|T(a)\|_{H[X] \to H[X]} \le \|P\|_{X \to X} \|a\|_{L^\infty}$.

This paper deals with extensions of two classical results on Toeplitz operators acting on the classical Hardy spaces H^p with $1 < p < \infty$. A fairly complete account on Toeplitz operators in this setting can be found in [2,3] (see also references given there). Lewis Coburn observed the following in the proof of [4, Theorem 4.1] (see also [5, Proposition 7.24]).

Lemma 1.1. (Coburn) *If $T(a)$ is a non-zero Toeplitz operator on H^2, then*

$$\operatorname{Ker} T(a) = \{0\} \quad \text{or} \quad \operatorname{Ker} T(\overline{a}) = \{0\}.$$

This result remains true for H^p with $1 < p < \infty$, and it can be rephrased as follows (see, e.g., [3, Theorem 2.38] or [2, Theorem 6.17]).

Theorem 1.2. *If $a \in L^\infty \setminus \{0\}$, then the Toeplitz operator $T(a)$ has a trivial kernel or a dense range on each Hardy space H^p with $1 < p < \infty$.*

Another basic result in the theory of Toeplitz operators on Hardy spaces H^p with $1 < p < \infty$ is usually attributed to Hartman-Wintner [11] and Simonenko [26]. It says the following (see [2, Theorem 6.20] and also [3, Theorem 2.30]).

Theorem 1.3. (Hartman–Wintner–Simonenko) *If $a \in L^\infty \setminus \{0\}$ and the Toeplitz operator $T(a)$ is normally solvable on a Hardy space H^p with $1 < p < \infty$, then $\frac{1}{a} \in L^\infty$.*

Note that the normal solvability of paired operators $aP + bQ$, where $Q := I - P$, is a more delicate matter (see [21, Theorem 2] for the case of L^2 and [12] for the case of L^p).

Let X be a Banach function space and X' be its associate space (see [1, Ch. 1] or Sect. 2.1 below). It follows from [1, Ch. 1, Corollaries 4.4 and 5.6] that a Banach function space X is reflexive if and only if the space X and its associate space X' are separable. Analogues of the above results for a reflexive Banach function space X, on which the Riesz projection P is bounded, were established in [13, Theorems 6.8–6.9] (under the additional assumption that the space X is rearrangement-invariant) and in [14, Theorems 6.11–6.12] (see also [15, Theorem 1.1]) (without this assumption) in the equivalent setting of singular integral operators $aP+Q$, where $a \in L^\infty$. Note also that the normal solvability of $aP + bQ$ with $a, b \in C$ on a separable rearrangement-invariant Banach function space, on which the Riesz projection P is bounded, was studied in [22].

In this paper, we do not assume that X is reflexive or separable. The possible lack of separability significantly complicates the matter because the Banach dual space X^* does not coincide with the associate (Köthe dual) space X' (see [1, Ch. 1, Corollaries 4.3 and 5.6]). In particular, a direct analogue of Theorem 1.2 is not true for the whole space X if X is not separable (see [19] and Sect. 6). It is only true when one replaces the space X by the subspace X_b, which is the closure the set of all simple functions with respect to the norm of X (see Theorem 6.1 below).

Below we only assume that the Riesz projection P is bounded on X. Then it is also bounded on the associate space X' (see Theorem 3.4 below). So, we can consider Toeplitz operators $T(a) : H[X] \to H[X]$ and $T(\overline{a}) : H[X'] \to H[X']$ simultaneously. Our first main result is the following extension of Lemma 1.1.

Theorem 1.4. *Let X be a Banach function space with the associate space X'. If the Riesz projection P is bounded on the space X and $a \in L^\infty \setminus \{0\}$, then the kernel of the Toeplitz operator $T(a) : H[X] \to H[X]$ or the kernel of the Toeplitz operator $T(\overline{a}) : H[X'] \to H[X']$ is trivial.*

Our second main result is the following generalisation of Theorem 1.3.

Theorem 1.5. *Let X be a Banach function space on which the Riesz projection P is bounded. If $a \in L^\infty \setminus \{0\}$ and the Toeplitz operator $T(a) : H[X] \to H[X]$ is normally solvable, then $\frac{1}{a} \in L^\infty$.*

The paper is organised as follows. In Sect. 2, we recall definitions of a Banach function space and its associate space X', of the subspace X_a of all functions of absolutely continuous norm and of the subspace X_b, which is the closure of the set of all simple functions in X. Further, we note that if $X_a = X_b$, then the set of trigonometric polynomials $\mathcal{P}$ is dense in X_b. We also need a few notions from the theory of analytic functions on the open unit disk $\mathbb{D}$. In Sect. 3, we first prove that if P is bounded from X_b to X, then $X_a = X_b$. Further, we show that if P is bounded from X_b to X, then it is also bounded from X to X (with the same norm) and from X' to X'. Sect. 4 is devoted to the proof of Theorem 1.4. In Sect. 5, we prove Theorem 1.5 and, as a consequence of it, establish the spectral inclusion theorem saying that the essential range of the symbol of a Toeplitz operator $T(a)$ is contained in its essential spectrum. Finally, in Sect. 6, we recall our recent results [19], which imply that there is no analogue of Theorem 1.2 for non-separable Banach function spaces X.

2. Preliminaries

2.1. Banach Function Spaces and Their Associate Spaces

Let $\mathcal{M}$ be the set of all measurable complex-valued functions on $\mathbb{T}$ equipped with the normalised measure $dm(t) = |dt|/(2\pi)$ and let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$.

Following [1, Ch. 1, Definition 1.1], a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in \mathcal{M}^+$ with $n \in \mathbb{N}$, and for all constants $a \geq 0$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,

(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),

(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\rho(1) < \infty$,

(A5) $\displaystyle\int_{\mathbb{T}} f(t)\, dm(t) \leq C\rho(f)$

with a constant $C \in (0, \infty)$ that may depend on ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X equipped with the natural linear space operations and this norm becomes a Banach space (see [1, Ch. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{T}} f(t)g(t)\, dm(t) \ : \ f \in \mathcal{M}^+, \ \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [1, Ch. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X. The associate space X' can be viewed as a subspace of the Banach dual space X^*. For $f \in X$ and $g \in X'$, put

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t)\overline{g(t)}\, dm(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta.$$

Let S_0 be the set of all simple functions on $\mathbb{T}$. The following lemma can be proved by a minor modification of the proof of [18, Lemma 2.10].

Lemma 2.1. *Let X be a Banach function space and X' be its associate space. For every $f \in X$,*

$$\|f\|_X = \sup\{|\langle f, s\rangle| \ : \ s \in S_0, \ \|s\|_{X'} \leq 1\}.$$

2.2. Density of Trigonometric Polynomials in the Subspace X_b

The characteristic (indicator) function of a measurable set $E \subset \mathbb{T}$ is denoted by $\mathbb{1}_E$. Following [1, Ch. 1, Definition 3.1], a function f in a Banach function space X is said to have absolutely continuous norm in X if $\|f\mathbb{1}_{\gamma_n}\|_X \to 0$ for every sequence $\{\gamma_n\}$ of measurable sets such that $\mathbb{1}_{\gamma_n} \to 0$ almost everywhere as $n \to \infty$. The set of all functions of absolutely continuous norm in X is denoted by X_a. If $X_a = X$, then one says that X has absolutely continuous norm. Following [1, Ch. 1, Definition 3.9], let X_b denote the closure of S_0 in the norm of X. By [1, Ch. 1, Proposition 3.10 and Theorem 3.11], X_b is the closure in X of the set of all bounded functions, and $X_a \subset X_b \subset X$.

For $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, a function of the form $\sum_{k=-n}^{n} \alpha_k \mathbf{e}_k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{-n, \dots, n\}$, is called a trigonometric polynomial of order n. The set of all trigonometric polynomials is denoted by $\mathcal{P}$.

Lemma 2.2. ([17, Lemma 2.1]) *Let X be a Banach function space. If $X_a = X_b$, then the set of trigonometric polynomials $\mathcal{P}$ is dense in X_b.*

2.3. Classes of Analytic Functions on the Open Unit Disk

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^p(\mathbb{D})$, $0 < p \le \infty$, if

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \quad 0 < p < \infty,$$
$$\|F\|_{H^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} |F(z)| < \infty.$$

Let g be a measurable function on $\mathbb{T}$ with $\log|g| \in L^1$. An outer function (of absolute value $|g|$) is a function $f = \lambda G$ with $|\lambda| = 1$ and

$$G(z) := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{,\theta} - z} \log|g(e^{i\theta})| \, d\theta \right), \quad z \in \mathbb{D}$$

(see, e.g., [24, Definition 3.1.1]). The Smirnov class $\mathcal{D}(\mathbb{D})$ consists of all functions f analytic in $\mathbb{D}$, which can be represented in the form $f = f_1/f_2$, where f_2 is outer and $f_1, f_2 \in \bigcup_{0 < p \le \infty} H^p(\mathbb{D})$ (see, e.g., [24, Definition 3.3.1]).

3. On the Boundendess of the Riesz Projection

3.1. Two Known Facts on the Riesz Projection

We start this section with two known results on the operator P which will be needed later.

Lemma 3.1. ([17, formula (1.4)]) *If $f \in L^1$ is such that $Pf \in L^1$, then*

$$(Pf)\widehat{\ }(n) = \begin{cases} \widehat{f}(n), & \text{if} \quad n \ge 0, \\ 0, & \text{if} \quad n < 0. \end{cases}$$

Lemma 3.2. ([16, Lemma 3.1]) *Let $f \in L^1$. Suppose there exists $g \in H^1$ such that $\widehat{f}(n) = \widehat{g}(n)$ for all $n \ge 0$. Then $Pf = g$.*

3.2. Necessary Condition for the Boundedness of P from X_b to X

We will need the following refinement of [17, Theorem 3.7].

Theorem 3.3. *Let X be a Banach function space. If the Riesz projection P is bounded from X_b to X, then $X_a = X_b$.*

Proof. The proof is similar to the proof of [17, Lemma 3.6]. For $f \in L^1$ consider its periodic Hilbert transform defined by

$$(\mathcal{H}f)\left(e^{i\vartheta}\right) := \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\cot\frac{\vartheta-\theta}{2}\,d\theta, \quad \vartheta \in [-\pi,\pi].$$

Then

$$Pf := \frac{1}{2}(f + i\mathcal{H}f) + \frac{1}{2}\widehat{f}(0) \tag{3.1}$$

(cf. [7, p. 104], [3, Section 1.43] and also [17, formula (1.3)]). Since X_b is a Banach space isometrically embedded into X (see [1, Ch. 1, Theorem 3.1]) and X is continuously embedded into L^1, the functional $f \mapsto \widehat{f}(0)$ is continuous on the space X_b. Then it follows from (3.1) that $P : X_b \to X$ is bounded if and only if $\mathcal{H} : X_b \to X$ is bounded. Taking into account that L^∞ is continuously embedded into X_b (see [1, Ch. 1, Proposition 3.10]), we conclude that $\mathcal{H} : L^\infty \to X$ is bounded. It follows from this observation and [17, Lemma 3.1 and Theorem 3.4] that $X_a = X_b$. $\square$

3.3. Boundedness of the Riesz Projection on the Associate Space

We are in a position to prove the main result of this section.

Theorem 3.4. *Let X be a Banach function space and X' be its associate space. If $P : X_b \to X$ is bounded, then $P : X \to X$ is bounded, P maps X_b into itself,*

$$\|P\|_{X\to X} = \|P\|_{X_b\to X_b}, \tag{3.2}$$

and the adjoint of the bounded operator $P : X_b \to X_b$ is the operator $P : X' \to X'$, which implies that the latter is also bounded.

Proof. Since $P : X_b \to X$ is bounded, by Theorem 3.3, we have $X_a = X_b$. Take any $f \in X_b$. In view of Lemma 2.2, there exist trigonometric polynomials p_n, $n \in \mathbb{N}$, such that $\|f - p_n\|_X \to 0$ as $n \to \infty$. Then

$$\|Pf - Pp_n\|_X \le \|P\|_{X_b\to X}\|f - p_n\|_X \to 0 \quad \text{as} \quad n \to \infty.$$

Since the trigonometric polynomials Pp_n are bounded, we conclude that $Pf \in X_b$ (see [1, Ch. 1, Proposition 3.10]). So, P maps X_b into itself, and the operator $P : X_b \to X_b$ is bounded.

On the other hand, it follows from $X_a = X_b$ and [1, Ch. 1, Corollary 4.2] that $(X_b)^* = X'$, so the adjoint operator $P^* : X' \to X'$ is bounded. We have

$$\langle Pf, h\rangle = \langle f, P^*h\rangle \quad \text{for all} \quad f \in X_b \quad \text{and} \quad h \in X'.$$

Taking $f = \mathbf{e}_n$, we get for $n \ge 0$,

$$\widehat{h}(n) = \langle h, \mathbf{e}_n\rangle = \overline{\langle \mathbf{e}_n, h\rangle} = \overline{\langle P\mathbf{e}_n, h\rangle} = \overline{\langle \mathbf{e}_n, P^*h\rangle} = \langle P^*h, \mathbf{e}_n\rangle = (P^*h)\widehat{\ }(n),$$

while for $n < 0$, we have

$$0 = \overline{\langle 0, h\rangle} = \overline{\langle P\mathbf{e}_n, h\rangle} = \overline{\langle \mathbf{e}_n, P^*h\rangle} = \langle P^*h, \mathbf{e}_n\rangle = (P^*h)\widehat{\ }(n).$$

So,

$$(P^*h)\widehat{\ }(n) = \widehat{h}(n) \quad \text{for} \quad n \geq 0, \qquad (P^*h)\widehat{\ }(n) = 0 \quad \text{for} \quad n < 0.$$

Since $P^*h \in X' \hookrightarrow L^1$, it follows from the above that $P^*h \in H^1$. Then Lemma 3.2 implies that $Ph = P^*h$ for all $h \in X'$. Therefore $P : X' \to X'$ is bounded and

$$\|P\|_{X'\to X'} = \|P^*\|_{(X_b)^*\to(X_b)^*} = \|P\|_{X_b\to X_b} = \|P\|_{X_b\to X}. \tag{3.3}$$

The operator $P : (X')_b \to X'$ is bounded as the restriction of the bounded operator $P : X' \to X'$ to the subspace $(X')_b$ and

$$\|P\|_{X'\to X'} \geq \|P\|_{(X')_b\to X'}. \tag{3.4}$$

Applying the above argument to the space X' in place of X, by analogy with (3.3), we get that $P : (X')' \to (X')'$ is bounded and

$$\|P\|_{(X')'\to(X')'} = \|P\|_{(X')_b\to(X')_b} = \|P\|_{(X')_b\to X'}. \tag{3.5}$$

Taking into account that $(X')' = X$ with equal norms (see [1, Ch. 1, Theorem 2.7]), we conclude that $P : X \to X$ is bounded and

$$\|P\|_{X\to X} = \|P\|_{(X')'\to(X')'}. \tag{3.6}$$

Since $P : X_b \to X_b$ is the restriction of $P : X \to X$ to X_b, we have

$$\|P\|_{X\to X} \geq \|P\|_{X_b\to X_b}. \tag{3.7}$$

Combining (3.3)–(3.7), we get

$$\begin{aligned}\|P\|_{X_b\to X_b} &= \|P\|_{X'\to X'} \geq \|P\|_{(X')_b\to X'} = \|P\|_{(X')'\to(X')'} \\ &= \|P\|_{X\to X} \geq \|P\|_{X_b\to X} = \|P\|_{X_b\to X_b},\end{aligned}$$

which implies (3.2). □

4. Coburn's Lemma

4.1. Duality Relations for the Riesz Projection *P*

We start this section with the duality relations for the Riesz projection P.

Lemma 4.1. *Let X be a Banach function space with the associate space X'. If the Riesz projection P is bounded on the space X and $Q := I - P$, then for all $f \in X$ and $h \in X'$,*

$$\langle Pf, h\rangle = \langle Pf, Ph\rangle = \langle f, Ph\rangle, \tag{4.1}$$

$$\langle Pf, Qh\rangle = 0 = \langle Qf, Ph\rangle. \tag{4.2}$$

Proof. It follows from Theorem 3.4 that P is bounded on X'. Hence Ph and Qh belong to X', whence all expressions in (4.1) and (4.2) are well defined. It is easy to see that (4.2) implies (4.1), so it is sufficient to prove the former. It follows from Lemma 3.1 that $(Qh)\widehat{\ }(n) = 0$ for $n \geq 0$. Hence $(\overline{Qh})\widehat{\ }(n) = \overline{(Qh)\widehat{\ }(-n)} = 0$ for $n \leq 0$. This implies that $\overline{Qh} \in H[X'] \subset H^1$.

For functions $Pf \in H[X] \subset H^1$ and $\overline{Qh} \in H^1$, let F and G denote their analytic extensions to the unit disk $\mathbb{D}$ by means of their Poisson integrals. Then $F, G \in H^1(\mathbb{D})$ and $G(0) = (\overline{Qh})\widehat{\ }(0) = 0$ (see, e.g., [6, Theorem 3.4]). Since $F, G \in H^1(\mathbb{D})$, by Hölder's inequality, $FG \in H^{1/2}(\mathbb{D})$. On the other hand, since $Pf \in X$ and $\overline{Qh} \in X'$, it follows from Hölder's inequality for Banach function spaces (see [1, Ch. 1, Theorem 2.4]) that $Pf\overline{Qh} \in L^1$. By the Smirnov theorem, $H^p(\mathbb{D}) = \mathcal{D} \cap L^p$ for $0 < p \leq \infty$ (see, e.g., [6, Theorem 2.11] or [24, Section 3.3.1 (a), (g)]). Therefore $FG \in H^1(\mathbb{D})$. Since $(FG)(0) = F(0)G(0) = 0$, we get

$$\begin{aligned} 0 &= (FG)(0) = (Pf\overline{Qh})\widehat{\ }(0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Pf\big(e^{i\theta}\big)\,\overline{Qh}\big(e^{i\theta}\big)\,d\theta = \langle Pf, Qh\rangle, \end{aligned} \tag{4.3}$$

which proves the first equality in (4.2). By the Lorentz-Luxemburg theorem (see [1, Ch. 1, Theorem 2.7]), one has $X'' = X$. Using (4.3) with $h \in X'$ and $f \in X'' = X$ in place of $f \in X$ and $h \in X'$, respectively, we get

$$\langle Qf, Ph\rangle = \overline{\langle Ph, Qf\rangle} = 0,$$

which completes the proof. □

4.2. Duality Relations for Toeplitz Operators

The following duality relations for Toeplitz operators will play a crucial role in the proof of our version of Coburn's lemma.

Lemma 4.2. *Let X be a Banach function space with the associate space X'. If the Riesz projection P is bounded on the space X and $a \in L^\infty$, then for all $u \in H[X]$ and $v \in H[X']$,*

$$\langle T(a)u, v\rangle = \langle u, T(\overline{a})\,v\rangle\,. \tag{4.4}$$

Proof. By Theorem 3.4, the operator P is bounded on the space X'. Hence all expressions in (4.4) are well defined. If $u \in H[X]$ and $v \in H[X']$, then in view of Lemma 3.2,

$$Pu = u, \quad Pv = v. \tag{4.5}$$

Using (4.1) and (4.5), one gets

$$\begin{aligned} \langle T(a)u, v\rangle &= \langle P(au), v\rangle = \langle au, Pv\rangle = \langle au, v\rangle = \langle u, \overline{a}v\rangle \\ &= \langle Pu, \overline{a}v\rangle = \langle u, P(\overline{a}v)\rangle = \langle u, T(\overline{a})\,v\rangle\,, \end{aligned}$$

which completes the proof. □

4.3. Proof of Theorem 1.4

Let $u \in H[X] \subset H^1$ and $v \in H[X'] \subset H^1$ be such that $T(a)u = 0$ and $T(\overline{a})\,v = 0$. Since $a \in L^\infty(\mathbb{T})$, $u \in X$, $v \in X'$, it follows from Hölder's inequality that $g := au\overline{v} \in L^1$.

Let $n \geq 0$. It is easy to check that $v\mathbf{e}_n \in H[X']$. Then Lemma 3.2 implies that $P(v\mathbf{e}_n) = v\mathbf{e}_n$. Using this observation and (4.1), we get

$$\begin{aligned} \widehat{g}(n) &= \langle g, \mathbf{e}_n\rangle = \langle au\overline{v}, \mathbf{e}_n\rangle = \langle au, v\mathbf{e}_n\rangle = \langle au, P(v\mathbf{e}_n)\rangle = \langle P(au), v\mathbf{e}_n\rangle \\ &= \langle T(a)u, v\mathbf{e}_n\rangle = 0 \quad \text{for all} \quad n \geq 0. \end{aligned}$$

Similarly, if $n \le 0$, then $\mathbf{e}_{-n}u \in H[X]$ and $P(\mathbf{e}_{-n}u) = \mathbf{e}_{-n}u$. Using this observation and applying (4.1) once again, we obtain

$$\begin{aligned}\widehat{g}(n) &= \langle a u \overline{v}, \mathbf{e}_n \rangle = \langle \mathbf{e}_{-n}u, \overline{a}v \rangle = \langle P(\mathbf{e}_{-n}u), \overline{a}v \rangle = \langle \mathbf{e}_{-n}u, P(\overline{a}v) \rangle \\ &= \langle \mathbf{e}_{-n}u, T(\overline{a})\, v \rangle = 0 \quad \text{for all} \quad n \le 0.\end{aligned}$$

So, all Fourier coefficients of g are equal to 0, i.e. $g = 0$ a.e. (see, e.g., [20, Ch. 1, Theorem 2.7]). Since $a \ne 0$, the product $u\overline{v}$ is equal to 0 on a set of positive measure. Then at least one of the functions $u \in H^1$ and $v \in H^1$ is equal to 0 on a set of positive measure and hence a.e. (see, e.g., [6, Theorem 2.2]). □

5. Normal Solvability of Toeplitz Operators

5.1. Relations Between the Range of $T(a)$ on $H[X_b]$ and the Kernel of $T(\overline{a})$ on $H[X']$

If P is bounded on X, then it maps X_b into itself according to Theorem 3.4. For any $a \in L^\infty$, the operator aI also maps X_b into itself. Hence $T(a) : H[X_b] \to H[X_b]$ is a bounded operator.

If $\mathcal{S}$ is a subset of of a Banach space E, then $\operatorname{clos}_E(\mathcal{S})$ denotes the closure of $\mathcal{S}$ in E.

Lemma 5.1. *Let X be a Banach function space with the associate space X'. If the Riesz projection P is bounded on the space X and $a \in L^\infty$, then $g \in H[X_b]$ belongs to the closure $\operatorname{clos}_{H[X_b]}\big(\operatorname{Ran} T(a)\big)$ of the range of the Toeplitz operator $T(a) : H[X_b] \to H[X_b]$ if and only if $\langle g, v\rangle = 0$ for every v in the kernel of the Toeplitz operator $T(\overline{a}) : H[X'] \to H[X']$.*

Proof. Since $(X_b)^* = X'$ and $P : X_b \to X_b$ (see the proof of Theorem 3.4), one can show in exactly the same way as for the classical Hardy spaces (see [6, Theorem 7.3]) that the dual of $H[X_b]$ is (non-isometrically) isomorphic to $H[X']$ and that the adjoint of $T(a) : H[X_b] \to H[X_b]$ can be identified with $T(\overline{a}) : H[X'] \to H[X']$ (see Lemma 4.2). Then the lemma follows from a standard fact about bounded linear operators (see, e.g. [25, formula (3.13)]).

One can rephrase this proof in such a way that it avoids explicitly using the isomorphism $(H[X_b])^* \rightleftarrows H[X']$.

Necessity. Suppose $g \in \operatorname{clos}_{H[X_b]}\big(\operatorname{Ran} T(a)\big)$. Then there exist $\varphi_n \in H[X_b]$, $n \in \mathbb{N}$ such that $\|g - T(a)\varphi_n\|_X \to 0$ as $n \to \infty$. By Hölder's inequality for Banach function spaces (see [1, Ch. 1, Theorem 2.4]), for every $v \in H[X']$ and every $n \in \mathbb{N}$,

$$|\langle g, v\rangle - \langle T(a)\varphi_n, v\rangle| \le \|g - T(a)\varphi_n\|_X \|v\|_{X'}.$$

Hence

$$\langle g, v\rangle = \lim_{n\to\infty} \langle T(a)\varphi_n, v\rangle.$$

Using Lemma 4.2, one gets $\langle T(a)\varphi_n, v\rangle = \langle \varphi_n, T(\overline{a})v\rangle$ for all $n \in \mathbb{N}$. Thus, for all $v \in \operatorname{Ker} T(\overline{a})$,

$$\langle g, v\rangle = \lim_{n\to\infty} \langle T(a)\varphi_n, v\rangle = \lim_{n\to\infty} \langle \varphi_n, T(\overline{a})\, v\rangle = 0,$$

which completes the proof of the necessity portion.

Sufficiency. Suppose that $g \in H[X_b] \setminus \operatorname{clos}_{H[X_b]} \left(\operatorname{Ran} T(a) \right)$ and

$$\langle g, u \rangle = 0 \quad \text{for all} \quad u \in \operatorname{Ker} T(\overline{a}). \tag{5.1}$$

Then, by the Hahn-Banach theorem, there exists $h \in (X_b)^* = X'$ such that $\langle g, h \rangle = 1$ and $\langle T(a)\varphi, h \rangle = 0$ for all $\varphi \in H[X_b]$. Lemma 3.2 implies that $T(a)\varphi = PT(a)\varphi$ and $T(\overline{a})v = PT(\overline{a})v$ for every $v \in H[X']$. Let $v := Ph \in H[X']$. Note that $\varphi := P\psi \in H[X_b]$ for every $\psi \in X_b$ (see Lemma 3.1). Combining this observations with (4.1) and Lemma 4.2, we see that

$$\begin{aligned}\langle \psi, T(\overline{a})\, v \rangle &= \langle \psi, PT(\overline{a})\, v \rangle = \langle P\psi, T(\overline{a})\, v \rangle = \langle \varphi, T(\overline{a})\, v \rangle = \langle T(a)\varphi, v \rangle \\ &= \langle T(a)\varphi, Ph \rangle = \langle PT(a)\varphi, h \rangle = \langle T(a)\varphi, h \rangle = 0.\end{aligned}$$

Hence for $n \in \mathbb{Z}$,

$$(T(\overline{a})v)\widehat{\ }(n) = \langle T(\overline{a})v, \mathbf{e}_n \rangle = \overline{\langle \mathbf{e}_n, T(\overline{a})v \rangle} = 0. \tag{5.2}$$

Thus, by the uniqueness theorem for Fourier series (see, e.g., [20, Ch. 1, Theorem 2.7]), $T(\overline{a})\, v = 0$, i.e. $v \in \operatorname{Ker} T(\overline{a})$. On the other hand, since $g \in H[X_b]$, Lemma 3.2 implies that $Pg = g$. Then, in view of (4.1), we see that

$$\langle g, v \rangle = \langle g, Ph \rangle = \langle Pg, h \rangle = \langle g, h \rangle = 1,$$

which contradicts (5.1). Thus $g \in \operatorname{clos}_{H[X_b]} \left(\operatorname{Ran} T(a) \right)$. □

5.2. Proof of Theorem 1.5

Suppose $\frac{1}{a} \notin L^\infty$. Let

$$\gamma_n := \{\zeta \in \mathbb{T} \ : \ |a(\zeta)| \le 1/n\}, \quad n \in \mathbb{N}.$$

Then $m(\gamma_n) > 0$ for all $n \in \mathbb{N}$. Let

$$g_n := \mathbb{1}_{\gamma_n} + \varepsilon_n \mathbb{1}_{\mathbb{T}\setminus\gamma_n}, \quad n \in \mathbb{N},$$

where ε_n are chosen so that

$$0 < \varepsilon_n \le \frac{\|\mathbb{1}_{\gamma_n}\|_Y}{n\|\mathbb{1}\|_Y}, \quad Y = X,\, X'.$$

Let $\varphi_n \in H^\infty \subset H[X]$ be an outer function such that $|\varphi_n| = g_n$ a.e. Then

$$\begin{aligned}\|T(a)\varphi_n\|_X &= \|P(a\varphi_n)\|_X \le \|P\|_{X\to X}\|a\varphi_n\|_X = \|P\|_{X\to X}\|ag_n\|_X \\ &\le \|P\|_{X\to X}\left(\frac{1}{n}\|\mathbb{1}_{\gamma_n}\|_X + \|a\|_{L^\infty}\varepsilon_n\|\mathbb{1}_{\mathbb{T}\setminus\gamma_n}\|_X\right) \\ &\le \|P\|_{X\to X}\left(\frac{1}{n}\|\mathbb{1}_{\gamma_n}\|_X + \|a\|_{L^\infty}\frac{\|\mathbb{1}_{\gamma_n}\|_X}{n\|\mathbb{1}\|_X}\|\mathbb{1}_{\mathbb{T}\setminus\gamma_n}\|_X\right) \\ &\le \frac{1}{n}\|P\|_{X\to X}\left(1 + \|a\|_{L^\infty}\right)\|g_n\|_X \\ &= \frac{1}{n}\|P\|_{X\to X}\left(1 + \|a\|_{L^\infty}\right)\|\varphi_n\|_X.\end{aligned} \tag{5.3}$$

By [9, Theorem IV.1.6], the operator $T(a) \in \mathcal{B}(H[X])$ is normally solvable if and only if its minimum modulus $\gamma(T(a))$ defined by

$$\gamma(T(a)) := \inf_{u \in H[X]} \frac{\|T(a)u\|_{H[X]}}{\operatorname{dist}(u, \operatorname{Ker} T(a))},$$

where

$$\operatorname{dist}(u, \operatorname{Ker} T(a)) := \inf_{v \in \operatorname{Ker} T(a)} \|u - v\|_{H[X]},$$

is positive.

If $\operatorname{Ker} T(a) = \{0\}$, then $\|\varphi_n\|_X = d(\varphi_n, \operatorname{Ker} T(a))$ and (5.3) implies that

$$0 \le \gamma(T(a)) \le \lim_{n\to\infty} \frac{\|T(a)\varphi_n\|_X}{\|\varphi_n\|_X} = 0.$$

Therefore the Toeplitz operator $T(a)$ cannot be normally solvable if

$$\operatorname{Ker} T(a) = \{0\}.$$

Suppose now that $\operatorname{Ker} T(a) \neq \{0\}$. Then, in view of Theorem 1.4, the kernel of $T(\overline{a}) : H[X'] \to H[X']$ is trivial. Hence, by Lemma 5.1, the range of the operator $T(a) : H[X_b] \to H[X_b]$ is dense in $H[X_b]$. Hence the Hardy space $H[X_b]$ is contained in the closure of the range of the operator $T(a) : H(X] \to H[X]$. Since the latter operator is normally solvable, $H[X_b]$ is contained in its range and $0 < \gamma(T(a))$. Therefore, for every $v \in H[X]$ there exists $s \in \operatorname{Ker} T(a) \subset H[X]$ such that

$$\|v - s\| \le \frac{2}{\gamma(T(a))} \|T(a)v\|_{H[X]}.$$

Since $H[X_b] \subset \operatorname{Ran} T(a)$, the above inequality implies that for every function $f \in H[X_b]$ there exist functions $v \in H[X]$ and $s \in \operatorname{Ker} T(a) \subset H[X]$ such that $u := v - s \in H[X]$,

$$T(a)u = f \quad \text{and} \quad \|u\|_X \le M\|f\|_X,$$

where $M := 2/\gamma(T(a))$.

We can show by analogy with (5.3) that

$$\|T(\overline{a})\,\varphi_n\|_{X'} \le \frac{1}{n}\|P\|_{X'\to X'}\,(1 + \|a\|_{L^\infty})\,\|\varphi_n\|_{X'}. \tag{5.4}$$

It follows from Lemma 2.1 and the Lorentz-Luxemburg theorem (see [1, Ch. 1, Theorem 2.7]) that for every $n \in \mathbb{N}$ there exists $s_n \in S_0$ such that $\|s_n\|_X \le 1$ and

$$|\langle \varphi_n, s_n\rangle| \ge \frac{\|\varphi_n\|_{X'}}{2}, \quad n \in \mathbb{N}. \tag{5.5}$$

Since $s_n \in X_b$, it follows from Theorem 3.4 that $h_n := Ps_n \in H[X_b]$. Then there exist $u_n \in H[X]$, $n \in \mathbb{N}$, such that $T(a)u_n = h_n$ and

$$\|u_n\|_X \le M\|h_n\|_X \le M\|P\|_{X\to X}\|s_n\|_X \le M\|P\|_{X\to X}. \tag{5.6}$$

Since $\varphi_n \in H^\infty$, it follows from Lemma 3.2 that $\varphi_n = P\varphi_n$ for $n \in \mathbb{N}$. Then using (5.4)–(5.6), (4.1), Lemma 4.2, and Hölder's inequality (see [1, Ch. 1, Theorem 2.4]), we get

$$\begin{aligned}\frac{\|\varphi_n\|_{X'}}{2} &\le |\langle \varphi_n, s_n\rangle| = |\langle P\varphi_n, s_n\rangle| = |\langle \varphi_n, Ps_n\rangle| = |\langle \varphi_n, h_n\rangle| \\ &= |\langle \varphi_n, T(a)u_n\rangle| = |\langle T(\overline{a})\,\varphi_n, u_n\rangle| \le \|T(\overline{a})\,\varphi_n\|_{X'}\|u_n\|_X \\ &\le \frac{1}{n}\|P\|_{X'\to X'}(1+\|a\|_{L^\infty})\,\|\varphi_n\|_{X'}M\|P\|_{X\to X},\end{aligned}$$

and hence

$$\frac{1}{2} \le \frac{M}{n}\|P\|_{X\to X}\|P\|_{X'\to X'}(\|a\|_{L^\infty}+1) \quad \text{for all} \quad n \in \mathbb{N}.$$

This contradiction shows that $T(a)$ cannot be normally solvable and completes the proof. □

5.3. The Spectral Inclusion Theorem

Let E be a Banach space. An operator $A \in \mathcal{B}(E)$ is called Fredholm if

$$\alpha(A) := \dim \operatorname{Ker} A < +\infty, \quad \beta(A) := \dim(X/\operatorname{Ran} A) < +\infty.$$

The integer number $\operatorname{Ind} A := \alpha(A) - \beta(A)$ is called the Fredholm index or, simply, the index of the operator A. The essential spectrum of $A \in \mathcal{B}(E)$ is the set

$$\operatorname{Spec}_{\mathrm{e}}(A;E) := \{\lambda \in \mathbb{C} \ : \ A - \lambda I \ \text{ is not Fredholm on } E\},$$

the essential spectral radius is defined by

$$r_{\mathrm{e}}(A;E) := \sup\{|\lambda| \ : \ \lambda \in \operatorname{Spec}_{\mathrm{e}}(A;E)\}.$$

The following theorem is an extension of [3, Theorem 2.30].

Theorem 5.2. *Let X be a Banach function space on which the Riesz projection P is bounded. If $a \in L^\infty$, then*

$$a(\mathbb{T})_{\mathrm{e}} := \left\{\lambda \in \mathbb{C} \ : \ \frac{1}{a-\lambda} \notin L^\infty\right\} \subseteq \operatorname{Spec}_{\mathrm{e}}(T(a); H[X]) \tag{5.7}$$

and

$$\|a\|_{L^\infty} \le r_{\mathrm{e}}(T(a); H[X]). \tag{5.8}$$

Proof. Let $\lambda \notin \operatorname{Spec}_{\mathrm{e}}(T(a); H[X])$. Then $T(a) - \lambda I = T(a-\lambda)$ is Fredholm on $H[X]$. In this case it is normally solvable (see, e.g., [9, Remark IV.2.5 and Corollary IV.1.13]). By Theorem 1.5, $1/(a-\lambda) \in L^\infty$. Thus $\lambda \notin a(\mathbb{T})_{\mathrm{e}}$, which completes the proof of (5.7). Inequality (5.8) is an immediate consequence of inclusion (5.7). □

6. Concluding Remarks

Theorems 3.4, 1.4 and Lemma 5.1 imply the following.

Theorem 6.1. *Let X be a Banach function space and $a \in L^\infty \setminus \{0\}$. If the operator $P : X_b \to X$ is bounded, then $T(a) : H[X_b] \to H[X_b]$ has a trivial kernel or a dense range.*

In general, an analogue of the above result is not true for $T(a) : H[X] \to H[X]$ if X is not separable. In order to illustrate this fact, we need the definition of the Hardy-Marcinkiewicz spaces $H[L^{p,\infty}]$ built upon the weak L^p-space $L^{p,\infty}$ (also called the Marcinkiewicz space).

The distribution function m_f of a measurable a.e. finite function $f : \mathbb{T} \to \mathbb{C}$ is given by

$$m_f(\lambda) := m(\{t \in \mathbb{T} : |f(t)| > \lambda\}), \quad \lambda \geq 0.$$

The non-increasing rearrangement of f is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \leq x\}, \quad x \geq 0.$$

We refer to [1, Ch. 2, Section 1] for properties of distribution functions and non-increasing rearrangements. For $1 < p < \infty$, the Marcinkiewicz space (or the weak-L^p space) $L^{p,\infty}$ consists of all measurable a.e. finite functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{p,\infty} := \sup_{x>0} \left(x^{1/p} f^*(x) \right)$$

is finite. Note that

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \left(\lambda m_f(\lambda)^{1/p} \right) \tag{6.1}$$

(see [10, Proposition 1.4.5(16)]). Although $\|\cdot\|_{p,\infty}$ is not a norm, it is equivalent to a norm. More precisely, by [1, Ch. 4, Lemma 4.5], for every measurable a.e. finite function $f : \mathbb{T} \to \mathbb{C}$, one has

$$\|f\|_{p,\infty} \leq \|f\|_{(p,\infty)} \leq \frac{p}{p-1} \|f\|_{p,\infty},$$

where

$$\|f\|_{(p,\infty)} := \sup_{x>0} \left(x^{1/p} f^{**}(x) \right)$$

and

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(y)\, dy, \quad x > 0.$$

In view of [1, Ch. 4, Theorem 4.6], $L^{p,\infty}$ is a Banach function space with respect to the norm $\|\cdot\|_{(p,\infty)}$. Marcinkiewicz spaces form a very interesting class of non-separable rearrangement-invariant Banach function spaces (see, e.g., [1, Ch. 4, Section 4]).

As usual, let C be the Banach space of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ with the supremum norm.

Theorem 6.2. ([19, Theorem 2]) *Let $1 < p < \infty$. Then there exists a function $a \in C\setminus\{0\}$ depending on p such that $a(-1) = 0$ and the following equalities hold for the kernel and the closure of the range of the Toeplitz operator $T(a)$ acting on the Hardy-Marcinkiewicz space $H[L^{p,\infty}]$:*

$$\dim(\operatorname{Ker} T(a)) = \infty, \quad \dim\left(H[L^{p,\infty}]/\operatorname{clos}_{H[L^{p,\infty}]}\left(\operatorname{Ran} T(a)\right)\right) = \infty.$$

The Toeplitz operator constructed in the proof of Theorem 6.2 is not normally solvable, since $a(-1) = 0$ (see Theorem 1.5). It would be interesting to find out whether there exists a normally solvable $T(a) : H[X] \to H[X]$ such that

$$\dim(\operatorname{Ker} T(a)) > 0, \quad \dim\big(H[X]/\operatorname{Ran} T(a)\big) > 0.$$

A normally solvable operator $A \in \mathcal{B}(E)$ is called semi-Fredholm if

$$\dim \operatorname{Ker} A < +\infty \quad \text{or} \quad \dim(X/\operatorname{Ran} A) < +\infty.$$

It follows from Coburn's lemma that every normally solvable Toeplitz operator $T(a) : H^p \to H^p$, $1 < p < \infty$ is semi-Fredholm. Unfortunately, we do not know whether the same is true for Toeplitz operators on (non-separable) abstract Hardy spaces $H[X]$. The proof of a version of Theorem 1.5 with "semi-Fredholm" in place of "normally solvable" is somewhat simpler than that given in Sect. 5.2. Indeed, if $a \in L^\infty\setminus\{0\}$ is equal to 0 on a set of positive measure, then using the F. and M. Riesz theorem (see, e.g., [7, Ch. II, Corollary 4.2]) one can easily prove that $\operatorname{Ker} T(a) = 0$. Then it follows from (a slightly simpler version of) (5.3) that $T(a) : H[X] \to H[X]$ is not normally solvable and hence not semi-Fredholm. If $a \in L^\infty\setminus\{0\}$ is such that $\frac{1}{a} \notin L^\infty$, then it can be approximated in the L^∞ norm by functions equal to 0 on sets of positive measure. So, $T(a) : H[X] \to H[X]$ can be approximated in the operator norm by Toeplitz operators that are not semi-Fredholm, and hence it cannot be semi-Fredholm (see [23, Ch. I, Theorem 3.9]). Therefore, if $a \in L^\infty\setminus\{0\}$, and $T(a) : H[X] \to H[X]$ is semi-Fredholm, then $\frac{1}{a} \in L^\infty$.

Funding Open access funding provided by FCT—FCCN (b-on). This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297 /2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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Received: October 11, 2022.
Accepted: January 3, 2023.