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A Simple Solution for the General Fractional Ambartsumian Equation

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Abstract: Fractionalisation and solution of the Ambartsumian equation is considered. The general approach to fractional calculus suitable for applications in physics and engineering is described. It is shown that Liouville-type derivatives are the necessary ones, because they fully preserve backward compatibility with classical results. Such derivatives are used to define and solve the fractional Ambartsumian equation. First, a solution in terms of a slowly convergent fractional Taylor series is obtained. Then, a simple solution expressed in terms of an infinite linear combination of Mittag–Leffler functions is deduced. A fast algorithm, based on a bilinear transformation and using the fast Fourier transform, is described and demonstrated for its approximate numerical realisation.

Keywords: Ambartsumian equation; Mittag–Leffler function; fractional derivative; Grünwald–Letnikov; bilinear transformation.

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1. Introduction

The original, integer order, Ambartsumian equation was introduced in the theory of surface brightness in the Milky Way [1]. It assumes the form:

$$x'(t) + x(t) = \frac{1}{q} x\left(\frac{t}{q}\right), \tag{1}$$

under a suitable initial condition $x_0 = x(0)$. The parameter q is a real greater than 1. This equation has been solved using two different methodologies. Patade and Bhalekar obtained a solution in terms of a Taylor series [2]. Alternatively, Bakodah and Ebaid, using the "Laplace-transform and decomposition method," got a solution expressed by a series with exponential terms [3]. Alharbi and Ebaid obtained a similar solution [4].

The fractional Ambartsumian equation was studied first by Kumar et al. [5] using the Caputo derivative. The solution they found was essentially a fractional Taylor series, generalising Patade and Bhalekar's formulation [2]. Two different papers obtained similar results generalising Bakodah and Ebaid's result with the exponentials substituted by Mittag–Leffler functions (MLF) [4,6,7].

Here, we will obtain a simpler formulation, also using MLF, but with a faster converging series. We will present an easy implementation of the MLF using a bilinear discrete-time equivalent [8] and the fast Fourier transform.

The paper outlines as follows. We start by considering the fractionalisation problem in Section 2. In Section 3 we obtain a solution in terms of a fractional Taylor series and refer to its drawbacks concerning computation difficulties. Alternatively, we present in Section 4 a new solution expressed as a linear combination of MLF. Numerical aspects are discussed in Section 5. Section 6 presents some conclusions.

2. On the Fractionalisation Problem

Fractional systems can be considered as the 21st century models for many natural and man-made phenomena [9]. However, they have already been used in many real applications with phenomena that require modelling beyond traditional tools. The well-known Curie—von Schweidler law, discovered in the 19th century, is a good example of a fractional behaviour modelled using fractional calculus (FC) [10,11]. Heaviside made some contributions to the application of FC in electromagnetics [12]; very curious is Heaviside's use of FC in the discussion about the age of the earth [13]. In the forties of the 20th century, S. Blair introduced FC in rheology and viscoelasticity [14,15]. More recently, the Schrödinger fractional equation was studied [16–19], as well as the fractional Maxwell equations [20–22]. Other applications in physics can be found in [20,23–25]. In fact, we are currently dealing with fractional phenomena such as $1/f$ noise, long-range dependence, fractional Gaussian noise, and fractional Brownian motion (fBm), which are ubiquitous in the scientific literature [26–28]. This raises a question: why use a fractional derivative? A first answer is obvious: we have another parameter to manipulate in order to get a better fit to the solution of a given problem. However, there may be another important reason: causality. In fact, we can enforce causality through an appropriate choice of the used derivatives. This reason is often overlooked.

A reading of the cited references shows a great diversity regarding the use of fractional operators, often leading to results that do not enjoy backward compatibility with classical theories. This is often the case in connection with the use of Riemann–Liouville (RL) or Caputo (C) derivatives. However, instead of replacing the derivative (D) with an appropriate one, this non-compatibility is accepted as a fatality of fractional calculus. The designation "metaphysical derivatives" has been coined [29]. In fact, these derivatives, which enjoy privilege among mathematicians, create several problems in scientific applications, by:

- not distinguishing between constant and Heaviside functions;
- not giving the derivative of the $\delta(t)$;
- the derivative of a sinusoid is not a sinusoid;
- non-validity of the index rule $D^\alpha D^\beta \neq D^{\alpha+\beta}$ for $\alpha, \beta > 0$ [30–32];
- together with the one-sided Laplace transform, introducing wrong initial-conditions (IC) [33–36].

These problems have been addressed and solved in a step-by-step deduction of the formulae from classical derivative definitions [37,38]. This approach makes clear the existence of different derivatives for (time) causal or (space) non-causal systems.

The classic derivative suitable for defining (1) must be given by:

$$Dx(t) = x'(t) = \lim_{h\to 0} \frac{x(t) - x(t-h)}{h}, \tag{2}$$

since it uses the present and past values of $x(t)$ (causality). This derivative has to be generalised, so that we are able to define a fractional Ambartsumian equation, given by

$$D^\alpha x(t) + x(t) = \frac{1}{q} x\left(\frac{t}{q}\right) \quad t \geq 0,\ q \geq 1, \tag{3}$$

under a suitable initial condition (IC) $x_0 = x(0)$.

Let the function $x(t)$ to be defined on $\mathbf{R}$ and α be any real number (the complex values lead to non-hermitian systems [39]). The above derivative can be generalized for any order by:

$$D^\alpha x(t) = \lim_{h\to 0^+} \frac{\sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(t-nh)}{h^\alpha}, \tag{4}$$

where $\frac{(-\alpha)_n}{n!}$, $n = 0, 1, \cdots$ represent the binomial coefficients [37] with $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$ denoting the Pochammer symbol for the raising factorial. This derivative is called "Grünwald–Letnikov" (GL), although it was proposed first by Liouville. Applying the bilateral Laplace transform (BLT),

$$X(s) = \int_{\mathbf{R}} x(t)e^{-st}\mathrm{d}t, \tag{5}$$

to (4), we obtain:

$$\mathcal{L}[x(t)] = s^{\alpha}X(s), \quad Re(s) > 0, \tag{6}$$

generalising a well-known property of the BLT [35]. It is important to remark that (4) and (6) are valid for any positive or negative real order. We will keep the designation "derivative" for positive orders. For negative ones, we will use "anti-derivative". It must be referred that, contrarily to the primitive, the anti-derivative is simultaneously the left and right inverse of the derivative $D\ D^{-1}x(t) = D^{-1}\ Dx(t) = x(t)$.

Let α be any real number and $\varepsilon(t)$ be the Heaviside-unit step function. The causal BLT inverse of s^{α},

$$\mathcal{L}^{-1}[s^{\alpha}] = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(t),$$

is the Green function of the derivative operator that, using the convolution property of the BLT, allows us to obtain three integral formulations for the fractional derivatives equivalent to (4). Let N be the positive integer greater than or equal to α ($\alpha \leq N$). Such derivatives are given by:

1. (Riemann–)Liouville derivative [30]

$$D^{\alpha}x(t) = D^{N}\left[\frac{1}{\Gamma(-\alpha+N)}\int_{0}^{\infty}\tau^{N-\alpha-1}x(t-\tau)d\tau\right]. \tag{7}$$

   The usual Riemann–Liouville derivative is a particular case of this one, obtained for functions null outside any interval $[a,b] \subset \mathbf{R}$.

2. Liouville–Caputo derivative [25]

$$D^{\alpha}x(t) = \frac{1}{\Gamma(-\alpha+N)}\int_{0}^{\infty}\tau^{N-\alpha-1}x^{(N)}(t-\tau)d\tau. \tag{8}$$

   The Caputo derivative is a particular case of this one as happens with the RL derivative.

3. Regularized Liouville derivative [40]

$$D^{\alpha}x(t) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{\infty}\tau^{-\alpha-1}\left[x(t-\tau) - \sum_{0}^{N-1}\frac{(-)^{m}x^{(m)}(t)}{m!}\tau^{m}\right]d\tau, \tag{9}$$

   where the summation is taken as being null, when $\alpha < 0$.

All the derivatives (4), (7) to (9) are equivalent for functions with BLT. As the Fourier transform is obtained from the BLT with a simple variable change $s = i\omega,\ \omega \in \mathbf{R},\ i = \sqrt{-1}$, the above derivatives exist also for functions with Fourier transform [35]. Necessary conditions for the existence of the above derivatives can be given [30,31,41]. It must be highlighted that we must take into account the behaviour of the function $x(t)$ along the half straight line $t - nh$, with $n \in \mathbf{Z}^{+}$. There are several properties exhibited by derivatives (4), (7)–(9) [37]:

- Linearity;
- Additivity and Commutativity of the orders;
- Neutral and inverse elements

$$D^{\alpha}D^{-\alpha}x(t) = D^0x(t) = x(t). \tag{10}$$

From (10) we conclude that there is always an inverse element; that is, for every α there is always the $-\alpha$ order that we called above the anti-derivative.
- Backward compatibility ($n \in \mathbf{N}$)
If $\alpha = n$, then:

$$D^n x(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^k\binom{n}{k}x(t-kh)}{h^n}.$$

We obtain this expression repeating the first order derivative.
If $\alpha = -n$, then:

$$D^{-n}f(t) = \lim_{h\to 0}\sum_{k=0}^{\infty}\frac{(n)_k}{k!}f(t-kh)\cdot h^n,$$

that corresponds to a n-th repeated summation [41,42]. For $n = -1$, we obtain the classic Riemann integral

$$D^{-1}f(t) = \lim_{h\to 0} h\sum_{k=0}^{\infty} f(t-kh).$$

- The generalized Leibniz rule
This rule gives the FD of the product of two functions and assumes the format [30]

$$D^{\alpha}[x(t)y(t)] = \sum_{k=0}^{\infty}\binom{\alpha}{k}D^k x(t)D^{\alpha-k}y(t). \tag{11}$$

This approach to fractional derivatives was used to obtain coherent formulations for linear systems, generalising the concept to non-commensurate and tempered ones, in full agreement with classic theory [9]. An interesting application to the fractional logistic equation can be found in [43].

3. Problem Formulation and a First Solution

Now, we are going into the solution of the fractional Ambartsumian Equation (3). For stability reasons, the derivative order (α) can only assume any real value in the interval $(0,2)$. The equation, as written above, corresponds to the representation of a system with null input, but having a non-null initial-condition (IC). Traditionally, the IC problems have been treated with the unilateral Laplace transform. However, this has proven to be inconsistent [35]. On the other hand, when used in connection with RL or C derivatives, it inserts wrong ICs [36]. In [36], an alternative approach into the IC problem is described. It is based on the fractional jump formula [44]. Basically, the procedure consists of modifying the original equation so that the initial-condition appears explicitly. In this particular case, we get the equation

$$D^{\alpha}x(t) + x(t) - x_0 D^{\alpha-1}\delta(t) = \frac{1}{q}x\left(\frac{t}{q}\right), \quad t \geq 0,\ q \geq 1, \tag{12}$$

where $\delta(t)$ is the Dirac impulse. This relation can be rewritten as

$$D^{\alpha}x(t) + x(t) - x_0 D^{\alpha}\varepsilon(t) = \frac{1}{q}x\left(\frac{t}{q}\right), \quad t \geq 0,\ q \geq 1,$$

since $\delta(t) = D\varepsilon(t)$. We modify it into

$$D^{\alpha}[x(t) - x_0\varepsilon(t)] + x(t) = \frac{1}{q}x\left(\frac{t}{q}\right) \quad t \geq 0,\ q \geq$$

that highlights the role of IC: to make continuous the function at the origin, before computing the derivative.

Remark 1. *The brightness, $x(t)$, is dimensionless. Therefore, having in mind that a derivative of order α introduces a division by a power of time, we must adjust the equation dimensions. Consequently, q must have units $[s^{\alpha}]$.*

Now, we are in a condition of looking for the solution of our problem. The case of $q = 1$ presents no particular difficulty. Therefore, we will not consider it.

Theorem 1. *Let $q > 1$ and $0 < \alpha < 2$. The solution of (3), under the IC x_0, is given by:*

$$x(t) = \sum_{n=0}^{\infty} C_n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad t > 0, \tag{13}$$

where $C_0 = x_0$ and

$$C_n = (-1)^n \prod_{k=0}^{n-1} \left[1 - q^{-k\alpha-1}\right] C_0, \tag{14}$$

for $n = 1, 2, \ldots$.

Proof. A somehow obvious approach to solve (12) is a search for a fractional Taylor solution as in [2,5] which corresponds, in the BLT domain, to assume that $X(s) = \mathcal{L}x(t)$ has the Laurent form

$$X(s) = \sum_{n=0}^{\infty} C_n s^{-n\alpha-1}, \tag{15}$$

where $C_n,\ n = 0, 1, 2\cdots$ are real parameters to be computed.

In terms of the bilateral Laplace transform [35], Equation (12) becomes

$$s^{\alpha}X(s) - x_0 s^{\alpha-1} + X(s) = X(qs), \quad Re(s) > 0, \tag{16}$$

where the scale property of the BLT was used.

We have,

$$s^{\alpha}\sum_{0}^{\infty} C_n s^{-n\alpha-1} + \sum_{0}^{\infty} C_n s^{-n\alpha-1} = x_0 s^{\alpha-1} + \sum_{0}^{\infty} C_n q^{-n\alpha-1} s^{-n\alpha-1},$$

that leads to

$$\sum_{n=-1}^{\infty} C_{n+1} s^{-n\alpha} + \sum_{0}^{\infty} C_n s^{-n\alpha} = x_0 s^{\alpha} + \sum_{0}^{\infty} C_n q^{-n\alpha-1} s^{-n\alpha}.$$

From this equation, we conclude immediately that

$$C_{n+1} = -C_n + C_n q^{-n\alpha-1} = -\left[1 - q^{-n\alpha-1}\right] C_n, \tag{17}$$

with $C_0 = x_0$. The solution of this difference equation is the q-shifted factorial (14) [41,45]. These coefficients constitute a decreasing sequence as $n \to \infty$. In fact, from (17) and, as $q > 1$,

$$\left|\frac{C_{n+1}}{C_n}\right| = \left[1 - q^{-n\alpha-1}\right] < 1.$$

On the other hand, from (14) and using well-known results, we obtain

$$\ln|C_n| = \sum_{k=0}^{n} \ln\left[1 - q^{-k\alpha-1}\right] + \ln C_0 \leq \sum_{k=0}^{n} q^{-k\alpha-1} + \ln C_0 = \ln C_0 + q^{-1}\frac{1 - q^{-(n+1)\alpha-1}}{1 - q^{-\alpha}},$$

showing the convergence of the infinite product in (14). According to these results, $|C_n| < |C_0|,\ n = 1, 2, \cdots$, and then,

$$|x(t)| < C_0 \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}.$$

Therefore, the series in (13) is absolutely convergent, due to the convergence of the MLF [46]. Alternatively, we can apply the ratio test

$$L = \left|\left[1 - q^{-\alpha}\right]\frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + \alpha + 1)}\right| < \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + \alpha + 1)} < 1.$$

The ratio of two gamma functions has the expansion

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}\left[1 + \sum_{1}^{N} c_k z^{-k} + O\left(z^{-N-1}\right)\right],$$

as $|z| \to \infty$, uniformly in every sector that excludes the negative real axis. Then, as $n \to \infty$, $\dfrac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + \alpha + 1)} \to n^{-\alpha}$, which leads to the same conclusion.

Assume that there are two solutions, $x_1(t)$ and $x_2(t)$, of (3) with the form (13). Therefore, both have to verify (12) which is a linear equation. Consequently, we can write

$$D^\alpha x_1(t) + x_1(t) - D^\alpha x_2(t) - x_2(t) = \frac{1}{q}x_1\left(\frac{t}{q}\right) - \frac{1}{q}x_2\left(\frac{t}{q}\right),$$

and, letting $y(t) = x_1(t) - x_2(t)$, we obtain

$$D^\alpha y(t) + y(t) = \frac{1}{q}y\left(\frac{t}{q}\right),$$

that represents the same Equation (3), but with null initial-condition. This implies that all the coefficients of the solution (13) are null. Therefore, the output is identically null: $y(t) \equiv 0,\ t > 0$. We conclude that Equation (3) has a unique solution. One could wonder about the existence of an irregular solution. It is immediate to conclude that, if it existed, it could not verify the IC. □

The solution of (3) expressed by (13), similar to the MLF, presents the same numerical difficulties [4,47], since the presence of the C_n coefficients does not contribute to accelerate the convergence, because, from a numerical point of view, for higher values of n, $C_{n+1} \approx -C_n$. This is illustrated in plots in Figure 1. Therefore, we need to look for another alternative.

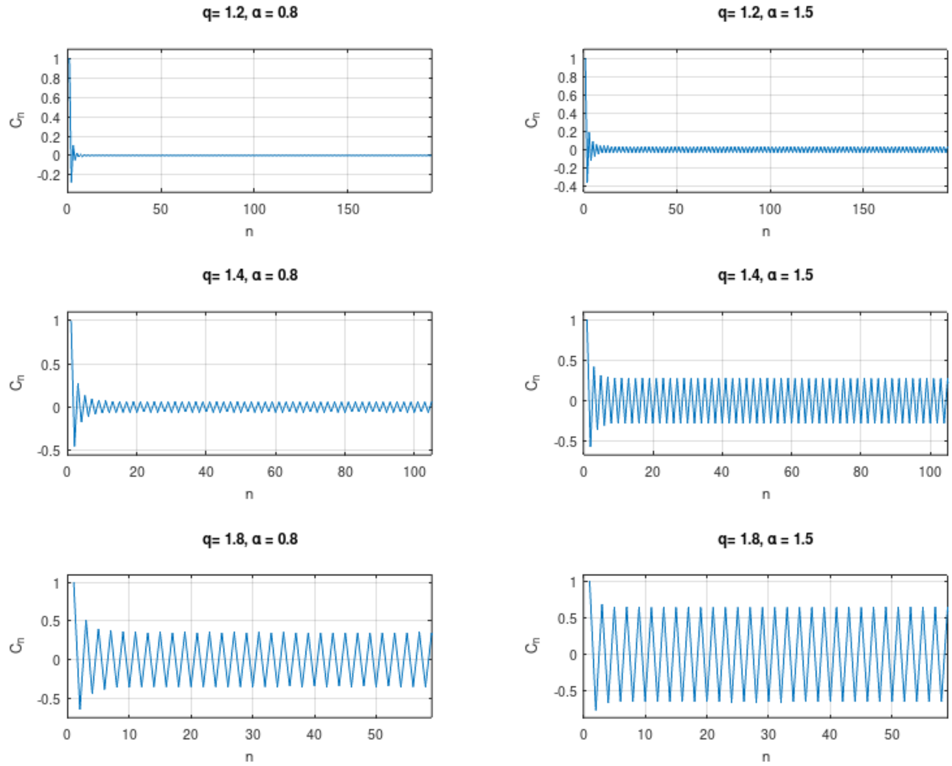

Figure 1. Evolution of the coefficients of the series (13) as function of the order: $C_n, n = 0, 1, \cdots$ given by (17).

4. Solution Using Mittag–Leffler Functions

In this Section, we will present the solution of Equation (3) in terms of a series involving Mittag–Leffler functions, instead of a Taylor series (13).

Theorem 2. *Let $q > 1$ and $0 < \alpha < 2$. The solution of (3), under the IC x_0, assumes the form*

$$x(t) = \sum_{n=0}^{\infty} B_n E_\alpha\left(-q^{-n\alpha} t^\alpha\right), \quad t > 0, \tag{18}$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function and

$$B_n = q^{-n} \prod_{k=1}^{n} \frac{1}{(1 - q^{-k\alpha})} B_0, \quad n = 1, 2, \cdots, \tag{19}$$

with

$$x_0 = \sum_{n=0}^{\infty} B_n. \tag{20}$$

Before going into the proof, let us make some steps into the solution.
Return to (16) and rewrite it as

$$X(s) = x_0 \frac{s^{\alpha-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1} X(qs) \quad Re(s) > 0. \tag{21}$$

From this equation, we get

$$X(q^n s) = x_0 \frac{q^{n\alpha-n} s^{\alpha-1}}{q^{n\alpha}s^\alpha+1} + \frac{1}{q^{n\alpha}s^\alpha+1} X(q^{n+1}s), \quad Re(s)>0,\ n=1,2,\dots, \tag{22}$$

or

$$X(q^n s) = x_0 \frac{q^{-n} s^{\alpha-1}}{s^\alpha+q^{-n\alpha}} + \frac{q^{-n\alpha}}{s^\alpha+q^{-n\alpha}} X(q^{n+1}s), \quad Re(s)>0,\ n=1,2,\dots. \tag{23}$$

With $n=1$, we obtain

$$X(qs) = x_0 \frac{q^{-1}s^{\alpha-1}}{s^\alpha+q^{-\alpha}} + \frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}} X(q^2 s),$$

and, using (21), we are led to

$$\begin{aligned} X(s) =& x_0 \frac{s^{\alpha-1}}{s^\alpha+1} + \frac{1}{s^\alpha+1}\left[x_0 q^{-1}\frac{s^{\alpha-1}}{s^\alpha+q^{-\alpha}} + \frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}X(q^2s)\right] = \\ & x_0\frac{s^{\alpha-1}}{s^\alpha+1} + x_0\frac{s^{\alpha-1}}{s^\alpha+1}\frac{q^{-1}}{s^\alpha+q^{-\alpha}} + \frac{1}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}X(q^2s). \end{aligned}$$

However,

$$\frac{1}{(s^\alpha+1)(s^\alpha+q^{-\alpha})} = \frac{1}{1-q^{-\alpha}}\left[-\frac{1}{s^\alpha+1} + \frac{1}{s^\alpha+q^{-\alpha}}\right],$$

which leads to

$$\begin{aligned} X(s) =& x_0\frac{s^{\alpha-1}}{s^\alpha+1} + \frac{x_0 q^{-1}}{1-q^{-\alpha}}\left[-\frac{s^{\alpha-1}}{s^\alpha+1} + \frac{s^{\alpha-1}}{s^\alpha+q^{-\alpha}}\right] + \frac{1}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}X(q^2s) = \\ & x_0\left[1-\frac{q^{-1}}{1-q^{-\alpha}}\right]\frac{s^{\alpha-1}}{s^\alpha+1} + \frac{x_0q^{-1}}{1-q^{-\alpha}}\frac{s^{\alpha-1}}{s^\alpha+q^{-\alpha}} + \frac{1}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}X(q^2s). \end{aligned}$$

Continuing from (23),

$$X(q^2s) = x_0 q^{-2}\frac{s^{\alpha-1}}{s^\alpha+q^{-2\alpha}} + \frac{q^{-2\alpha}}{s^\alpha+q^{-2\alpha}}X(q^3s),$$

which gives

$$\begin{aligned} X(s) =& x_0\left[1-\frac{q^{-1}}{1-q^{-\alpha}}\right]\frac{s^{\alpha-1}}{s^\alpha+1} + \frac{x_0q^{-1}}{1-q^{-\alpha}}\frac{s^{\alpha-1}}{s^\alpha+q^{-\alpha}} + \\ & \frac{x_0}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}\frac{q^{-2}s^{\alpha-1}}{s^\alpha+q^{-2\alpha}} + \frac{1}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}\frac{q^{-2\alpha}}{s^\alpha+q^{-2\alpha}}X(q^3s). \end{aligned}$$

However,

$$\frac{q^{-\alpha}}{(s^\alpha+1)(s^\alpha+q^{-\alpha})(s^\alpha+q^{-2\alpha})} = \frac{A}{(s^\alpha+1)} + \frac{B}{(s^\alpha+q^{-\alpha})} + \frac{C}{(s^\alpha+q^{-2\alpha})},$$

where the constants, A, B, and C are residues (their expressions are not important here). Therefore, we can write

$$\begin{aligned} X(s) =& x_0\left[1-\frac{q^{-1}}{1-q^{-\alpha}}+A\right]\frac{s^{\alpha-1}}{s^\alpha+1} + x_0\left[\frac{q^{-1}}{1-q^{-\alpha}}+B\right]\frac{s^{\alpha-1}}{s^\alpha+q^{-\alpha}} + Cx_0q^{-2}\frac{s^{\alpha-1}}{s^\alpha+q^{-2\alpha}} + \\ & \frac{1}{s^\alpha+1}\frac{q^{-\alpha}}{s^\alpha+q^{-\alpha}}\frac{q^{-2\alpha}}{s^\alpha+q^{-2\alpha}}X(q^3s). \end{aligned}$$

We could continue the process, but this relation suggests we set

$$X(s) = \sum_{n=0}^{\infty} B_n \frac{s^{\alpha-1}}{s^{\alpha} + q^{-n\alpha}}, \tag{24}$$

where B_n are real parameters to be computed.

Proof. Begin by inserting (24) into (21) to get

$$(s^{\alpha}+1)\sum_{n=0}^{\infty} B_n \frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}} = x_0 s^{\alpha-1} + q^{-1}\sum_{n=0}^{\infty} B_n \frac{s^{\alpha-1}}{s^{\alpha}+q^{-(n+1)\alpha}},$$

and

$$(s^{\alpha}+1)\sum_{n=0}^{\infty} B_n \frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}} = x_0 s^{\alpha-1} + q^{-1}\sum_{n=1}^{\infty} B_{n-1} \frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}}.$$

On the other hand,

$$\frac{s^{\alpha}+1}{s^{\alpha}+q^{-n\alpha}} = \frac{s^{\alpha}+q^{-n\alpha}-q^{-n\alpha}+1}{s^{\alpha}+q^{-n\alpha}} = 1 + \frac{1-q^{-n\alpha}}{s^{\alpha}+q^{-n\alpha}},$$

that leads to

$$s^{\alpha-1}\sum_{n=0}^{\infty} B_n + \sum_{n=1}^{\infty} B_n\left(1-q^{-n\alpha}\right)\frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}} = x_0 s^{\alpha-1} + q^{-1}\sum_{n=1}^{\infty} B_{n-1}\frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}},$$

from where we obtain

$$B_n = \frac{1}{q(1-q^{-n\alpha})} B_{n-1}, \quad n = 1, 2, \cdots, \tag{25}$$

that leads to (19). The relation (20) is obtained from the initial value theorem [48].

To finish, we must study the convergence of (18). First, we note that each function

$$\frac{s^{\alpha-1}}{s^{\alpha}+q^{-n\alpha}}, \quad Re(s) > 0,$$

has a pseudo-pole [49], since the zero of the denominator is in the second Riemann surface in the complex plane, due to the fact that q is a positive parameter. Therefore, it does not have a pole. Consequently, the inverse Laplace transform, $E_{\alpha}(-q^{-n\alpha}t^{\alpha})$, is bounded [46,49]. In fact, it is a continuous function for $t > 0$ [46] with initial and final values 1 and 0, respectively. Therefore, it has a maximum and minimum. This implies absolute and uniform convergence of (18), since the $|B_n|$ sequence decreases exponentially to 0. □

Remark 2. *The recursion (25) leaves undetermined the value of B_0. Therefore, we start with a "seed" $\hat{B}_0 = 1$ to obtain a sequence $\hat{B}_n,\ n = 1, 2, \cdots$ that has to be normalised so that the corresponding sum equals x_0. Let $S = \sum_{n=0}^{\infty} \hat{B}_n$. Define a normalising factor $F = \frac{x_0}{S}$. The normalised coefficients are given by $B_n = \hat{B}_n \cdot F$.*

The exponential decreasing character of the series coefficients (19) explains why the series in (18) has a fast convergence, as we illustrate in the following.

In Figure 2 we depict some plots illustrating the variation of the B_n sequence for several values of α and q. As seen, above a given n_{th}, the sequence becomes negligible, implying that the series solutions (18), or (24), have a finite number of meaningful terms, so that they can be truncated, disregarding the terms for $n > n_{th}$. This means that the solution of (3) can be approximated by a finite linear combination of Mittag–Leffler functions.

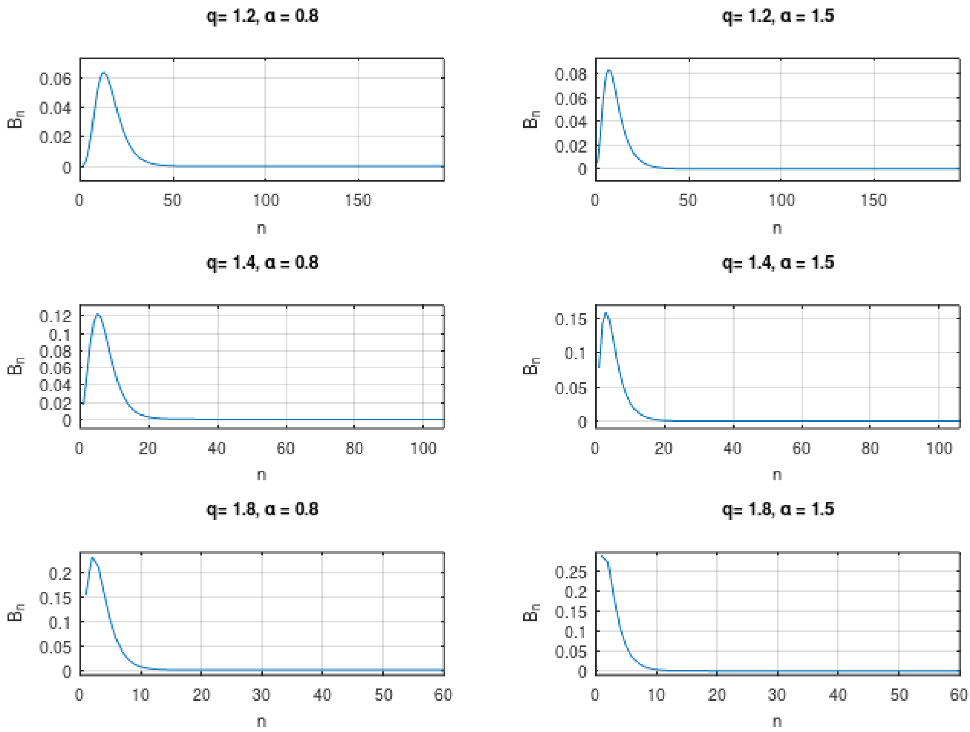

Figure 2. Evolution of the coefficients in the series (18) as a function of the order: $B_n, n = 0, 1, \cdots$ in (19).

These considerations may lead us to conclude that the solutions stated in (13) and in (18) are different and could lead to a loss of the unicity of the solution of (3), which is impossible, since it is a linear equation. In fact, (13) and (18) present two different ways of expressing such a solution as we can verify in the following. For simplicity, we will assume that $x_0 = 1$. Equating the two solutions, we obtain

$$\sum_{n=0}^{\infty}(-1)^n \prod_{k=0}^{n-1}\left[1-q^{-k\alpha-1}\right]\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} q^{-n}\frac{1}{\prod_{k=1}^{n}(1-q^{-k\alpha})}E_\alpha\left(-q^{-n\alpha}t^\alpha\right), \quad t>0, \tag{26}$$

and

$$\sum_{n=0}^{\infty}(-1)^n \prod_{k=0}^{n-1}\left[1-q^{-k\alpha-1}\right]\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{m=0}^{\infty}\frac{q^{-m}}{\prod_{k=1}^{m}(1-q^{-k\alpha})}\sum_{n=0}^{\infty}(-1)^n q^{-nm\alpha}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \quad t>0. \tag{27}$$

Permuting the summations on the right hand side, we get

$$\sum_{n=0}^{\infty}(-1)^n \prod_{k=0}^{n-1}\left[1-q^{-k\alpha-1}\right]\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty}(-1)^n\left[\sum_{m=0}^{\infty}\frac{q^{-m}}{\prod_{k=1}^{m}(1-q^{-k\alpha})}q^{-nm\alpha}\right]\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \quad t>0, \tag{28}$$

that leads to a well-known relation in the theory of q-hypergeometric series [50]

$$\prod_{k=0}^{n-1}\left[1-q^{-k\alpha-1}\right] = \sum_{m=0}^{\infty} q^{-m}\frac{1}{\prod_{k=1}^{m}(1-q^{-k\alpha})}q^{-nm\alpha}, \quad n\in\mathbb{Z}^+. \tag{29}$$

Therefore, the solutions we found are different representations of the same entity.

5. Numerical Aspects

The plots presented in the above figures point to (24) as the easiest way of computing the solution of our problem. As stated, the coefficients decrease to zero. From a numerical point of view, this means that they become comparable to the relative spacing between any two adjacent numbers in the machine's floating point system, "eps" in Octave/Matlab. We set $n_{th} = \lfloor \frac{-\ln(eps)}{ln(q)\alpha} \rfloor$. Therefore, the series in (18) becomes a polynomial.

We have two different numerically equivalent approaches to compute such summations. The most obvious is the use of (18) together with a suitable algorithm for the computation of the needed MLF [47]. We performed this, but the algorithm has a very severe drawback: the computational burden. For example, taking $q = 1.2$, with a sampling interval equal to 0.01, it took around 17 minutes to compute $x(t)$. Therefore, we looked for a different approach. This consisted of approximating a continuous-time function to a discrete-time one [8]. To accomplish this, we worked in frequency domain, using (24) and the bilinear (Tustin) transformation, followed by a (discrete) Fourier-transform inversion. We proceeded according to the following steps.

1. Use the $s \to z$ bilinear transformation [8]

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}},$$

to pass from a continuous-time to a sampled (discrete) function. T is the sampling interval. We obtain the function

$$X_d(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \sum_{n=0}^{n_{th}} C_n \frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha}}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha} + q^{-n\alpha}}. \tag{30}$$

2. Set $z = e^{i\omega}$, $-\pi < \omega \leq \pi$, to get a discrete-time Fourier transform $X_d(e^{i\omega})$ of $x(nT)$ [39]. To avoid the effect of the branchcut points at $z = \pm 1$, we found it better to move them slightly into the unit circle. This is accomplished by setting

$$s = \frac{2}{T}\frac{1-0.99e^{-i\omega}}{1+0.99e^{-i\omega}}.$$

We can use a simple procedure that consists of the following steps:

   (a) Set $N(e^{i\omega}) = \mathcal{F}([1, -0.99])$, where $\mathcal{F}$ represents the discrete Fourier transform that is implemented by the fast Fourier transform algorithm;
   (b) Similarly, set $D(e^{i\omega}) = \mathcal{F}([1, +0.99])$;
   (c) Compute $s(\omega) = \frac{2}{T}\frac{N(e^{i\omega})}{D(e^{i\omega})}$;
   (d) Obtain the approximation to $X(s)$

$$X_d(s(\omega)) = \frac{T}{2}\frac{1}{s(\omega)} \sum_{n=0}^{n_{th}} C_n \frac{s^{\alpha}(\omega)}{s^{\alpha}(\omega) + q^{-n\alpha}}. \tag{31}$$

3. Sample $X_d(e^{i\omega})$ in a uniform grid, making $\omega_k = \frac{2\pi}{N}k$, $k = 0, 1, \cdots, N-1$, where N is in high enough agreement with T and the interval where we want to compute the function. In our simulations, we used $T = 0.001$ and $N = 2^{16}$.
4. We could use the fast Fourier transform to compute the inverse of $X_d(e^{i\omega_k})$, $x(nT)$. However, this is not very good from numerical aspects, due to the singularity at the origin. We found it was better to separate the computation in three steps:

   (a) Computation of the summation in (32), making a slight transformation

$$\bar{X}_d(e^{i\omega}) = \sum_{n=0}^{n_{th}} C_n \frac{1}{1 + q^{-n\alpha}s^{-\alpha}(\omega)}; \tag{32}$$

(b) Invert $\bar{X}_d(e^{i\omega})$ to obtain $\bar{x}(nT)$. We must remark that this procedure, based on the inversion of a transform, leads to $\frac{x(0)}{2}$ at the origin, implying a correction;

(c) To get $x(nT)$, we use the following difference equation [8]

$$x(nT) = x((n-1)T) + \frac{T}{2}(\bar{x}(nT) + \bar{x}((n-1)T)).$$

This procedure was very fast (≈8 s) and the result was not visibly different from the approach based on numerical integration for the computation of MLF. In Figures 3 and 4, we illustrate the solution for $\alpha = 0.8, 1, 1.6, 1.9$ and $q = 1.2, 1.8$.

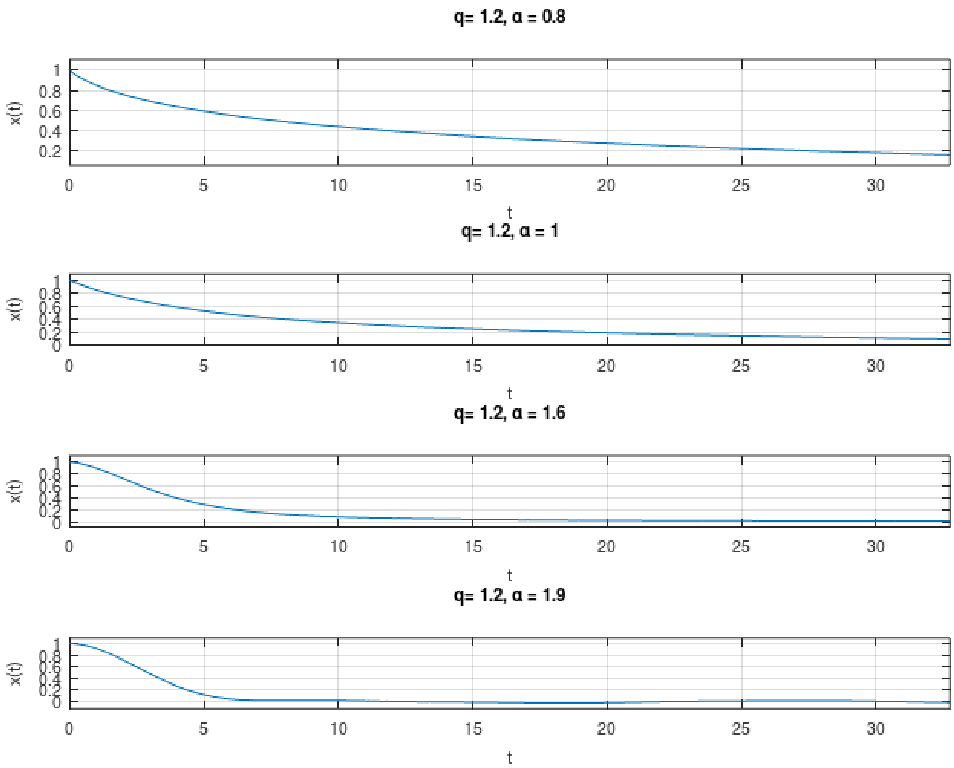

Figure 3. Numerical computation of $x(t)$ for $q = 1.2$.

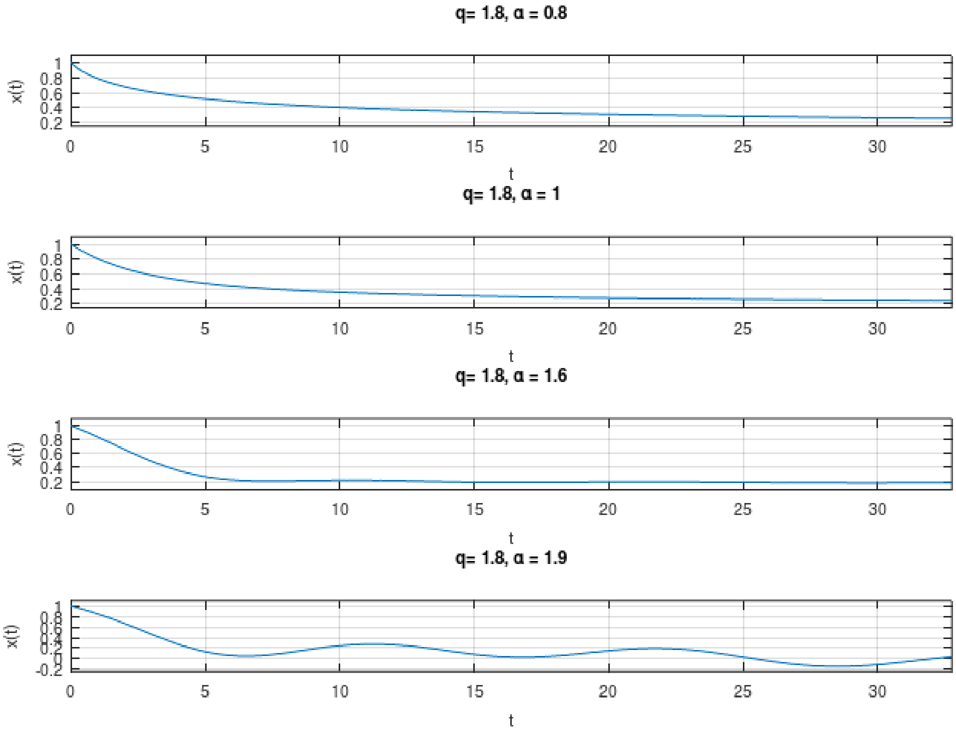

Figure 4. Numerical computation of $x(t)$ for $q = 1.8$.

6. Conclusions

We presented and solved the fractional Ambartsumian equation. We introduced a general framework without the usual constraints on the derivative order so that our solutions have wider validity. Two solutions were obtained taking the form of a power series and a series involving a sequence of Mittag–Leffler functions. In fact, we actually did an accelerating transformation, so that the second one converged faster than the first. We showed that it is possible to compute an approximate numerical solution using a simple implementation having as its base the discrete Fourier transform and its fast implementation (FFT). Several examples were presented.

In passing, we introduced a fractional calculus framework suitable for our purposes. The theoretical development we presented leaves an unsolved problem: finding the solution of the equation when the initial condition is not taken at $t = 0$, but at any other instant $t_0 > 0$. One wonders if a similar procedure could be found and applied to other problems, mainly nonlinear, as is the case of the fractional logistic equation that has a solution also expressed in the form of a Taylor series. These two questions will be considered later in a future paper.

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References

1. Ambartsumian, V. On the fluctuation of the brightness of the Milky Way. *Dokl. Akad. Nauk SSSR* **1944**, *44*, 223–226.
2. Patade, J.; Bhalekar, S. On analytical solution of Ambartsumian equation. *Natl. Acad. Sci. Lett.* **2017**, *40*, 291–293. [CrossRef]
3. Bakodah, H.O.; Ebaid, A. Exact solution of Ambartsumian delay differential equation and comparison with Daftardar-Gejji and Jafari approximate method. *Mathematics* **2018**, *6*, 331. [CrossRef]
4. Alharbi, F.M.; Ebaid, A. New analytic solution for Ambartsumian equation. *J. Math. Syst. Sci.* **2018**, *8*, 182–186.
5. Kumar, D.; Singh, J.; Baleanu, D.; Rathore, S. Analysis of a fractional model of the Ambartsumian equation. *Eur. Phys. J. Plus* **2018**, *133*, 1–7. [CrossRef]
6. Ebaid, A.; Cattani, C.; Al Juhani, A.S.; El-Zahar, E.R. A novel exact solution for the fractional Ambartsumian equation. *Adv. Differ. Equations* **2021**, *2021*, 1–18. [CrossRef]
7. Alharbi, W.; Hristova, S. New Series Solution of the Caputo Fractional Ambartsumian Delay Differential Equationation by Mittag-Leffler Functions. *Mathematics* **2021**, *9*, 157. [CrossRef]
8. Ortigueira, M.D.; Machado, J.A.T. New discrete-time fractional derivatives based on the bilinear transformation: Definitions and properties. *J. Adv. Res.* **2020**, *25*, 1–10. [CrossRef]
9. Ortigueira, M.D.; Machado, J.T. The 21st century systems: An updated vision of continuous-time fractional models. *IEEE Circuits Syst. Mag.* **2022**, *22*, 36–56. [CrossRef]
10. Westerlund, S.; Ekstam, L. Capacitor Theory. *IEEE Trans. Dielectr. Electr. Insul.* **1994**, *1*, 826–839. [CrossRef]
11. Westerlund, S. *Dead Matter has Memory!* Causal Consulting: Kalmar, Sweden, 2002.
12. Heaviside, O. *Electromagnetic Theory 2*; Ernest Benn Limited: London, UK, 1925; Volume 2.
13. Nahin, P.J. Oliver Heaviside, Fractional Operators, and the Age of the Earth. *IEEE Trans. Educ.* **1985**, *28*, 94–104. [CrossRef]
14. Rogosin, S.; Mainardi, F. George William Scott Blair–the pioneer of factional calculus in rheology. *arXiv* **2014**, arXiv:1404.3295.
15. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*; Imperial College Press: London, UK, 2010.
16. Laskin, N. Fractional quantum mechanics. *Phys. Rev. E* **2000**, *62*, 3135. [CrossRef] [PubMed]
17. Laskin, N. Fractional Schrödinger equation. *Phys. Rev. E* **2002**, *66*, 056108. [CrossRef] [PubMed]
18. Naber, M. Time fractional Schrödinger equation. *J. Math. Phys.* **2004**, *45*, 3339–3352. [CrossRef]
19. Laskin, N. Time fractional quantum mechanics. *Chaos, Solitons Fractals* **2017**, *102*, 16–28. [CrossRef]
20. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Nonlinear Physical Science; Springer: Beijing, China; Heidelberg, Germany, 2011.
21. Ortigueira, M.D.; Rivero, M.; Trujillo, J.J. From a generalised Helmholtz decomposition theorem to fractional Maxwell equations. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *22*, 1036–1049. [CrossRef]
22. Gulgowski, J.; Kwiatkowski, D.; Stefański, T.P. Signal Propagation in Electromagnetic Media Modelled by the Two-Sided Fractional Derivative. *Fractal Fract.* **2021**, *5*, 10. [CrossRef]
23. Uchaikin, V.V. *Fractional Derivatives for Physicists and Engineers. Vol. I: Background and Theory. Vol. II: Applications*; Nonlinear Physical Science; Springer and Higher Education Press: Heidelberg, Germany, 2012.
24. Uchaikin, V.; Sibatov, R. *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors, Dielectrics and Nanosystems*; World Scientific Publishing Company: Singapore, 2013.
25. Herrmann, R. *Fractional Calculus*, 3rd ed.; World Scientific: Singapore, 2018.
26. Mandelbrot, B.; Van Ness, J. Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Rev.* **1968**, *10*, 422–437.
27. Nossenson, N.; Bobrovsky, B.Z. Analysis of direct spectrum measurement of a sinusoidal signal impaired by either fractional gaussian phase noise or fractional Brownian phase motion. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **2009**, *56*, 2351–2362. [CrossRef]
28. Lilly, J.M.; Sykulski, A.M.; Early, J.J.; Olhede, S.C. Fractional Brownian motion, the Matérn process, and stochastic modeling of turbulent dispersion. *Nonlinear Process. Geophys.* **2017**, *24*, 481–514. [CrossRef]
29. Dugowson, S. Les Différentielles Métaphysiques (Histoire et Philosophie de la Généralisation de L'ordre de Dérivation). Ph.D. Thesis, Université Paris Nord, Paris, France, 1994.
30. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach: Yverdon, Switzerland, 1993.
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
32. Ortigueira, M.D.; Tenreiro Machado, J. What is a fractional derivative? *J. Comput. Phys.* **2015**, *293*, 4–13. [CrossRef]
33. Trigeassou, J.C.; Maamri, N. Initial conditions and initialization of linear fractional differential equations. *Signal Process.* **2011**, *91*, 427–436. [CrossRef]
34. Sabatier, J.; Farges, C. Comments on the description and initialization of fractional partial differential equations using Riemann–Liouville's and Caputo's definitions. *J. Comput. Appl. Math.* **2018**, *339*, 30–39. [CrossRef]
35. Ortigueira, M.D.; Machado, J.T. Revisiting the 1D and 2D Laplace Transforms. *Mathematics* **2020**, *8*, 1330. [CrossRef]
36. Ortigueira, M.D. A New Look at the Initial Condition Problem. *Mathematics* **2022**, *10*, 1771. [CrossRef]
37. Ortigueira, M.D.; Machado, J.T. Which Derivative? *Fractal Fract.* **2017**, *1*. [CrossRef]

38. Ortigueira, M.D.; Machado, J.T. Fractional derivatives: The perspective of system theory. *Mathematics* **2019**, *7*, 150. [CrossRef]
39. Ortigueira, M.D.; Valério, D. *Fractional Signals and Systems*; De Gruyter: Berlin, Germany, 2020.
40. Ortigueira, M.D. Two-sided and regularised Riesz-Feller derivatives. *Math. Methods Appl. Sci.* **2021**, *44*, 8057–8069. [CrossRef]
41. Ortigueira, M.D. *Fractional Calculus for Scientists and Engineers*; Lecture Notes in Electrical Engineering; Springer: Dordrecht, The Netherlands; Heidelberg, Germany, 2011.
42. Oppenheim, A.V.; Schafer, R.W. *Discrete-Time Signal Processing*, 3rd ed.; Prentice Hall Press: Upper Saddle River, NJ, USA, 2009.
43. Ortigueira, M.; Bengochea, G. A new look at the fractionalisation of the logistic equation. *Phys. A Stat. Mech. Appl.* **2017**, *467*, 554–561. [CrossRef]
44. Ortigueira, M.; Coito, F. System initial conditions vs derivative initial conditions. *Comput. Math. Appl.* **2010**, *59*, 1782–1789. [CrossRef]
45. Kac, V.G.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2002; Volume 113.
46. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2014; Second Edition in Preparation.
47. Ortigueira, M.D.; Lopes, A.M.; Machado, J.A.T. On the Numerical Computation of the Mittag-Leffler function. *Int. J. Nonlinear Sci. Numer. Simul.* **2019**, *20*. [CrossRef]
48. Zemanian, A.H. *Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications*; Lecture Notes in Electrical Engineering, 84, Dover Publications: New York, NY, USA, 1987.
49. Ortigueira, M.D.; Machado, J.T.; Rivero, M.; Trujillo, J.J. Integer/fractional decomposition of the impulse response of fractional linear systems. *Signal Process.* **2015**, *114*, 85–88. [CrossRef]
50. Andrews, G. *q-Hypergeometric and Related Functions, NIST Handbook of Mathematical Functions, 419–433. US Dept*; Cambridge University Press: New York, NY, USA, 2010.