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On the weak convergence of shift operators to zero on rearrangement-invariant spaces

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Abstract Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ tending to infinity and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators given by $(T_{h_n}f)(x) = f(x-h_n)$ for $x \in \mathbb{R}^d$. We prove that $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$ on a separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ if and only if its fundamental function φ_X satisfies $\varphi_X(t)/t \to 0$ as $t \to \infty$. On the other hand, we show that $\{T_{h_n}\}$ does not converge weakly to the zero operator as $n \to \infty$ on all Marcinkiewicz endpoint spaces $M_\varphi(\mathbb{R}^d)$ and on all non-separable Orlicz spaces $L^\Phi(\mathbb{R}^d)$. Finally, we prove that if $\{h_n\}$ is an arithmetic progression: $h_n = nh$, $n \in \mathbb{N}$ with an arbitrary $h \in \mathbb{R}^d \setminus \{0\}$, then $\{T_{nh}\}$ does not converge weakly to the zero operator on any non-separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ as $n \to \infty$.

Keywords Rearrangement-invariant Banach function space · Marcinkiewicz endpoint space · non-separable Orlicz space · shift operator · weak convergence to zero · fundamental function · limit operator.

Mathematics Subject Classification (2010) 46E30

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1 Introduction

One of the powerful methods for the study of Fredholm properties of convolution type operators with oscillating symbols on Lebesgue spaces L^p is the so-called method of limit operators (see [5, 15, 21] and also [6]). It is based on the observation that for a given bounded linear operator A and a cleverly chosen sequence of isometries $\{V_n\}$, the strong limit of the sequence $V_n^{-1}AV_n$ (if it exists) preserves some important information about A and can be much simpler than the original operator A. This strong limit is called the limit operator of the operator A with respect to the sequence of isometries $\{V_n\}$. To give a simple example illustrating this idea, let us consider the shift operator T_h on $L^p(\mathbb{R}^d)$ with $d \in \mathbb{N}$ and $1 \le p \le \infty$ given for $h \in \mathbb{R}^d$ by

$$(T_h f)(x) := f(x-h), \quad x \in \mathbb{R}^d. \tag{1.1}$$

It is clear that T_h is an isometry on $L^p(\mathbb{R}^d)$.

Lemma 1 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on the Lebesgue space $L^p(\mathbb{R}^d)$, $1 < p < \infty$. If K is a compact operator on $L^p(\mathbb{R}^d)$, then the strong limit of the sequence $\{T_{h_n}^{-1} K T_{h_n}\}$ as $n \to \infty$ is the zero operator.*

Thus the limit operators of a compact operator with respect to the sequences $\{T_{h_n}\}$ with $\{h_n\}$ tending to infinity are all equal to the zero operator. The proof of the above lemma is contained in [6, Lemma 18.9] for $d = 1$. For other values of $d \in \mathbb{N}$ the proof is essentially the same. It is reduced to the proof of the weak convergence of $\{T_{h_n}\}$ to the zero operator as $n \to \infty$.

Weak convergence of the sequence $\{T_{h_n}\}$ to the zero operator (that is, convergence to the zero operator in the weak operator topology) on more general rearrangement-invariant Banach function spaces does not seem to have been studied before. Since this is not an entirely trivial question, we address it in the present paper.

For definitions of a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ and its fundamental function φ_X we refer the reader to [3, Chap. 2] and Section 2.1 below (see also [20, Chap. 7], [13, Chap. 2], and [23]). Let $h \in \mathbb{R}^d$ and $f \in X(\mathbb{R}^d)$. Since f and $T_h f$ are equimeasurable, the shift operator defined by (1.1) is an isometry on $X(\mathbb{R}^d)$.

Our first main result gives necessary and sufficient conditions for the weak convergence of the sequence $\{T_{h_n}\}$ to the zero operator as $n \to \infty$ on a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$.

Theorem 1 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ with fundamental function φ_X.*

(a) *If the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$ on the space $X(\mathbb{R}^d)$, then*

$$\varphi_X(t)/t \to 0 \quad \text{as} \quad t \to \infty. \tag{1.2}$$

(b) *If the space* $X(\mathbb{R}^d)$ *is separable and* (1.2) *is fulfilled, then the sequence* $\{T_{h_n}\}$ *converges weakly to the zero operator as* $n \to \infty$ *on the space* $X(\mathbb{R}^d)$.

Note that condition (1.2) is not fulfilled for the space $L^1(\mathbb{R}^d)$. Hence the sequence $\{T_{h_n}\}$ does not converge weakly to the zero operator on $L^1(\mathbb{R}^d)$. We will show that condition (1.2) is fulfilled if the space $X(\mathbb{R}^d)$ is reflexive (see Corollary 1) or its upper Zippin index is non-trivial, that is, $q_X < 1$ (see Lemma 3).

On the other hand, we will show that there are non-separable rearrangement-invariant Banach function spaces satisfying (1.2), in which the sequence $\{T_{h_n}\}$ fails to converge weakly to the zero operator as $n \to \infty$. For instance, these are the Marcinkiewicz endpoint spaces $M_\varphi(\mathbb{R}^d)$ built upon quasi-concave functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(t)/t \to 0$ as $t \to \infty$ and non-separable Orlicz spaces $L^\Phi(\mathbb{R}^d)$ (see Section 2.2 below).

Our second main result is the following.

Theorem 2 *Let* $\{h_n\}$ *be a sequence in* $\mathbb{R}^d$ *such that* $|h_n| \to +\infty$ *as* $n \to \infty$ *and let* $\{T_{h_n}\}$ *be the corresponding sequence of shift operators on a rearrangement-invariant Banach function space* $X(\mathbb{R}^d)$ *with fundamental function* φ_X. *Then the sequence* $\{T_{h_n}\}$ *does not converge weakly to the zero operator as* $n \to \infty$ *on the space* $X(\mathbb{R}^d)$ *if one of the following conditions is satisfied:*

(a) $\lim_{t\to 0} \varphi_X(t) > 0$ *or* $\lim_{t\to\infty} \varphi_X(t) < \infty$;
(b) *the space* $X(\mathbb{R}^d)$ *is the Marcinkiewicz endpoint space* $M_\varphi(\mathbb{R}^d)$ *built upon a quasi-concave function* φ;
(c) $X(\mathbb{R}^d)$ *is a non-separable Orlicz space* $L^\Phi(\mathbb{R}^d)$ *built upon a Young's function* Φ.

Note that $L^\infty(\mathbb{R}^d)$ satisfies both conditions in (a) in the above theorem. Hence the sequence $\{T_{h_n}\}$ does not converge weakly to the zero operator on $L^\infty(\mathbb{R}^d)$.

The above theorem suggests the following question, which we were unable to answer.

Question 1 Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$. Is there a non-separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ on which the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$?

Nevertheless, our last main result shows that the answer to the above question is negative in the case when $\{h_n\}$ is an arithmetic progression: $h_n = nh$, $n \in \mathbb{N}$ with an arbitrary $h \in \mathbb{R}^d \setminus \{0\}$. In this case, the sequence of shift operators $\{T_{nh}\}$ does not converge weakly to the zero operator on any non-separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ as $n \to \infty$.

Theorem 3 *Let* $X(\mathbb{R}^d)$ *be a non-separable rearrangement-invariant Banach function space and* $h \in \mathbb{R}^d \setminus \{0\}$. *Then there exist* $f \in X(\mathbb{R}^d)$ *and* $F \in X^*(\mathbb{R}^d)$ *such that*

$$\inf_{n\in\mathbb{N}} |F(T_{nh}f)| > 0, \tag{1.3}$$

and hence the sequence $\{T_{nh}\}$ *does not converge weakly to the zero operator as* $n \to \infty$ *on the space* $X(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2, we recall definitions of a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ and its fundamental function φ_X, as well as definitions of Marcinkiewicz endpoint spaces and Orlicz spaces, which are prominent examples of rearrangement-invariant Banach function spaces. Then we discuss the measure preserving transformation $\Theta(x) = \omega_d|x|^d$ of $\mathbb{R}^d$ to $\mathbb{R}_+$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Further, we recall the definitions of the Boyd and Zippin indices of a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ and state some sufficient conditions for (1.2) in terms of these indices. Finally, we present a version of [13, Chap. II, Theorem 4.8]. In Section 3, we prove that, given a compact set F of positive measure, the condition $\varphi_X(t) \to \infty$ as $t \to \infty$ is equivalent to the condition $\int_{h_n+F} f(x)\,dx = o(1)$ as $n \to \infty$ for every function $f \in X(\mathbb{R}^d)$. Using this auxiliary fact, we prove Theorem 1. In Section 4, we first state an extension of Lemma 1 to the setting of separable rearrangement-invariant Banach function spaces satisfying (1.2). Further, we show that condition (1.2) is fulfilled if the space $X(\mathbb{R}^d)$ is reflexive or if its upper Zippin index satisfies $q_X < 1$. We believe that these two cases will arise more frequently in expected applications of our analogue of Lemma 1. Section 5 is devoted to the proof of Theorem 2. We start it with the proof of part (a). Further, we show that if the space $X(\mathbb{R}^d)$ contains a function g, whose non-increasing rearrangement cannot be approximated by the right truncations $g^*\chi_{[0,N]}$, then the sequence of shift operators $\{T_{h_n}\}$ cannot converge weakly to the zero operator as $n \to \infty$ on the space $X(\mathbb{R}^d)$. Using this result, we prove parts (b) and (c) of Theorem 2. In Section 6, we give a proof of Theorem 3. In Section 7, we show that the closure $X_c(\mathbb{R}^d)$ of the set of all compactly supported (not necessarily bounded) functions in a non-separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ with respect to the norm of $X(\mathbb{R}^d)$ is a proper subspace of the space $X(\mathbb{R}^d)$. We prove that if the upper Boyd index satisfies $\beta_X < 1$, then the sequences $\{T_{h_n} f\}$ converge weakly to the zero function as $n \to \infty$ in the space $X(\mathbb{R}^d)$ for all functions $f \in X_c(\mathbb{R}^d)$.

2 Preliminaries

2.1 Rearrangement-invariant Banach function spaces

By m_d and $\overline{m}$ denote the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{R}_+ := [0,\infty)$, respectively. Let $(\mathbb{S},\mu)$ be one of the measure spaces $(\mathbb{R}^d, m_d)$ or $(\mathbb{R}_+, \overline{m})$. The set of all μ-measurable extended complex-valued functions on $\mathbb{S}$ is denoted by $\mathcal{M}(\mathbb{S},\mu)$. Let $\mathcal{M}^+(\mathbb{S},\mu)$ be the subset of all functions in $\mathcal{M}(\mathbb{S},\mu)$ whose values lie in $[0,\infty]$. The measure and the characteristic (indicator) function of a measurable set $E \subset \mathbb{S}$ are denoted by $\mu(E)$ and χ_E, respectively. Following [3, Chap. 1, Definition 1.1], a mapping $\rho : \mathcal{M}^+(\mathbb{S},\mu) \to [0,\infty]$ is called a *Banach function norm* if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathcal{M}^+(\mathbb{S},\mu)$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{S}$, the following axioms hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f+g) \le \rho(f) + \rho(g)$,
(A2) $0 \le g \le f$ a.e. $\Rightarrow$ $\rho(g) \le \rho(f)$ (the lattice property),
(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),
(A4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$,
(A5) $\mu(E) < \infty \Rightarrow \int_E f(x)\,dx \le C_E \rho(f)$

with $C_E \in (0,\infty)$, which may depend on E and ρ but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{S})$ of all functions $f \in \mathcal{M}(\mathbb{S},\mu)$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X(\mathbb{S})$, the norm of f is defined by $\|f\|_{X(\mathbb{S})} := \rho(|f|)$. Under the natural linear space operations and under this norm, the set $X(\mathbb{S})$ becomes a Banach space (see [3, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+(\mathbb{S},\mu)$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{S}} f(x)g(x)\,dx : f \in \mathcal{M}^+(\mathbb{S},\mu),\ \rho(f) \le 1 \right\}, \quad g \in \mathcal{M}^+(\mathbb{S},\mu).$$

It is a Banach function norm itself [3, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{S})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ is naturally identified with a subspace of the (Banach) dual space $X^*(\mathbb{S})$.

Let $\mathcal{M}_0(\mathbb{S},\mu)$ and $\mathcal{M}_0^+(\mathbb{S},\mu)$ be the classes of a.e. finite functions in $\mathcal{M}(\mathbb{S},\mu)$ and $\mathcal{M}^+(\mathbb{S},\mu)$, respectively. The distribution function μ_f of $f \in \mathcal{M}_0(\mathbb{S},\mu)$ is given by

$$\mu_f(\lambda) := \mu\{x \in \mathbb{S} : |f(x)| > \lambda\}, \quad \lambda \ge 0.$$

The non-increasing rearrangement of $f \in \mathcal{M}_0(\mathbb{S},\mu)$ is the function defined by

$$f^*(t) := \inf\{\lambda : \mu_f(\lambda) \le t\}, \quad t \ge 0.$$

We use here the standard convention that $\inf \emptyset = +\infty$. Now let $(\mathbb{S},\mu), (\mathbb{T},\nu) \in \{(\mathbb{R}^d, m_d), (\mathbb{R}_+, \overline{m})\}$. Two functions $f \in \mathcal{M}_0(\mathbb{S},\mu)$ and $g \in \mathcal{M}_0(\mathbb{T},\nu)$ are said to be equimeasurable if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \ge 0$.

A Banach function norm $\rho : \mathcal{M}^+(\mathbb{S},\mu) \to [0,\infty]$ is called *rearrangement-invariant* if for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+(\mathbb{S},\mu)$, the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{S})$ generated by ρ is said to be a rearrangement-invariant Banach function space (or simply a rearrangement-invariant space). Lebesgue spaces $L^p(\mathbb{S})$, $1 \le p \le \infty$, Orlicz spaces $L^\Phi(\mathbb{S})$, and Lorentz spaces $L^{p,q}(\mathbb{S})$ are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [3] and the references therein). By [3, Chap. 2, Proposition 4.2], if a Banach function space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is also rearrangement-invariant.

Following [3, Chap. 2, Definition 5.1], for each finite value t, let $E \subset \mathbb{S}$ be such that $\mu(E) = t$ and let

$$\varphi_X(t) := \|\chi_E\|_{X(\mathbb{S})}.$$

The function φ_X so defined is called the *fundamental function* of the rearrangement-invariant Banach function space $X(\mathbb{S})$.

2.2 Marcinkiewicz endpoint spaces and Orlicz spaces

Let us recall the definition of Marcinkiewicz endpoin spaces $M_\varphi(\mathbb{R}^d)$. Following [3, Chap. 2, Definition 5.6], [13, Chap. II, Definition 1.1], and [20, Definition 7.9.10], a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ is said to be *quasi-concave* if (i) $\varphi(0) = 0$; (ii) $\varphi(t) > 0$ for $t > 0$; (iii) $t/\varphi(t)$ is non-decreasing on $(0,\infty)$. For a given quasi-concave function $\varphi : [0,\infty) \to [0,\infty)$, the Marcinkiewicz endpoint space $M_\varphi(\mathbb{R}^d)$ consists of all functions $f \in \mathcal{M}_0(\mathbb{R}^d, m_d)$ such that

$$\|f\|_{M_\varphi(\mathbb{R}^d)} := \sup_{0<t<\infty} (\varphi(t) f^{**}(t))$$

is finite, where

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(x)\,dx$$

(see [3, Chap. 2, Definition 5.7], [20, Definition 7.10.1] and also [13, Chap. II, § 5]). It is well known that if $\varphi : [0,\infty) \to [0,\infty)$ is quasi-concave, then the Marcinkiewicz endpoint space $M_\varphi(\mathbb{R}^d)$ is a rearrangement-invariant Banach function space, whose fundamental function is φ (see [3, Chap. 2, Proposition 5.8] and [20, Proposition 7.10.2]). Note that if $1 < p < \infty$, then the function $\varphi(t) = t^{1/p}$ is concave and $M_\varphi(\mathbb{R}^d)$ is nothing but the weak L^p-space $L^{p,\infty}(\mathbb{R}^d)$, also known as the Marcinkiewicz space. It is well known that the Marcinkiewicz endpoint space is separable if and only if it coincides with $L^1(\mathbb{R}^d)$ up to equivalence of the norms (see, e.g., [9, p. 256], [12]), i.e. if and only if $\varphi(t) \asymp t$. As it was communicated to us by F. Sukochev, this fact can be obtained from [13, Chap. II, Theorem 4.8 and Lemma 5.4] (see also [20, Theorem 7.10.23]).

We will also need Orlicz spaces. Let $\phi : [0,\infty) \to [0,\infty]$ be a non-decreasing and left-continuous function such that $\phi(0) = 0$. Suppose that ϕ is neither identically zero nor identically infinite on $(0,\infty)$. Then the function

$$\Phi(t) := \int_0^t \phi(s)\,ds$$

is said to be a Young's function (see, e.g., [3, Chap. 4, Definition 8.1]). The Orlicz space $L^\Phi(\mathbb{R}^d)$ is the set of all measurable function $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$M_\Phi(f/k) := \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{k}\right) dx < \infty$$

for some $k = k(f) > 0$. It is well known that $L^{\Phi}(\mathbb{R}^d)$ is a rearrangement-invariant Banach function space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^d)} := \inf\{k > 0 \;:\; M_{\Phi}(f/k) \le 1\}$$

(see, e.g., [3, Chap. 4, Theorem 8.9] or [17, Chap. II, §2, Theorem 1]).

2.3 A measure preserving transformation from $\mathbb{R}^d$ to $\mathbb{R}_+$

Throughout the paper, we assume that $\mathbb{R}^d$ and $\mathbb{R}_+$ are equipped with the standard metrics and the measures m_d and $\overline{m}$, respectively.

Recall that a mapping $\sigma : \mathbb{R}^d \to \mathbb{R}_+$ is said to be a measure preserving transformation if, whenever E is an $\overline{m}$-measurable subset of $\mathbb{R}_+$, the set

$$\sigma^{-1}(E) := \{x \in \mathbb{R}^d \;:\; \sigma(x) \in E\}$$

is an m_d-measurable subset of $\mathbb{R}^d$ and $m_d(\sigma^{-1}(E)) = \overline{m}(E)$ (see [3, Chap. 2, Definition 7.1]).

Let

$$\omega_d := m_d\left\{x \in \mathbb{R}^d \;:\; |x| < 1\right\} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

be the volume of the unit ball in $\mathbb{R}^d$.

Lemma 2 *The mapping $\Theta : \mathbb{R}^d \to \mathbb{R}_+$, given by*

$$\Theta(x) := \omega_d |x|^d, \quad x \in \mathbb{R}^d, \tag{2.1}$$

is a measure preserving transformation from $\mathbb{R}^d$ to $\mathbb{R}_+$.

Proof Since Θ is continuous, $\Theta^{-1}(B)$ is a Borel set in $\mathbb{R}^d$ for every Borel set B in $\mathbb{R}_+$. Hence, we can define a Borel measure ν on $\mathbb{R}_+$ by

$$\nu(B) := m_d\left(\Theta^{-1}(B)\right)$$

for every Borel set $B \subset \mathbb{R}_+$ (see, [25, Theorem 3.21]). If $0 \le \alpha \le \beta$, then

$$\begin{aligned} m_d\left(\Theta^{-1}([\alpha,\beta])\right) &= m_d\left\{x \in \mathbb{R}^d \;:\; \alpha \le \omega_d|x|^d \le \beta\right\} \\ &= m_d\left\{x \in \mathbb{R}^d \;:\; \left(\frac{\alpha}{\omega_d}\right)^{1/d} \le |x| \le \left(\frac{\beta}{\omega_d}\right)^{1/d}\right\} \\ &= m_d\left\{x \in \mathbb{R}^d \;:\; |x| \le \left(\frac{\beta}{\omega_d}\right)^{1/d}\right\} - m_d\left\{x \in \mathbb{R}^d \;:\; |x| < \left(\frac{\alpha}{\omega_d}\right)^{1/d}\right\} \\ &= \beta - \alpha. \end{aligned} \tag{2.2}$$

So, $\nu([\alpha,\beta]) = \beta - \alpha = \overline{m}([\alpha,\beta])$. Then, for every Borel set $B \subset \mathbb{R}_+$,

$$\nu(B) = \overline{m}(B) \tag{2.3}$$

(see [4, Theorem 1.5.6 and Corollary 1.5.9]; note that [4, Corollary 1.5.9] deals with the case of $\mathbb{R}$, but the proof is essentially the same in the case of $\mathbb{R}_+$).

Let $Z \subset \mathbb{R}_+$ be a measurable set such that $\overline{m}(Z) = 0$. Then there exists a Borel set $H \subset \mathbb{R}_+$ such that $Z \subseteq H$ and $\overline{m}(H) = 0$ (see [25, Theorem 3.8 and (3.11)]). Then $\Theta^{-1}(Z) \subseteq \Theta^{-1}(H)$, and $m_d\left(\Theta^{-1}(H)\right) = 0$ according to (2.3). Hence $\Theta^{-1}(Z)$ is measurable and $m_d\left(\Theta^{-1}(Z)\right) = 0$ (see [25, Sect. 3.2, Example 2]).

For any measurable set $E \subset \mathbb{R}_+$, there exist a Borel set $B \subset \mathbb{R}_+$ and a measurable set $Z \subset \mathbb{R}_+$ such that $E = B \cup Z$ and $\overline{m}(Z) = 0$ (see, [25, Theorem 3.28]). It follows from the above that $\Theta^{-1}(E) = \Theta^{-1}(B) \cup \Theta^{-1}(Z)$ is measurable and

$$m_d\left(\Theta^{-1}(E)\right) = m_d\left(\Theta^{-1}(B)\right) = \overline{m}(B) = \overline{m}(E).$$

This completes the proof. □

2.4 Submultiplicative functions and their indices

A measurable function $\varrho : (0,\infty) \to (0,\infty)$ is said to be submultiplicative if $\varrho(x_1x_2) \le \varrho(x_1)\varrho(x_2)$ for all $x_1, x_2 \in (0,\infty)$. The behavior of a measurable submultiplicative function ϱ in neighborhoods of zero and infinity is described by the quantities

$$\alpha(\varrho) := \sup_{x\in(0,1)} \frac{\log \varrho(x)}{\log x} = \lim_{x\to 0} \frac{\log \varrho(x)}{\log x},$$

$$\beta(\varrho) := \inf_{x\in(1,\infty)} \frac{\log \varrho(x)}{\log x} = \lim_{x\to \infty} \frac{\log \varrho(x)}{\log x},$$

where $-\infty < \alpha(\varrho) \le \beta(\varrho) < \infty$ (see [13, Chap. II, Theorem 1.3]). The numbers $\alpha(\varrho)$ and $\beta(\varrho)$ are called the lower and upper indices of the measurable submultiplicative function ϱ.

2.5 Zippin indices

Following [26, p. 271] (see also [18, p. 28]), for a given rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ with fundamental function φ_X, let us consider the function

$$M(t,X) := \sup_{0<x<\infty} \frac{\varphi_X(tx)}{\varphi_X(x)}, \quad t \in (0,\infty).$$

It is easy to check that this function is nondecreasing (and hence, measurable) and submultiplicative on $(0,\infty)$. The indices of $M(\cdot,X)$ are called the *Zippin (or fundamental) indices* of the rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ and are denoted by

$$p_X := \alpha(M(\cdot,X)), \quad q_X := \beta(M(\cdot,X)).$$

They satisfy the inequalities $0 \le p_X \le q_X \le 1$ (see [18, formula (4.14)]).

Lemma 3 *If $X(\mathbb{R}^d)$ is a rearrangement-invariant Banach function space with the upper Zippin index satisfying $q_X < 1$, then condition* (1.2) *is fulfilled.*

Proof It is clear that for $t > 0$,

$$\frac{\varphi_X(t)}{t} \le \frac{\varphi_X(1)}{t} \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)} = \varphi_X(1)\frac{M(t,X)}{t}. \tag{2.4}$$

Since $q_X < 1$, there exists $\varepsilon > 0$ such that $q_X + \varepsilon < 1$. Then in view of [13, Chap. II, Theorem 1.3], there exists $t_0 = t_0(\varepsilon) > 1$ such that

$$M(t,X) \le t^{q_X+\varepsilon}, \quad t > t_0. \tag{2.5}$$

Now condition (1.2) immediately follows from (2.4)–(2.5) and $q_X + \varepsilon < 1$. □

2.6 Dilation operators on the Luxemburg representation and Boyd indices

Let $X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm ρ over $(\mathbb{R}^d, m_d)$. By the Luxemburg representation theorem (see [3, Chap. 2, Theorem 4.10] or [20, Theorem 7.8.3]), there exists a unique rearrangement-invariant Banach function norm $\overline{\rho}$ over $(\mathbb{R}_+, \overline{m})$ such that

$$\rho(f) = \overline{\rho}(f^*), \quad f \in \mathcal{M}_0^+(\mathbb{R}^d, m_d).$$

The rearrangement-invariant Banach function space over $(\mathbb{R}_+, \overline{m})$ generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{R}_+)$ and is called the Luxemburg representation of $X(\mathbb{R}^d)$. For $t > 0$, let E_t be the dilation operator defined on the set $\mathcal{M}_0(\mathbb{R}_+, \overline{m})$ by

$$(E_t\varphi)(s) = \varphi(ts), \quad 0 < s < \infty. \tag{2.6}$$

It follows from [3, Chap. 3, Proposition 5.11] that the operators E_t are bounded on $\overline{X}(\mathbb{R}_+)$ for all $t > 0$. The operator norm of the operator $E_{1/t}$ on the Luxemburg representation $\overline{X}(\mathbb{R}_+)$ will be denoted by

$$h(t,X) := \|E_{1/t}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}, \quad t > 0.$$

By [3, Chap. 3, Proposition 5.11], the function $h(\cdot, X)$ is nondecreasing (and hence, measurable) and submultiplicative on $(0, \infty)$. The indices of $h(\cdot, X)$ are called the *Boyd indices* [7] of the rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ and are denoted by

$$\alpha_X := \alpha(h(\cdot, X)), \quad \beta_X := \beta(h(\cdot, X)).$$

The Boyd and Zippin indices of $X(\mathbb{R}^d)$ satisfy

$$0 \le \alpha_X \le p_X \le q_X \le \beta_X \le 1$$

(see [18, formula (4.14)]). We refer to the survey paper [18] and the monographs [3,13] for the properties of the Zippin indices p_X, q_X and the Boyd indices α_X, β_X of rearrangement-invariant Banach function spaces. The following statement is a consequence of [13, Chap. II, Theorem 1.3 and Corollary 2].

Lemma 4 *Let $X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space. Then its upper Boyd index satisfies $\beta_X < 1$ if and only if*

$$\lim_{\tau\to 0} \tau \|E_\tau\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))} = 0.$$

2.7 Separability of a rearrangement-invariant Banach function space

The results in this subsection are probably well known, but we could not find suitable references for them. We provide details here for the sake of completeness.

Let $\overline{X}(\mathbb{R}_+)$ be a rearrangement-invariant Banach function space. For a function $w \in \overline{X}(\mathbb{R}_+)$ and $N > 0$, let

$$\mathrm{E}_N = \mathrm{E}_{w,N} := \{s \in \mathbb{R}_+ : \ |w(s)| > N\}$$

and

$$w^N(t) := \begin{cases} w(t), & \text{if } t \in \mathbb{R}_+ \setminus \mathrm{E}_N, \\ N\,\dfrac{w(t)}{|w(t)|}, & \text{if } t \in \mathrm{E}_N. \end{cases}$$

Lemma 5 *For a rearrangement-invariant Banach function space $\overline{X}(\mathbb{R}_+)$, the following statements are equivalent:*

$$\lim_{t\to 0} \varphi_{\overline{X}}(t) = 0, \qquad \lim_{N\to\infty} \left\|w - w^N\right\|_{\overline{X}(\mathbb{R}_+)} = 0 \quad \text{for all} \quad w \in \overline{X}(\mathbb{R}_+), \tag{2.7}$$

and

$$\lim_{\tau\to 0} \left\|w^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} = 0 \quad \text{for all} \quad w \in \overline{X}(\mathbb{R}_+). \tag{2.8}$$

Proof Suppose (2.7) holds. Take any $w \in \overline{X}(\mathbb{R}_+)$. For any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left\|w - w^N\right\|_{\overline{X}(\mathbb{R}_+)} < \frac{\varepsilon}{2}.$$

For this N, there exists $t_N > 0$ such that

$$\varphi_{\overline{X}}(t) < \frac{\varepsilon}{2N}$$

for all $t \in (0, t_N)$. Since $|w| \le \left|w - w^N\right| + N$, we have

$$w^* \le \left(\left|w - w^N\right| + N\right)^* = \left(w - w^N\right)^* + N \quad \text{for all} \quad N > 0.$$

Then for all $\tau \in (0, t_N)$,

$$\begin{aligned} \left\|w^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} &\le \left\|\left(w - w^N\right)^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} + \left\|N\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} \\ &\le \left\|w - w^N\right\|_{\overline{X}(\mathbb{R}_+)} + N\varphi_{\overline{X}}(\tau) < \frac{\varepsilon}{2} + N\,\frac{\varepsilon}{2N} = \varepsilon, \end{aligned}$$

which proves (2.8).

Suppose now (2.8) holds. Applying it to $w = \chi_{(0,1]}$, we get

$$\lim_{t\to 0}\varphi_{\overline{X}}(t) = \lim_{t\to 0}\left\|\chi_{(0,t]}\right\|_{\overline{X}(\mathbb{R}_+)} = \lim_{t\to 0}\left\|\chi_{(0,1]}\chi_{(0,t]}\right\|_{\overline{X}(\mathbb{R}_+)}$$
$$= \lim_{t\to 0}\left\|\chi^*_{(0,1]}\chi_{(0,t]}\right\|_{\overline{X}(\mathbb{R}_+)} = 0.$$

It is easy to see that, for any $w \in \overline{X}(\mathbb{R}_+)$,

$$\left|w - w^N\right| = (|w| - N)\chi_{\mathrm{E}_N} \le |w|\chi_{\mathrm{E}_N}, \qquad |w| \ge N\chi_{\mathrm{E}_N}. \tag{2.9}$$

Let

$$\tau(N) := \overline{m}(\mathrm{E}_N), \qquad \tau_0 := \lim_{N\to\infty}\tau(N).$$

If $\tau_0 > 0$, then the second inequality in (2.9) implies

$$\|w\|_{\overline{X}(\mathbb{R}_+)} \ge \|N\chi_{\mathrm{E}_N}\|_{\overline{X}(\mathbb{R}_+)} = N\varphi_X(\tau(N)) \ge N\varphi_X(\tau_0) \to \infty \quad \text{as} \quad N\to\infty,$$

which contradicts $w \in \overline{X}(\mathbb{R}_+)$. Hence $\tau_0 = 0$, i.e. $\tau(N) \to 0$ as $N \to \infty$. Then it follows from the inequality $(|w|\chi_{\mathrm{E}_N})^* \le w^*\chi_{(0,\tau(N)]}$, the first inequality in (2.9), and equality (2.8) that

$$\lim_{N\to\infty}\left\|w - w^N\right\|_{\overline{X}(\mathbb{R}_+)} \le \lim_{N\to\infty}\||w|\chi_{\mathrm{E}_N}\|_{\overline{X}(\mathbb{R}_+)} = \lim_{N\to\infty}\left\|(|w|\chi_{\mathrm{E}_N})^*\right\|_{\overline{X}(\mathbb{R}_+)}$$
$$\le \lim_{N\to\infty}\left\|w^*\chi_{(0,\tau(N)]}\right\|_{\overline{X}(\mathbb{R}_+)} = \lim_{\tau\to 0}\left\|w^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} = 0,$$

which completes the proof. □

The following result is a reformulation of [13, Chap. II, Theorem 4.8]. A similar result is stated in [23, Note 6.5.4] without proof. For the reader's convenience, we give a proof here.

Theorem 4 *A rearrangement-invariant Banach function space $\overline{X}(\mathbb{R}_+)$ is separable if and only if for all $w \in \overline{X}(\mathbb{R}_+)$,*

$$\lim_{\tau\to 0}\left\|w^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} = 0 = \lim_{N\to\infty}\|w^*\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)}. \tag{2.10}$$

Proof Suppose $\overline{X}(\mathbb{R}_+)$ is separable. Then it follows from [13, Chap. II, Theorem 4.8] (and Lemma 5) that (2.10) holds.

Suppose now (2.10) holds. Take any non-negative function $w \in \overline{X}(\mathbb{R}_+)$ and $\varepsilon > 0$. There exist $0 < \tau < N$ such that

$$\left\|w^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} < \frac{\varepsilon}{2}, \qquad \|w^*\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} < \frac{\varepsilon}{2}.$$

The second equality in (2.10) implies also that $\lim_{t\to\infty} w^*(t) = 0$. Indeed, if $\omega_\infty := \lim_{t\to\infty} w^*(t) > 0$, then

$$\|w^*\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} \ge \omega_\infty\|\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} \ge \omega_\infty\varphi_{\overline{X}}(1) \not\to 0 \quad \text{as} \quad N\to\infty,$$

which contradicts (2.10). Since $\lim_{t\to\infty} w^*(t) = 0$, there exists a measure preserving transformation σ from the support of w onto the support of w^* such that $w = w^* \circ \sigma$ (see [3, Chap. 2, Corollary 7.6]).

Let

$$G_{\tau,N} := \sigma^{-1}\big((\tau,N)\cap \operatorname{supp} w^*\big).$$

Then

$$\overline{m}\,(G_{\tau,N}) \le \overline{m}\big((\tau,N)\big) = N-\tau, \quad w\chi_{G_{\tau,N}} = \big(w^*\chi_{(\tau,N)}\big)\circ\sigma,$$

and $w\chi_{G_{\tau,N}}$ is a bounded function supported in a set of finite measure. Hence

$$w - w\chi_{G_{\tau,N}} = \big(w^* - w^*\chi_{(\tau,N)}\big)\circ\sigma,$$

which implies that the functions $w - w\chi_{G_{\tau,N}}$ and $w^* - w^*\chi_{(\tau,N)}$ are equimeasurable (see [3, Chap. 2, Proposition 7.2]), and

$$\begin{aligned}\big\|w - w\chi_{G_{\tau,N}}\big\|_{\overline{X}(\mathbb{R}_+)} &= \big\|w^* - w^*\chi_{(\tau,N)}\big\|_{\overline{X}(\mathbb{R}_+)} \\ &\le \big\|w^*\chi_{(0,\tau]}\big\|_{\overline{X}(\mathbb{R}_+)} + \|w^*\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Let $\overline{X}_b(\mathbb{R}_+)$ denote the closure in $\overline{X}(\mathbb{R}_+)$ of the set of bounded functions supported in sets of finite measure. The above inequality implies that $w \in \overline{X}_b(\mathbb{R}_+)$ for every non-negative $w \in \overline{X}(\mathbb{R}_+)$. Since every complex-valued function $w \in \overline{X}(\mathbb{R}_+)$ is a linear combination of four non-negative functions from $\overline{X}(\mathbb{R}_+)$, we conclude that $\overline{X}(\mathbb{R}_+) = \overline{X}_b(\mathbb{R}_+)$. It follows from the first equality in (2.10) and Lemma 5 that $\lim_{t\to 0}\varphi_{\overline{X}}(t) = 0$. Then [3, Chap. 2, Theorem 5.5; Chap. 1, Corollary 5.6] imply that the space $\overline{X}(\mathbb{R}_+)$ is separable. $\square$

3 Proof of the result about the weak convergence

3.1 Integrals over translations of a compact set

Let $\mathbb{1}$ be the constant function identically equal to 1. In order to prove Theorem 1, we will need the following auxiliary result.

Lemma 6 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to \infty$ as $n\to\infty$ and $F \subset \mathbb{R}^d$ be a compact set of positive measure. Suppose $X(\mathbb{R}^d)$ is a rearrangement-invariant Banach function space with fundamental function φ_X. Then*

$$\int_{h_n+F} f(x)\,dx \to 0 \quad \textit{as} \quad n\to\infty \quad \textit{for every} \quad f\in X(\mathbb{R}^d) \tag{3.1}$$

if and only if $\varphi_X(t)\to\infty$ as $t\to\infty$.

Proof Necessity. Suppose (3.1) holds and

$$\kappa := \lim_{t\to\infty}\varphi_X(t) < \infty.$$

Let $\{B_n\}$ be the sequence of the closed balls of radii $n\in\mathbb{N}$ centered at $0\in\mathbb{R}^d$. Since $\chi_{B_n}\uparrow \mathbb{1}$ as $n\to\infty$ and $\|\chi_{B_n}\|_{X(\mathbb{R}^d)} \le \varphi_X(m_d(B_n)) \le \kappa$ for all $n\in\mathbb{N}$, by

Fatou's lemma for $X(\mathbb{R}^d)$ (see [3, Chap. 1, Lemma 1.5]), we have $\mathbb{1} \in X(\mathbb{R}^d)$. It is clear that

$$\int_{h_n+F} \mathbb{1}(x)\,dx = m_d(F) > 0 \quad \text{for all} \quad n \in \mathbb{N},$$

and (3.1) does not hold, so we arrive at a contradiction.

Sufficiency. Suppose $\varphi_X(t) \to \infty$ as $t \to \infty$ and (3.1) does not hold. Then there exist a function $f \in X(\mathbb{R}^d)$, a number $\delta > 0$, and a strictly increasing sequence of natural numbers $\{n_j\}$ such that

$$\left| \int_{h_{n_j}+F} f(x)\,dx \right| \geq \delta \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.2}$$

Extracting a subsequence if necessary, we can assume that the sets $h_{n_j} + F$ are pairwise disjoint. Let

$$g := \sum_{j=1}^{\infty} \left(\frac{1}{m_d(F)} \int_{h_{n_j}+F} f(x)\,dx \right) \chi_{h_{n_j}+F}.$$

By [3, Chap. 2, Theorem 4.8], $\|g\|_{X(\mathbb{R}^d)} \leq \|f\|_{X(\mathbb{R}^d)} < \infty$. On the other hand, it follows from (3.2) that for all $N \in \mathbb{N}$,

$$|g| \geq \frac{\delta}{m_d(F)} \chi_{E_N}, \quad \text{where} \quad E_N := \bigcup_{j=1}^{N} (h_{n_j} + F).$$

Hence

$$\varphi_X(N m_d(F)) = \|\chi_{E_N}\|_{X(\mathbb{R}^d)} \leq \frac{m_d(F)}{\delta} \|g\|_{X(\mathbb{R}^d)} < \infty,$$

which contradicts the condition $\varphi_X(t) \to \infty$ as $t \to \infty$. □

3.2 Proof of Theorem 1

(a) *Necessity.* By [3, Chap. 1, Theorem 2.9], the associate space $X'(\mathbb{R}^d)$ is canonically isometrically isomorphic to a closed subspace of the Banach space dual $X^*(\mathbb{R}^d)$ of $X(\mathbb{R}^d)$. This implies that if the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$ on the space $X(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f(x - h_n) g(x)\,dx \to 0 \quad \text{as} \quad n \to \infty \tag{3.3}$$

for every $f \in X(\mathbb{R}^d)$ and $g \in X'(\mathbb{R}^d)$. Assume that (1.2) does not hold. Then

$$\varlimsup_{t\to\infty} \frac{\varphi_X(t)}{t} > 0 \tag{3.4}$$

(note that the above limit exists because the function $\varphi_X(t)/t$ is nonincreasing in view of [3, Chap. 2, Corollary 5.3]). It follows from (3.4) and [3, Chap. 2, Theorem 5.2] that

$$\lim_{t\to\infty} \varphi_{X'}(t) = \lim_{t\to\infty} \frac{t}{\varphi_X(t)} < \infty.$$

Then one can prove as in the proof of the necessity portion of Lemma 6 that $\mathbb{1} \in X'(\mathbb{R}^d)$. Let $F \subset \mathbb{R}^d$ be a measurable set of finite positive measure and $f := \chi_F$. Then $f \in X(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(x-h_n)\mathbb{1}(x)\,dx = m_d(F) > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence (3.3) does not hold and the sequence $\{T_{h_n}\}$ does not converge weakly to the zero operator. The obtained contradiction completes the proof of part (a).

(b) *Sufficiency.* Since $X(\mathbb{R}^d)$ is separable, its Banach space dual is canonically isometrically isomorphic to its associate space $X'(\mathbb{R}^d)$ (see [3, Chap. 1, Corollaries 4.3 and 5.6]). Hence it is enough to prove that (1.2) implies that (3.3) is fulfilled for all $f \in X(\mathbb{R}^d) \setminus \{0\}$ and $g \in X'(\mathbb{R}^d) \setminus \{0\}$.

It follows from [3, Chap. 1, Propositions 3.6 and 3.10, Theorem 3.11, and Corollary 5.6] that the set $L_c^\infty(\mathbb{R}^d)$ of all bounded compactly supported functions is dense in the separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$.

Take any $\varepsilon > 0$. There exists $f_0 \in L_c^\infty(\mathbb{R}^d)$ such that

$$\|f - f_0\|_{X(\mathbb{R}^d)} < \frac{\varepsilon}{2\|g\|_{X'(\mathbb{R}^d)}}.$$

Then it follows from Hölder's inequality for $X(\mathbb{R}^d)$ (see [3, Chap. 1, Theorem 2.4]), the fact that T_{h_n} is an isometry on $X(\mathbb{R}^d)$, and the above inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} f(x-h_n)g(x)\,dx\right| \\
&\quad\le \left|\int_{\mathbb{R}^d} f_0(x-h_n)g(x)\,dx\right| + \|T_{h_n}(f-f_0)\|_{X(\mathbb{R}^d)}\|g\|_{X'(\mathbb{R}^d)} \\
&\quad= \left|\int_{\mathbb{R}^d} f_0(x-h_n)g(x)\,dx\right| + \|f-f_0\|_{X(\mathbb{R}^d)}\|g\|_{X'(\mathbb{R}^d)} \\
&\quad< \left|\int_{\mathbb{R}^d} f_0(x-h_n)g(x)\,dx\right| + \frac{\varepsilon}{2}.
\end{aligned} \tag{3.5}$$

Let F be the support of f_0. Since $\varphi_{X'}(t) = t/\varphi_X(t) \to \infty$ as $t \to \infty$, it follows from Lemma 6 applied to $|g| \in X'(\mathbb{R}^d)$ that there exists $n_0 \in \mathbb{N}$ such that

$$\int_{h_n+F} |g(x)|\,dx < \frac{\varepsilon}{2\|f_0\|_{L^\infty(\mathbb{R}^d)}} \quad \text{for all} \quad n \ge n_0. \tag{3.6}$$

Inequalities (3.5)–(3.6) imply that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f(x-h_n)g(x)\,dx\right| &< \left|\int_{\mathbb{R}^d} f_0(x-h_n)g(x)\,dx\right| + \frac{\varepsilon}{2}\\ &= \left|\int_{h_n+F} f_0(x-h_n)g(x)\,dx\right| + \frac{\varepsilon}{2}\\ &\le \|f_0\|_{L^\infty(\mathbb{R}^d)} \int_{h_n+F} |g(x)|\,dx + \frac{\varepsilon}{2}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for all $n \ge n_0$. So, (3.3) holds for all $f \in X(\mathbb{R}^d) \setminus \{0\}$ and $g \in X'(\mathbb{R}^d) \setminus \{0\}$. This completes the proof. □

4 Some corollaries of the result about the weak convergence

4.1 On the strong convergence of the sequence $\{T_{h_n}^{-1} K T_{h_n}\}$ to zero for a compact operator K

Now we are in a position to extend Lemma 1 to the setting of separable rearrangement-invariant Banach function spaces.

Lemma 7 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on a separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$ such that $\varphi_X(t)/t \to 0$ as $t \to \infty$. If K is a compact operator on $X(\mathbb{R}^d)$, then the sequence $\{T_{h_n}^{-1} K T_{h_n}\}$ converges strongly to the zero operator as $n \to \infty$ on the space $X(\mathbb{R}^d)$.*

Proof Since the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$ in view of Theorem 1 and the operator K is compact, it follows from [22, Lemma 1.4.6] that the sequence $\{KT_{h_n}\}$ converges strongly to the zero operator as $n \to \infty$. Now taking into account that $\|T_{h_n}^{-1}\| = 1$ for all $n \in \mathbb{N}$, the desired result follows from [22, Lemma 1.4.4]. □

4.2 Condition (1.2) is fulfilled if $X(\mathbb{R}^d)$ is reflexive

In the following two subsections, we give some sufficient conditions for (1.2). We start with the following result, which follows easily from the known ones. Since we were not able to provide a suitable reference, we give its proof below.

Lemma 8 *Let $X(\mathbb{R}^d)$ be a separable rearrangement-invariant Banach function space. Then $\varphi_X(t) \to \infty$ as $t \to \infty$.*

Proof If $X(\mathbb{R}^d)$ is separable, then its Luxemburg representation $\overline{X}(\mathbb{R}_+)$ is also separable. If $\lim\limits_{t\to\infty} \varphi_X(t) < \infty$, then one can show as in the proof of Lemma 6 that $\mathbb{1} \in \overline{X}(\mathbb{R}_+)$. For every $N > 0$, the functions $\mathbb{1}$ and $\mathbb{1} - \chi_{(0,N)}\mathbb{1}$ are equimeasurable. Therefore,

$$\lim_{N\to\infty} \|\mathbb{1} - \chi_{(0,N)}\mathbb{1}\|_{\overline{X}(\mathbb{R}_+)} = \|\mathbb{1}\|_{\overline{X}(\mathbb{R}_+)} > 0,$$

which contradicts [13, Chap. II, Theorem 4.8(2)]. □

Corollary 1 *If $X(\mathbb{R}^d)$ is a reflexive rearrangement-invariant Banach function space, then* (1.2) *holds.*

Proof Since the space $X(\mathbb{R}^d)$ is reflexive, both $X(\mathbb{R}^d)$ and $X'(\mathbb{R}^d)$ are separable (see [3, Chap. 1, Corollaries 4.4 and 5.6]). It follows from Lemma 8 that $\varphi_X(t) \to \infty$ and $\varphi_{X'}(t) \to \infty$ as $t \to \infty$. Then $\varphi_X(t)/t = 1/\varphi_{X'}(t) \to 0$ as $t \to \infty$. □

Theorem 1 and Corollary 1 yield the following.

Corollary 2 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$. If the space $X(\mathbb{R}^d)$ is reflexive, then the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$.*

4.3 The case when the upper Zippin index of $X(\mathbb{R}^d)$ satisfies $q_X < 1$

Theorem 1 and Lemma 3 immediately imply the following.

Corollary 3 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on a separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$. If the upper Zippin index of $X(\mathbb{R}^d)$ satisfies $q_X < 1$, then the sequence $\{T_{h_n}\}$ converges weakly to the zero operator as $n \to \infty$.*

Note that there exist reflexive rearrangement-invariant Banach function spaces $X(\mathbb{R}^d)$ with both Zippin indices being trivial, that is, $p_X = 0$ and $q_X = 1$ (see [11]). On the other hand, the Lorentz space $L^{p,1}(\mathbb{R}^d)$ is a separable and non-reflexive rearrangement-invariant Banach function space with the Zippin indices $p_{L^{p,1}} = q_{L^{p,1}} = 1/p < 1$ (see [3, Chap. 4, Theorem 4.6], [8, p. 83], and [18, formula (4.14)]). So, neither of Corollaries 2 and 3 implies the other one.

5 Proofs of the results about the absence of the weak convergence

5.1 Proof of Theorem 2(a)

If $\lim\limits_{t\to\infty} \varphi_X(t) < \infty$, then $\mathbb{1} \in X(\mathbb{R}^d)$ (see the proof of the necessity portion of Lemma 6). Let B be the unit ball in $\mathbb{R}^d$. Then $\chi_B \in X'(\mathbb{R}^d)$. It follows from Hölder's inequality (see [3, Chap. 1, Theorem 2.4]) that

$$G(f) := \int_{\mathbb{R}^d} f(x)\chi_B(x)\,dx, \quad f \in X(\mathbb{R}^d),$$

is a bounded linear functional on $X(\mathbb{R}^d)$. Since

$$\inf_{h\in\mathbb{R}^d} |G(T_h\mathbb{1})| = \int_B dx > 0,$$

the functions $T_{h_n}\mathbb{1}$ cannot converge weakly to the zero function as $n \to \infty$.

Now suppose that $\lim\limits_{t\to 0} \varphi_X(t) > 0$. Let $X_0(\mathbb{R}^d)$ be the linear subspace of $X(\mathbb{R}^d)$ consisting of all functions $f \in X(\mathbb{R}^d)$ that are continuous in a neighborhood of the origin (which may depend on f). Consider the linear functional $F_0 : X_0(\mathbb{R}^d) \to \mathbb{C}$ defined by $F_0(f) = f(0)$ for all $f \in X_0(\mathbb{R}^d)$. The condition $\lim\limits_{t\to 0} \varphi_X(t) > 0$ implies that $X(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ (see, e.g., [24, Lemma 4(d)]). Then

$$|F_0(f)| = |f(0)| \le \|f\|_{L^\infty(\mathbb{R}^d)} \le \text{const}\,\|f\|_{X(\mathbb{R}^d)}, \quad f \in X_0(\mathbb{R}^d).$$

By the Hahn-Banach theorem, F_0 admits an extension $F \in X^*(\mathbb{R}^d)$. Let

$$B_n := \left\{x \in \mathbb{R}^d : \ |x + h_n| < 2^{-n}\right\}, \quad n \in \mathbb{N}, \quad B := \bigcup_{n=1}^{\infty} B_n. \tag{5.1}$$

Since B has finite measure, $\chi_B \in X(\mathbb{R}^d)$. It is easy to see that $T_{h_n}\chi_B \in X_0(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$,

$$F\left(T_{h_n}\chi_B\right) = F_0\left(T_{h_n}\chi_B\right) = \left(T_{h_n}\chi_B\right)(0) = \chi_B(-h_n) = 1,$$

the sequence $\{T_{h_n}\chi_B\}$ does not converge weakly to 0 as $n \to \infty$. □

5.2 A sufficient condition for the absence of the weak convergence of shift operators to the zero operator

Theorem 5 *Let $X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space and let $\overline{X}(\mathbb{R}_+)$ be its Luxemburg representation. Suppose that there exists $g \in X(\mathbb{R}^d)$ such that*

$$\kappa(g) := \lim_{N\to\infty} \|g^*\chi_{(N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} > 0.$$

Then there exists a functional $F \in X^*(\mathbb{R}^d)$ such that

$$\inf_{h\in\mathbb{R}^d} |F(T_h f)| > 0,$$

where $f := g^* \circ \Theta$ and Θ is given by (2.1).

Proof Let $\mathfrak{F}$ be the closed convex hull of the set

$$\left\{f - T_h f \ : \ h \in \mathbb{R}^d\right\} \subset X(\mathbb{R}^d).$$

Let us show that $f \notin \mathfrak{F}$. Take any $h_1, \dots, h_n \in \mathbb{R}^d$ and any $c_1, \dots, c_n \in [0,1]$ such that $\sum_{k=1}^n c_k = 1$. If

$$|x| \ge R := \max\{|h_1|, \dots, |h_n|\},$$

then

$$|x - h_k| \le 2|x|, \quad k = 1, \dots, n.$$

Since $f(x) = g^*(\omega_d |x|^d)$ and g^* is non-negative and non-increasing, we have for $|x| \ge R$,

$$\begin{aligned}
&\left| f(x) - \sum_{k=1}^n c_k \left(f(x) - (T_{h_k} f)(x)\right) \right| = \left| f(x) - f(x) + \sum_{k=1}^n c_k (T_{h_k} f)(x) \right| \\
&= \sum_{k=1}^n c_k (T_{h_k} f)(x) = \sum_{k=1}^n c_k g^* \left(\omega_d |x - h_k|^d\right) \ge \sum_{k=1}^n c_k g^* \left(\omega_d |2x|^d\right) \\
&= g^*\left(2^d \Theta(x)\right) = (E_{2^d} g^*)(\Theta(x)),
\end{aligned}$$

where the dilation operator E_t is defined by (2.6). Let $r := \omega_d R^d$. Then

$$\begin{aligned}
\left| f - \sum_{k=1}^n c_k (f - T_{h_k} f) \right| &\ge \chi_{\{x \in \mathbb{R}^d :\, |x| \ge R\}} (E_{2^d} g^*) \circ \Theta \\
&= \left(\chi_{[r,\infty)} (E_{2^d} g^*)\right) \circ \Theta. \qquad (5.2)
\end{aligned}$$

By Lemma 2, Θ is a measure preserving transformation. Then, in view of [3, Chap. 2, Proposition 7.2], we obtain $f^* = g^*$ and

$$\left(\left(\chi_{[r,\infty)} (E_{2^d} g^*)\right) \circ \Theta\right)^* = \left(\chi_{[r,\infty)} (E_{2^d} g^*)\right)^*. \qquad (5.3)$$

Taking into account (5.2), (5.3), and the following equalities that hold for every nonincreasing function $u : (0,\infty) \to [0,\infty)$,

$$u^*(t) = u(t), \quad \left(\chi_{[r,\infty)} u\right)^*(t) = u(t + r), \quad t, r > 0, \qquad (5.4)$$

one gets for $t > 0$,

$$\begin{aligned}
&\left(f - \sum_{k=1}^n c_k (f - T_{h_k} f) \right)^* (t) \ge \left(\chi_{[r,\infty)} (E_{2^d} g^*)\right)^* (t) = (E_{2^d} g^*)^* (t + r) \\
&= (E_{2^d} g^*)(t + r) \ge \chi_{[r,\infty)}(t) (E_{2^d} g^*)(t + r) \ge \chi_{[r,\infty)}(t) (E_{2^d} g^*)(2t) \\
&= \left(E_{2^{d+1}} \left(\chi_{[2^{d+1} r,\infty)} g^*\right)\right)(t). \qquad (5.5)
\end{aligned}$$

Since the operator $E_{2^{-(d+1)}}$ is bounded on the space $\overline{X}(\mathbb{R}_+)$, we see that

$$\begin{aligned}
\left\| \chi_{[2^{d+1} r,\infty)} g^* \right\|_{\overline{X}(\mathbb{R}_+)} &= \left\| E_{2^{-(d+1)}} E_{2^{d+1}} \left(\chi_{[2^{d+1} r,\infty)} g^*\right) \right\|_{\overline{X}(\mathbb{R}_+)} \\
&\le \left\| E_{2^{-(d+1)}} \right\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))} \left\| E_{2^{d+1}} \left(\chi_{[2^{d+1} r,\infty)} g^*\right) \right\|_{\overline{X}(\mathbb{R}_+)}.
\end{aligned}$$

Therefore

$$\begin{aligned}\left\|E_{2^{d+1}}\left(\chi_{[2^{d+1}r,\infty)}g^*\right)\right\|_{\overline{X}(\mathbb{R}_+)} &\geq \frac{\left\|\chi_{[2^{d+1}r,\infty)}g^*\right\|_{\overline{X}(\mathbb{R}_+)}}{\|E_{2^{-(d+1)}}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}} \\ &\geq \frac{\lim\limits_{N\to\infty}\left\|\chi_{(N,\infty)}g^*\right\|_{\overline{X}(\mathbb{R}_+)}}{\|E_{2^{-(d+1)}}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}} = \frac{\kappa(g)}{\|E_{2^{-(d+1)}}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}} > 0. \qquad (5.6)\end{aligned}$$

Combining (5.5) and (5.6), we see that

$$\left\|f - \sum_{k=1}^{n} c_k \left(f - T_{h_k} f\right)\right\|_{X(\mathbb{R}^d)} \geq \frac{\kappa(g)}{\|E_{2^{-(d+1)}}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}} > 0.$$

Then

$$\|f - w\|_{X(\mathbb{R}^d)} \geq \frac{\kappa(g)}{\|E_{2^{-(d+1)}}\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))}} > 0 \quad \text{for all} \quad w \in \mathfrak{F}.$$

Since $\mathfrak{F}$ is closed and convex, and $f \notin \mathfrak{F}$, the Hahn-Banach separation theorem implies the existence of a functional $F \in X^*(\mathbb{R}^d)$ and numbers $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\operatorname{Re} F(f) > \gamma_1 > \gamma_2 > \operatorname{Re} F(w) \quad \text{for all} \quad w \in \mathfrak{F}.$$

Hence, for all $h \in \mathbb{R}^d$,

$$\operatorname{Re} F(T_h f) = \operatorname{Re} F(f) - \operatorname{Re} F\big(f - T_h f\big) > \gamma_1 - \gamma_2 > 0$$

and

$$\inf_{h\in\mathbb{R}^d} |F(T_h f)| \geq \inf_{h\in\mathbb{R}^d} \operatorname{Re} F(T_h f) \geq \gamma_1 - \gamma_2 > 0,$$

which completes the proof. □

5.3 Proof of Theorem 2(b)

Let

$$\kappa_\infty := \lim_{t\to\infty} \frac{\varphi(t)}{t}.$$

If $\kappa_\infty > 0$, then it follows from Theorem 1(a) that the sequence $\{T_{h_n}\}$ does not converge to the zero operator as $n \to \infty$ on the space $M_\varphi(\mathbb{R}^d)$.

Let us show that if $\kappa_\infty = 0$, then there exists $g \in M_\varphi(\mathbb{R}^d)$ such that

$$\kappa(g) := \lim_{N\to\infty} \sup_{0<t<\infty} \Big(\varphi(t) \left(g^* \chi_{(N,\infty)}\right)^{**}(t)\Big) \geq 1 > 0. \qquad (5.7)$$

Let ψ be the quasi-concave function given by

$$\psi(t) = \frac{t}{\varphi(t)} \quad \text{for} \quad t > 0, \quad \psi(0) = 0. \qquad (5.8)$$

Then there exists a smallest concave function $\widetilde{\psi}$, which dominates ψ. The function $\widetilde{\psi}$ is called the least concave majorant of ψ. It satisfies

$$\frac{1}{2}\widetilde{\psi}(t) \le \psi(t) \le \widetilde{\psi}(t), \quad t \in [0,\infty). \tag{5.9}$$

(see [3, Chap. 2, Proposition 5.10], [13, Chap. II, inequalities (1.7)], or [20, Proposition 7.10.10]). Fix $t > 0$ and take $\varepsilon \in (0,t)$. Since the least concave majorant $\widetilde{\psi}$ of ψ is concave and positive for $t > 0$ in view of (5.9), its right derivative $\widetilde{\psi}'_+$ is right-continuous, non-negative, non-increasing and

$$\widetilde{\psi}(t) - \widetilde{\psi}(\varepsilon) = \int_\varepsilon^t \widetilde{\psi}'_+(s)\,ds, \quad 0 < \varepsilon < t \tag{5.10}$$

(see [19, Theorems 1.4.2 and 1.5.2]), where these facts are proved for convex functions). By [3, Chap. 2, Corollary 7.8], there exists $g \in \mathcal{M}_0(\mathbb{R}^d, m_d)$ such that $\widetilde{\psi}'_+(s) = g^*(s)$ for $s > 0$. It follows from this observation and (5.8)–(5.10) that

$$\int_\varepsilon^t g^*(s)\,ds \le 2\psi(t) - \psi(\varepsilon) \le 2\psi(t) = \frac{2t}{\varphi(t)}, \quad 0 < \varepsilon < t.$$

Passing to the limit as $\varepsilon \to 0+$ and multiplying by $\varphi(t)/t$, we get $\varphi(t)g^{**}(t) \le 2$ for $0 < t < \infty$, which implies that $g \in M_\varphi(\mathbb{R}^d)$.

If $N > 0$, then

$$\left(g^*\chi_{(N,\infty)}\right)^*(s) = g^*(s+N), \quad s > 0$$

(see (5.4)). Then, similarly to (5.10), we get for $t > 0$,

$$\begin{aligned}\int_0^t \left(g^*\chi_{(N,\infty)}\right)^*(s)\,ds &= \int_0^t g^*(s+N)\,ds = \int_N^{t+N} g^*(s)\,ds \\ &= \int_N^{t+N} \widetilde{\psi}'_+(s)\,ds = \widetilde{\psi}(t+N) - \widetilde{\psi}(N).\end{aligned} \tag{5.11}$$

It follows from (5.8), (5.9), and (5.11) that for $t > 0$ and $N \in \mathbb{N}$,

$$\left(g^*\chi_{(N,\infty)}\right)^{**}(t) \ge \frac{\psi(t+N)}{t} - \frac{2\psi(N)}{t} \ge \frac{\psi(t)}{t} - \frac{2\psi(N)}{t} = \frac{1}{\varphi(t)} - \frac{2\psi(N)}{t}.$$

Taking into account that $\kappa_\infty = 0$, we deduce from the later inequality that for $N > 0$,

$$\sup_{0<t<\infty}\Big(\varphi(t)\left(g^*\chi_{(N,\infty)}\right)^{**}(t)\Big) \ge 1 - 2\psi(N)\lim_{t\to\infty}\frac{\varphi(t)}{t} = 1,$$

which immediately implies (5.7).

It follows from (5.7) and Theorem 5 that there exists a functional $F \in M_\varphi^*(\mathbb{R}^d)$ such that

$$\inf_{h\in\mathbb{R}^d} |F(T_h f)| > 0, \tag{5.12}$$

where $f = g^* \circ \Theta$ and Θ is given by (2.1). In view of Lemma 2 and [3, Chap. 2, Proposition 7.2], we see that $f \in M_\varphi(\mathbb{R}^d)$. Therefore, (5.12) implies that $\{T_{h_n} f\}$ does not converge weakly to the zero function as $n \to \infty$ in $M_\varphi(\mathbb{R}^d)$. □

5.4 Proof of Theorem 2(c)

Following [17, Chap. I, §3, Definition 3], let $L_b^\Phi(\mathbb{R}^d)$ be the closure in $L^\Phi(\mathbb{R}^d)$ of the set of all bounded functions $\varphi \in \mathcal{M}_0(\mathbb{R}^d, m_d)$ with compact support. Further, let $L_0^\Phi(\mathbb{R}^d)$ be the set of all finite elements of $L^\Phi(\mathbb{R}^d)$, that is, the set of all functions $\varphi \in L^\Phi(\mathbb{R}^d)$ such that $M_\Phi(\varphi/k) < \infty$ for all $k > 0$ (see [17, Chap. II, §3, Definition 1]). Since $L^\Phi(\mathbb{R}^d)$ is non-separable, by [17, Chap. I, §3, Theorem 7], $L^\Phi(\mathbb{R}^d) \setminus L_b^\Phi(\mathbb{R}^d) \neq \emptyset$. On the other hand, $L_0^\Phi(\mathbb{R}^d)$ is a closed linear subspace of $L^\Phi(\mathbb{R}^d)$ and $L_0^\Phi(\mathbb{R}^d) = L_b^\Phi(\mathbb{R}^d)$ (see [17, Chap. II, §3, Lemma 1 and Theorem 1]). Thus, there exists $g \in L^\Phi(\mathbb{R}^d) \setminus L_0^\Phi(\mathbb{R}^d)$.

Let $f := g^* \circ \Theta$, where Θ is defined by (2.1). It follows from Lemma 2 and [3, Chap. 2, Proposition 7.2] that $f^* = g^*$. Since

$$\int_{\mathbb{R}^d} \Phi(|\varphi(x)|)\, dx = \int_{\mathbb{R}_+} \Phi(\varphi^*(t))\, dt, \quad \varphi \in \mathcal{M}_0(\mathbb{R}^d, m_d) \tag{5.13}$$

(see, e.g., [3, Chap. 2, Exercise 3]), we observe that the Luxemburg representation $\overline{L^\Phi}(\mathbb{R}_+)$ of $L^\Phi(\mathbb{R}^d)$ is generated by the Banach function norm

$$\overline{\rho}(\varphi) = \inf \left\{ k > 0 \;:\; \int_{\mathbb{R}_+} \Phi\left(\frac{\varphi^*(t)}{k}\right) dy \le 1 \right\}, \quad \varphi \in \mathcal{M}_0^+(\mathbb{R}_+, \overline{m}).$$

If

$$\varkappa(g) := \lim_{N\to\infty} \left\| g^* \chi_{(N,\infty)} \right\|_{\overline{L^\Phi}(\mathbb{R}_+)} > 0, \tag{5.14}$$

then it follows from Theorem 5 that there exists a functional $F \in (L^\Phi(\mathbb{R}^d))^*$ such that

$$\inf_{n\in\mathbb{N}} |F(T_{h_n} f)| > 0.$$

Thus, the sequence $\{T_{h_n} f\}$ does not converge weakly to the zero function as $n \to \infty$ in the Orlicz space $L^\Phi(\mathbb{R}^d)$.

Now assume that (5.14) does not hold. It follows from (5.13) and $f^* = g^*$ that for $k > 0$,

$$\int_{\mathbb{R}^d} \Phi\left(\frac{|g(x)|}{k}\right) dx = \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{k}\right) dx.$$

Therefore $f \in L^\Phi(\mathbb{R}^d) \setminus L_0^\Phi(\mathbb{R}^d)$ and

$$\delta := \inf_{\varphi \in L_0^\Phi(\mathbb{R}^d)} \|f - \varphi\|_{L^\Phi(\mathbb{R}^d)} > 0. \tag{5.15}$$

Since (5.14) does not hold, there exists $N \in \mathbb{N}$ such that

$$\left\| g^* \chi_{(N,\infty)} \right\|_{\overline{L^\Phi}(\mathbb{R}_+)} < \frac{\delta}{2}.$$

Put

$$f_N := \left(g^* \chi_{[0,N]}\right) \circ \Theta.$$

Then $f - f_N = \left(g^*\chi_{(N,\infty)}\right) \circ \Theta$ and $g^*\chi_{(N,\infty)}$ are equimeasurable in view of Lemma 2 and [3, Chap. 2, Proposition 7.2]. Hence

$$\|f - f_N\|_{L^\Phi(\mathbb{R}^d)} = \left\|g^*\chi_{(N,\infty)}\right\|_{\overline{L^\Phi}(\mathbb{R}_+)} < \frac{\delta}{2}. \tag{5.16}$$

It follows from (5.15) and (5.16) that $f_N \in L^\Phi(\mathbb{R}^d) \setminus L_0^\Phi(\mathbb{R}^d)$. The support of f_N lies in the closed ball $B_N \subset \mathbb{R}^d$ centered at the origin and such that $m_d(B_N) = N$. Therefore, the radius of B_N is equal to

$$R_N = \frac{1}{\sqrt{\pi}}\left(N\Gamma\left(\frac{d}{2}+1\right)\right)^{1/d}. \tag{5.17}$$

Since $f_N \in L^\Phi(\mathbb{R}^d)$, there exists $\alpha > 0$ such that $f_0 := f_N/\alpha$ satisfies

$$M_\Phi(f_0) = \int_{\mathbb{R}^d} \Phi(f_0(x))\,dx = \int_{B_N} \Phi(f_0(x))\,dx < \infty. \tag{5.18}$$

On the other hand, $f_N \notin L_0^\Phi(\mathbb{R}^d)$ implies that there exists $\beta > 0$ such that $M_\Phi(f_N/\beta) = \infty$, which yields $M_\Phi(f_0/\gamma) = \infty$ for $\gamma = \beta/\alpha > 0$. Thus $f_0 \notin L_0^\Phi(\mathbb{R}^d) = L_b^\Phi(\mathbb{R}^d)$.

It follows from (5.18) that for every $n \in \mathbb{N}$ there exists a ball $B_n \subset B_N$ centered at 0 such that

$$\int_{B_n} \Phi(f_0(x))\,dx \le 2^{-n}. \tag{5.19}$$

Since $|h_n| \to +\infty$ as $n \to \infty$, we can extract a subsequence $\left\{h_n^{(1)}\right\}$ of $\{h_n\}$ such that

$$\min_{n\in\mathbb{N}}\left|h_n^{(1)}\right| > 2R_N, \quad \inf_{n\neq k}\left|h_n^{(1)} - h_k^{(1)}\right| > 2R_N, \tag{5.20}$$

where R_N is given by (5.17).

Set

$$\widetilde{f} := f_0 + \sum_{n=1}^{\infty} T_{-h_n^{(1)}}\left(f_0\,\chi_{B_n}\right).$$

The support of $T_{-h_n^{(1)}}\left(f_0\,\chi_{B_n}\right)$ lies in $-h_n^{(1)} + B_n \subset -h_n^{(1)} + B_N$. It follows from (5.20) that $-h_n^{(1)} + B_N$ are pairwise disjoint and disjoint form B_N. Therefore, taking into account (5.19), we see that

$$\begin{aligned}
\int_{\mathbb{R}^d} \Phi\left(\widetilde{f}(x)\right)\,dx &= \int_{\mathbb{R}^d} \Phi(f_0(x))\,dx + \sum_{n=1}^{\infty}\int_{\mathbb{R}^d} \Phi\left(\left(T_{-h_n^{(1)}}\left(f_0\,\chi_{B_n}\right)\right)(x)\right)\,dx \\
&= \int_{B_N} \Phi(f_0(x))\,dx + \sum_{n=1}^{\infty}\int_{\mathbb{R}^d} \Phi\left(\left(f_0\,\chi_{B_n}\right)(x)\right)\,dx \\
&= \int_{B_N} \Phi(f_0(x))\,dx + \sum_{n=1}^{\infty}\int_{B_n} \Phi(f_0(x))\,dx
\end{aligned}$$

$$\leq \int_{B_N} \Phi(f_0(x))\,dx + \sum_{n=1}^{\infty} 2^{-n} = \int_{B_N} \Phi(f_0(x))\,dx + 1 < \infty.$$

Hence $\widetilde{f} \in L^\Phi(\mathbb{R}^d)$.

Let $L^\Phi_{b,N}(\mathbb{R}^d)$ be the closure of the set of all functions in $L^\Phi(\mathbb{R}^d)$ that are bounded on the ball B_N. It is easy to see that for all $m \in \mathbb{N}$,

$$\begin{aligned}\left(\widetilde{f} - T_{h_m^{(1)}}\widetilde{f}\right)\chi_{B_N} &= (f_0 - f_0\,\chi_{B_m})\,\chi_{B_N} = f_0\,\chi_{B_N\setminus B_m}\\ &= \frac{1}{\alpha}\left(\left(g^*\chi_{[0,N]}\right)\circ\Theta\right)\chi_{B_N\setminus B_m}.\end{aligned}$$

Hence the function $\widetilde{f} - T_{h_m^{(1)}}\widetilde{f}$ is bounded on B_N. Thus $\widetilde{f} - T_{h_m^{(1)}}\widetilde{f} \in L^\Phi_{b,N}(\mathbb{R}^d)$ for all $m \in \mathbb{N}$.

We claim that $\widetilde{f} \notin L^\Phi_{b,N}(\mathbb{R}^d)$. Indeed, if $\widetilde{f} \in L^\Phi_{b,N}(\mathbb{R}^d)$, then there exists a sequence $\{f_j\}$ of functions in $L^\Phi(\mathbb{R}^d)$ such that $f_j\chi_{B_N} \in L^\infty(\mathbb{R}^d)$ and

$$\left\|\widetilde{f} - f_j\right\|_{L^\Phi(\mathbb{R}^d)} \to 0 \text{ as } j \to \infty.$$

Since $\widetilde{f}\chi_{B_N} = f_0\chi_{B_N} = f_0$, we obtain

$$\|f_0 - f_j\chi_{B_N}\|_{L^\Phi(\mathbb{R}^d)} = \left\|\chi_{B_N}(\widetilde{f} - f_n)\right\|_{L^\Phi(\mathbb{R}^d)} \leq \left\|\widetilde{f} - f_j\right\|_{L^\Phi(\mathbb{R}^d)} \to 0 \text{ as } j \to \infty.$$

Hence $f_0 \in L^\Phi_b(\mathbb{R}^d)$, which is not the case. Thus, $\widetilde{f} \in L^\Phi(\mathbb{R}^d) \setminus L^\Phi_{b,N}(\mathbb{R}^d)$.

By the Hahn-Banach theorem, there exists a functional $G \in (L^\Phi(\mathbb{R}^d))^*$ such that

$$G(\widetilde{f}) = 1 \quad \text{and} \quad G(w) = 0 \quad \text{for all} \quad w \in L^\Phi_{b,N}(\mathbb{R}^d).$$

Then for all $m \in \mathbb{N}$,

$$G\left(T_{h_m^{(1)}}\widetilde{f}\right) = G\left(\widetilde{f} - \left(\widetilde{f} - T_{h_m^{(1)}}\widetilde{f}\right)\right) = G(\widetilde{f}) - G\left(\widetilde{f} - T_{h_m^{(1)}}\widetilde{f}\right) = 1 - 0 = 1.$$

So, the sequence $\left\{T_{h_n}\widetilde{f}\right\}$ does not converge weakly to the zero function as $n \to \infty$ in the non-separable Orlicz space $L^\Phi(\mathbb{R}^d)$, which completes the proof in the case when (5.14) does not hold. □

6 Proof of Theorem 3

Since $X(\mathbb{R}^d)$ is non-separable, its Luxemburg representation $\overline{X}(\mathbb{R}_+)$ is also non-separable. Therefore, at least one equality in (2.10) does not hold. If the second equality there does not hold, then (1.3) follows from Theorem 5. Suppose now the first equality in (2.10) does not hold, i.e. there exists $g \in X(\mathbb{R}^d)$, for which

$$\kappa_0(g) := \lim_{\tau\to 0} \left\|g^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} > 0.$$

Then $E_4 g^* \in \overline{X}(\mathbb{R}_+)$ and for every $\tau > 0$,

$$\begin{aligned}\kappa_0(g) &\le \left\| g^* \chi_{(0,4\tau]} \right\|_{\overline{X}(\mathbb{R}_+)} = \left\| E_{\frac{1}{4}} E_4 \left(g^* \chi_{(0,4\tau]} \right) \right\|_{\overline{X}(\mathbb{R}_+)} \\ &\le \left\| E_{\frac{1}{4}} \right\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)} \left\| E_4 \left(g^* \chi_{(0,4\tau]} \right) \right\|_{\overline{X}(\mathbb{R}_+)} \\ &= \left\| E_{\frac{1}{4}} \right\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)} \left\| \left(E_4 g^* \right) \chi_{(0,\tau]} \right\|_{\overline{X}(\mathbb{R}_+)} .\end{aligned}$$

So,

$$\left\| \left(E_4 g^* \right) \chi_{(0,\tau]} \right\|_{\overline{X}(\mathbb{R}_+)} \ge \frac{\kappa_0(g)}{\left\| E_{\frac{1}{4}} \right\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)}} > 0 \quad \text{for all} \quad \tau > 0. \tag{6.1}$$

Let

$$s_m := \omega_d |h|^d 2^{-d} \sum_{j=m}^{\infty} \frac{1}{j^2}, \qquad \varDelta_m := \left[g^*(s_m), g^*(s_{m+1}) \right], \qquad m \in \mathbb{N}. \tag{6.2}$$

Note that

$$s_m - s_{m+1} = \frac{\omega_d |h|^d}{2^d m^2} .$$

Let

$$g_m(t) := \begin{cases} g^*(t + s_{m+1}) \text{ if } 0 \le t \le \dfrac{\omega_d |h|^d}{2^d m^2} , \\ 0 \qquad\qquad\quad \text{ if } t > \dfrac{\omega_d |h|^d}{2^d m^2} , \end{cases}$$

and

$$f_m := T_{-mh} \left(g_m \circ \Theta \right), \qquad f := \sum_{m=1}^{\infty} f_m .$$

It follows from Lemma 2 and [3, Chap. 2, Proposition 7.2] that the functions f_m and g_m are equimeasurable. By construction, f_m is supported in the ball B_m of volume $\frac{\omega_d |h|^d}{2^d m^2}$, i.e. of radius $\frac{|h|}{2m^{2/d}}$, centred at $-mh$, and its nonzero values belong to $\varDelta_m$ (see (6.2)). Since the supports of the functions f_m, $m \in \mathbb{N}$ are pairwise disjoint, the functions f and $g^* \chi_{(0,s_1]}$ are equimeasurable. In particular, $f \in X(\mathbb{R}^d)$.

Applying the same procedure as above to $E_4 g^*$, one gets a function $\widetilde{f} \in X(\mathbb{R}^d)$, which is equimeasurable with $\left(E_4 g^* \right) \chi_{(0,s_1/4]}$, is supported in the union of the balls $\widetilde{B}_m$ of volume $\frac{\omega_d |h|^d}{2^d (2m)^2}$ centred at $-mh$, $m \in \mathbb{N}$, and whose values on each of these balls belong to the corresponding $\varDelta_m$.

Take any $n \in \mathbb{N}$. It is easy to see that the values of the function $T_{nh} f$ on the ball $B_{m,n}$ of volume $\frac{\omega_d |h|^d}{2^d (m+n)^2}$ centred at $-mh$ belong to $\varDelta_{m+n}$. If $m \ge n$, i.e $m + n \le 2m$, then $B_{m,n} \supseteq \widetilde{B}_m$, and it follows from $\min \varDelta_{m+n} \ge \max \varDelta_m$ that

$$\left(T_{nh} f \right)(x) \ge \widetilde{f}(x), \quad x \in \widetilde{B}_m .$$

Hence

$$T_{nh}f \geq \widetilde{f}\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|\geq\left(n-\frac{1}{2}\right)|h|\right\}}. \tag{6.3}$$

The remaining part of the proof is similar to the argument used in the proof of Theorem 5. Let $\mathfrak{F}$ be the closed convex hull of the set

$$\{f - T_{nh}f \ : \ n \in \mathbb{N}\} \subset X(\mathbb{R}^d).$$

Let us show that $f \notin \mathfrak{F}$. Take any $n_1, \dots, n_\ell \in \mathbb{N}$ and any $c_1, \dots, c_\ell \in [0,1]$ such that $\sum_{k=1}^{\ell} c_k = 1$. Let

$$N := \max\{n_1, \dots, n_\ell\}.$$

Using (6.3), one gets

$$\begin{aligned}\left|f - \sum_{k=1}^{\ell} c_k\left(f - T_{n_k h}f\right)\right| &= \left|f - f + \sum_{k=1}^{\ell} c_k T_{n_k h}f\right| = \sum_{k=1}^{\ell} c_k T_{n_k h} f\\ &\geq \sum_{k=1}^{\ell} c_k \widetilde{f}\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|\geq\left(N-\frac{1}{2}\right)|h|\right\}} = \widetilde{f}\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|\geq\left(N-\frac{1}{2}\right)|h|\right\}}.\end{aligned}$$

It is easy to see that the functions $\widetilde{f}\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|\geq\left(N-\frac{1}{2}\right)|h|\right\}}$ and $\left(E_4 g^*\right)\chi_{(0,s_N/4]}$ equimeasurable. Then it follows from (6.1) that

$$\begin{aligned}\left\|f - \sum_{k=1}^{\ell} c_k\left(f - T_{n_k h}f\right)\right\|_{X(\mathbb{R}^d)} &\geq \left\|\widetilde{f}\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|\geq\left(N-\frac{1}{2}\right)|h|\right\}}\right\|_{X(\mathbb{R}^d)}\\ &= \left\|\left(E_4 g^*\right)\chi_{(0,s_N/4]}\right\|_{\overline{X}(\mathbb{R}_+)} \geq \frac{\kappa_0(g)}{\left\|E_{\frac{1}{4}}\right\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)}}.\end{aligned}$$

So,

$$\|f - w\|_{X(\mathbb{R}^d)} \geq \frac{\kappa_0(g)}{\left\|E_{\frac{1}{4}}\right\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)}} > 0 \quad \text{for all} \quad w \in \mathfrak{F}.$$

Since $\mathfrak{F}$ is closed and convex, and $f \notin \mathfrak{F}$, the Hahn-Banach separation theorem implies the existence of a functional $F \in X^*(\mathbb{R}^d)$ and numbers $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\operatorname{Re} F(f) > \gamma_1 > \gamma_2 > \operatorname{Re} F(w) \quad \text{for all} \quad w \in \mathfrak{F}.$$

Hence, for all $n \in \mathbb{N}$,

$$\operatorname{Re} F(T_{nh}f) = \operatorname{Re} F(f) - \operatorname{Re} F\big(f - T_{nh}f\big) > \gamma_1 - \gamma_2 > 0.$$

So, (1.3) holds. □

7 Weak convergence of shifts of compactly supported functions to zero

7.1 Compactly supported functions are not dense in non-separable rearrangement-invariant spaces

Let $X(\mathbb{R}^d)$ be a rearrangement-invariant Banach function space. By $X_c(\mathbb{R}^d)$ denote the closure with respect to the norm of $X(\mathbb{R}^d)$ of the set of all compactly supported (not necessarily bounded) functions in $X(\mathbb{R}^d)$. Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$. In this section, we will find some conditions guaranteeing that the sequences $\{T_{h_n} f\}$ converge weakly to the zero function as $n \to \infty$ for all functions $f \in X_c(\mathbb{R}^d)$ even if the sequence $\{T_{h_n}\}$ does not converge weakly to the zero operator as $n \to \infty$ on the whole space $X(\mathbb{R}^d)$.

We start with the following result, which shows that the set of compactly supported functions is not dense in a non-separable rearrangement-invariant Banach function space $X(\mathbb{R}^d)$, that is, $X_c(\mathbb{R}^d) \neq X(\mathbb{R}^d)$.

Theorem 6 *If $X(\mathbb{R}^d)$ is a non-separable rearrangement-invariant Banach function space, then there exists $f \in X(\mathbb{R}^d)$ such that*

$$\lim_{R\to\infty} \left\|\chi_{\{x\in\mathbb{R}^d\,:\,|x|>R\}} f\right\|_{X(\mathbb{R}^d)} > 0. \tag{7.1}$$

Proof Since $X(\mathbb{R}^d)$ is non-separable, its Luxemburg representation $\overline{X}(\mathbb{R}_+)$ is non-separable. Therefore there exists $g \in \overline{X}(\mathbb{R}_+)$ such that

$$\kappa_0(g) := \lim_{\tau\to 0} \left\|g^*\chi_{(0,\tau]}\right\|_{\overline{X}(\mathbb{R}_+)} > 0 \tag{7.2}$$

or

$$\lim_{N\to\infty} \|g^*\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} > 0 \tag{7.3}$$

(see Theorem 4). Then there exists $g_0 \in \overline{X}(\mathbb{R}_+)$ such that

$$\lim_{N\to\infty} \|g_0\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} > 0. \tag{7.4}$$

Indeed, there is nothing to prove if (7.3) holds as one can simply take $g_0 = g^*$ in this case. So, suppose that (7.2) holds. Let $\tau_1 := 1$. Since $g^*\chi_{(1/n,1]} \uparrow g^*\chi_{(0,1]}$ as $n \to \infty$, we have

$$\left\|g^*\chi_{(1/n,1]}\right\|_{\overline{X}(\mathbb{R}_+)} \uparrow \left\|g^*\chi_{(0,1]}\right\|_{\overline{X}(\mathbb{R}_+)} \geq \kappa_0(g)$$

as $n \to \infty$. Hence there exists $\tau_2 \in (0,1)$ such that

$$\left\|g^*\chi_{(\tau_2,1]}\right\|_{\overline{X}(\mathbb{R}_+)} \geq \frac{\kappa_0(g)}{2}.$$

Arguing similarly, we can construct a sequence $\{\tau_n\}$ such that $\tau_n \downarrow 0$ as $n \to \infty$ and

$$\left\|g^*\chi_{(\tau_{n+1},\tau_n]}\right\|_{\overline{X}(\mathbb{R}_+)} \geq \frac{\kappa_0(g)}{2}, \quad n \in \mathbb{N}.$$

The supports of the functions

$$g_n(x) := \left(g^*\chi_{(\tau_{n+1},\tau_n]}\right)(x-n), \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N},$$

lie in $(n, n+1]$, and hence are pairwise disjoint. It is easy to see that the functions $g_0 := \sum_{n=1}^{\infty} g_n$ and $g^*\chi_{(0,1]}$ are equimeasurable. So, $g_0 \in \overline{X}(\mathbb{R}_+)$. For any $N > 0$, we have

$$\|g_0\chi_{[N,\infty)}\|_{\overline{X}(\mathbb{R}_+)} \geq \|g_{[N]+1}\|_{\overline{X}(\mathbb{R}_+)} \geq \frac{\kappa_0(g)}{2}.$$

Hence (7.4) holds.

Let $f := g_0 \circ \Theta$, where Θ is given by (2.1). It follows from Lemma 2 and [3, Chap. 2, Proposition 7.2] that $f^* = g_0^*$ and

$$\begin{aligned}\left(\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|>\omega_d^{-1/d}N^{1/d}\right\}}f\right)^* &= \left((g_0\circ\Theta)\chi_{\{x\in\mathbb{R}^d\,:\,\Theta(x)>N\}}\right)^* \\ &= \left(\left(g_0\chi_{(N,\infty)}\right)\circ\Theta\right)^* = \left(g_0\chi_{(N,\infty)}\right)^*.\end{aligned}$$

Hence $f \in X(\mathbb{R}^d)$ and

$$\begin{aligned}\lim_{R\to\infty}\left\|\chi_{\{x\in\mathbb{R}^d\,:\,|x|>R\}}f\right\|_{X(\mathbb{R}^d)} &= \lim_{N\to\infty}\left\|\chi_{\left\{x\in\mathbb{R}^d\,:\,|x|>\omega_d^{-1/d}N^{1/d}\right\}}f\right\|_{X(\mathbb{R}^d)} \\ &= \lim_{N\to\infty}\left\|g_0\chi_{(N,\infty)}\right\|_{\overline{X}(\mathbb{R}_+)} > 0\end{aligned}$$

(see (7.4)), which completes the proof of (7.1). □

Remark 1 One can prove the above theorem with the help of an argument used in the proof of [1, Theorem 1]. If (7.2) holds, then $\overline{X}(0,1)$ is non-separable and hence contains a subspace that is order isomorphic to $\ell^\infty(\mathbb{N})$ according to a theorem by G.Ja. Lozanovskiĭ and A.A. Mekler ([16], see also [2, Theorem 4.51] or [14, Vol. II, Proposition 1.a.7]). Moreover, the images of the standard unit vectors of $\ell^\infty(\mathbb{N})$ under this isomorphism have pairwise disjoint supports. Taking their shifts, one gets analogues of the functions g_n used in the proof of Theorem 6. Their sum is an analogue of the function g_0 constructed above and is equimeasurable with the image of the element $(1, 1, 1, \dots) \in \ell^\infty(\mathbb{N})$.

Remark 2 Condition (7.1) plays an important role in the proof of [10, Theorem 1.4] saying that a certain algebra $\mathcal{A}_{L^{p,1}}(\mathbb{R})$ of convolution type operators with continuous data on the separable non-reflexive Lorentz space $L^{p,1}(\mathbb{R})$, $1 < p < \infty$, does not contain all rank one operators. In that proof, we constructed a function f, lying in the non-separable Marcinkiewicz space $L^{p',\infty}(\mathbb{R})$, where $1/p + 1/p' = 1$, such that (7.1) holds for the norm $\|\cdot\|_{L^{p',\infty}(\mathbb{R})}$. Theorem 6 allows one to improve [10, Theorem 1.4], replacing the separable non-reflexive Lorentz space $L^{p,1}(\mathbb{R})$, $1 < p < \infty$, by an arbitrary separable non-reflexive rearrangement-invariant Banach function space $X(\mathbb{R})$ with the Boyd indices satisfying $\alpha_X, \beta_X \in (0,1)$. This follows from the Lorentz-Shimogaki theorem [3, Chap. 3, Theorem 5.17], the fact that the associate space $X'(\mathbb{R}^d)$ of $X(\mathbb{R}^d)$ is non-separable (see [3, Chap. 1, Corollaries 4.4 and 5.6]), our result [10, Theorem 4.3], and Theorem 6 applied to the space $X'(\mathbb{R}^d)$.

7.2 Sufficient condition for the weak convergence of $\{T_{h_n} f\}$ to the zero function for $f \in X_c(\mathbb{R}^d)$

We are now in a position to prove the main result of this section.

Theorem 7 *Let $\{h_n\}$ be a sequence in $\mathbb{R}^d$ such that $|h_n| \to +\infty$ as $n \to \infty$ and let $\{T_{h_n}\}$ be the corresponding sequence of shift operators on a rearrangement-invariant Banach function space $X(\mathbb{R}^d)$. If the space $X(\mathbb{R}^d)$ is non-separable and its upper Boyd index satisfies $\beta_X < 1$, then the sequence $\{T_{h_n} f\}$ converges weakly to the zero function as $n \to \infty$ in the space $X(\mathbb{R}^d)$ for every function $f \in X_c(\mathbb{R}^d)$.*

Proof By Lemma 4,

$$\lim_{\tau \to 0} \tau \|E_\tau\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}_+)\right)} = 0. \tag{7.5}$$

Let us prove by contradiction that if $g \in X(\mathbb{R}^d)$ is compactly supported, then the sequence $\{T_{h_n} g\}$ converges weakly to the zero function as $n \to \infty$ in the space $X(\mathbb{R}^d)$. Suppose that there exists a functional $G \in X^*(\mathbb{R}^d)$ such that the sequence $\{G(T_{h_n} f)\}$ does not converge to 0 as $n \to \infty$. Then there exist $\delta > 0$ and a subsequence $\left\{h_n^{(1)}\right\}$ of $\{h_n\}$ such that

$$\left|G\left(T_{h_n^{(1)}} g\right)\right| \geq \delta \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly, there exists a subsequence $\left\{h_n^{(2)}\right\}$ of $\left\{h_n^{(1)}\right\}$ such that the numbers $G\left(T_{h_n^{(2)}} g\right)$, $n \in \mathbb{N}$, belong to the same quadrant of $\mathbb{C}$. Then the closed convex hull $\mathfrak{C}$ of the set $\left\{G\left(T_{h_n^{(2)}} g\right) : n \in \mathbb{N}\right\}$ lies outside the triangle with the vertices at 0 and the two points where the circle $\{\zeta \in \mathbb{C} : \ |\zeta| = \delta\}$ meets the sides of the quadrant. Hence

$$\inf_{\zeta \in \mathfrak{C}} |\zeta| \geq \frac{\delta}{\sqrt{2}}. \tag{7.6}$$

Let B be a ball containing the support of g and let R be the radius of B. Since $|h_n^{(2)}| \to +\infty$ as $n \to \infty$, there is a subsequence $\left\{h_n^{(3)}\right\}$ of $\left\{h_n^{(2)}\right\}$ such that

$$\inf_{n \neq k} \left|h_n^{(3)} - h_k^{(3)}\right| > 2R.$$

Then the supports of the functions $T_{h_n^{(3)}} g$, $n \in \mathbb{N}$, are pairwise disjoint. Therefore, we have for all $N \in \mathbb{N}$ and $\lambda > 0$,

$$\begin{aligned} m_d & \left\{x \in \mathbb{R}^d \ : \ \left|\sum_{n=1}^N \left(T_{h_n^{(3)}} g\right)(x)\right| > \lambda\right\} \\ &= m_d \left(\bigcup_{n=1}^N \left\{x \in \operatorname{supp} T_{h_n^{(3)}} g \ : \ \left|\left(T_{h_n^{(3)}} g\right)(x)\right| > \lambda\right\}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} m_d \left\{ x \in \operatorname{supp} T_{h_n^{(3)}} g \ : \ \left| \left(T_{h_n^{(3)}} g \right)(x) \right| > \lambda \right\} \\
&= N m_d \left\{ x \in \mathbb{R}^d : \ |g(x)| > \lambda \right\}.
\end{aligned}
$$

So,

$$
\left(\sum_{n=1}^{N} T_{h_n^{(3)}} g \right)^* (t) = g^*(t/N), \quad t > 0,
$$

and

$$
\left\| \frac{1}{N} \sum_{n=1}^{N} T_{h_n^{(3)}} g \right\|_{X(\mathbb{R}^d)} = \frac{1}{N} \left\| E_{1/N} g^* \right\|_{\overline{X}(\mathbb{R}_+)} \le \frac{1}{N} \left\| E_{1/N} \right\|_{\mathcal{B}(\overline{X}(\mathbb{R}_+))} \| g^* \|_{\overline{X}(\mathbb{R}_+)}.
$$

It follows from this inequality and (7.5) that

$$
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T_{h_n^{(3)}} g \right\|_{X(\mathbb{R}^d)} = 0. \tag{7.7}
$$

On the other hand, since $\frac{1}{N} \sum_{n=1}^{N} G \left(T_{h_n^{(3)}} g \right) \in \mathfrak{C}$ for all $N \in \mathbb{N}$, inequality (7.6) implies that for all $N \in \mathbb{N}$,

$$
\begin{aligned}
\|G\|_{X^*(\mathbb{R}^d)} \left\| \frac{1}{N} \sum_{n=1}^{N} T_{h_n^{(3)}} g \right\|_{X(\mathbb{R}^d)} &\ge \left| G \left(\frac{1}{N} \sum_{n=1}^{N} T_{h_n^{(3)}} g \right) \right| \\
&= \left| \frac{1}{N} \sum_{n=1}^{N} G \left(T_{h_n^{(3)}} g \right) \right| \ge \frac{\delta}{\sqrt{2}},
\end{aligned}
$$

which contradicts (7.7). Thus, we have proved that the sequence $\{T_{h_n} g\}$ converges weakly to the zero function as $n \to \infty$ in the space $X(\mathbb{R}^d)$ for every compactly supported function $g \in X(\mathbb{R}^d)$.

Let $f \in X_c(\mathbb{R}^d)$ and $F \in X^*(\mathbb{R}^d)$. Take any $\varepsilon > 0$. Then there exists a compactly supported function $g \in X(\mathbb{R}^d)$ such that

$$
\|f - g\|_{X(\mathbb{R}^d)} < \frac{\varepsilon}{2(\|F\|_{X^*(\mathbb{R}^d)} + 1)}. \tag{7.8}
$$

By what has been already proved, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$
|F(T_{h_n} g)| < \frac{\varepsilon}{2}. \tag{7.9}
$$

It follows from (7.8) and (7.9) that for all $n > n_0$,

$$
\begin{aligned}
|F(T_{h_n} f)| &\le |F(T_{h_n} g)| + |F(T_{h_n}(f - g))| \\
&< \frac{\varepsilon}{2} + \|F\|_{X^*(\mathbb{R}^d)} \|T_{h_n}(f - g)\|_{X(\mathbb{R}^d)} \\
&= \frac{\varepsilon}{2} + \|F\|_{X^*(\mathbb{R}^d)} \|f - g\|_{X(\mathbb{R}^d)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, the sequence $\{T_{h_n} f\}$ converges weakly to the zero function as $n \to \infty$ in the space $X(\mathbb{R}^d)$ for every function $f \in X_c(\mathbb{R}^d)$. □

The above theorem is meaningful only if the space $X(\mathbb{R}^d)$ is non-separable because if the space $X(\mathbb{R}^d)$ is separable, then Theorem 7 follows from Theorem 1(b), Lemma 3 and the inequality $q_X \le \beta_X < 1$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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