PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 9, Pages 60–70 (March 22, 2022)
https://doi.org/10.1090/bproc/118

ON THE ESSENTIAL NORMS OF SINGULAR INTEGRAL OPERATORS WITH CONSTANT COEFFICIENTS AND OF THE BACKWARD SHIFT

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(Communicated by Javad Mashreghi)

ABSTRACT. Let X be a rearrangement-invariant Banach function space on the unit circle $\mathbb{T}$ and let $H[X]$ be the abstract Hardy space built upon X. We prove that if the Cauchy singular integral operator $(Hf)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau$ is bounded on the space X, then the norm, the essential norm, and the Hausdorff measure of non-compactness of the operator $aI + bH$ with $a, b \in \mathbb{C}$, acting on the space X, coincide. We also show that similar equalities hold for the backward shift operator $(Sf)(t) = (f(t) - \widehat{f}(0))/t$ on the abstract Hardy space $H[X]$. Our results extend those by Krupnik and Polonskiĭ [Funkcional. Anal. i Priložen. 9 (1975), pp. 73-74] for the operator $aI + bH$ and by the second author [J. Funct. Anal. 280 (2021), p. 11] for the operator S.

1. INTRODUCTION

For a Banach space E, let $\mathcal{B}(E)$ and $\mathcal{K}(E)$ denote the sets of bounded linear and compact linear operators on E, respectively. The norm of an operator $A \in \mathcal{B}(E)$ is denoted by $\|A\|_{\mathcal{B}(E)}$. The essential norm of $A \in \mathcal{B}(E)$ is defined by

$$\|A\|_{\mathcal{B}(E),\mathrm{e}} := \inf\{\|A - K\|_{\mathcal{B}(E)} : K \in \mathcal{K}(E)\}.$$

For a bounded subset Ω of the space E, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r. For $A \in \mathcal{B}(E)$, set

$$\|A\|_{\mathcal{B}(E),\chi} := \chi\left(A(B_E)\right),$$

where B_E denotes the closed unit ball in E. The quantity $\|A\|_{\mathcal{B}(E),\chi}$ is called the (Hausdorff) measure of non-compactness of the operator A. It follows from the definition of the essential norm and [24, inequality (3.29)] that for every $A \in \mathcal{B}(E)$ one has

$$\|A\|_{\mathcal{B}(E),\chi} \le \|A\|_{\mathcal{B}(E),\mathrm{e}} \le \|A\|_{\mathcal{B}(E)}. \tag{1.1}$$

We refer to the monographs [1–3] for the general theory of measures of non-compactness.

Received by the editors November 5, 2021.
2020 *Mathematics Subject Classification.* Primary 45E05, 46E30, 47B38.
Key words and phrases. Rearrangement-invariant Banach function space, abstract Hardy singular integral operator, backward shift operator, norm, essential norm, measure of non-compactness.
This work was supported by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P. (Portuguese Foundation for Science and Technology) within the scope of the project UIDB/00297/2020 (Centro de Matemática e Aplicações).

In this paper, we deal with the norm, the essential norm, and the measure of noncompactness of the following two operators: the singular integral operator $aI+bH$, where I is the identity operator, $a, b \in \mathbb{C}$ and H is the Cauchy singular integral operator (sometimes called the Hilbert transform), acting on rearrangement-invariant Banach function spaces X (see Section 2.2), and the backward shift operator S, acting on the abstract Hardy spaces $H[X]$ built upon rearrangement-invariant Banach function spaces X.

For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the standard Lebesgue space on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$ with respect to the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. For $f \in L^1$, the Cauchy singular integral Hf is defined by

$$(Hf)(t) := \frac{1}{\pi i}\, p.v. \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \mathbb{T},$$

where the integral is understood in the Cauchy principal value sense. The Riesz projection is defined by $P := \frac{1}{2}(I + H)$. It is well known that $H \in \mathcal{B}(L^p)$ if $1 < p < \infty$ (see, e.g., [13, Chap. 1, Lemma 2.1]). It follows from the results by Gohberg and Krupnik [12, 15], Pichorides [27], Hollenbeck and Verbitsky [17] that for all $1 < p < \infty$,

$$\|H\|_{\mathcal{B}(L^p)} = \|H\|_{\mathcal{B}(L^p),\mathrm{e}} = \cot\left(\frac{\pi}{2}\lambda_{L^p}\right) \tag{1.2}$$

and

$$\|P\|_{\mathcal{B}(L^p)} = \|P\|_{\mathcal{B}(L^p),\mathrm{e}} = \frac{1}{\sin(\pi\lambda_{L^p})}, \tag{1.3}$$

where

$$\lambda_{L^p} = \min\{1/p, 1 - 1/p\}. \tag{1.4}$$

Rearrangement-invariant Banach function spaces are far-reaching generalizations of Lebesgue spaces L^p, $1 \le p \le \infty$ (see [4, Chap. 2] and Section 2.2). Krupnik and Polonskiĭ proved [23, Theorem 1] that if X is a reflexive rearrangement-invariant Banach function space on the unit circle $\mathbb{T}$ (see Section 2.2) such that $H \in \mathcal{B}(X)$, then

$$\|H\|_{\mathcal{B}(X)} = \|H\|_{\mathcal{B}(X),\mathrm{e}}. \tag{1.5}$$

See also [21, Example 4.1], where the equality

$$\|aI + bH\|_{\mathcal{B}(L^p)} = \|aI + bH\|_{\mathcal{B}(L^p),\mathrm{e}} \tag{1.6}$$

is stated for $1 < p < \infty$ and $a, b \in \mathbb{C}$.

Further, the first author obtained in [18, Theorem 4.5 and Corollary 4.6] lower estimates for $\|H\|_{\mathcal{B}(X),\mathrm{e}}$ and $\|P\|_{\mathcal{B}(X),\mathrm{e}}$ in the case of a reflexive rearrangement-invariant Banach function space X such that $H \in \mathcal{B}(X)$. The lower bounds for the essential norms are defined exactly as the right-hand sides of (1.2) and (1.3) with λ_{L^p} given by (1.4) replaced by $\lambda_X = \min\{p_X, 1 - q_X\}$, where $0 < p_X \le q_X < 1$ are the so-called Zippin indices of the rearrangement-invariant Banach function space X. Note that for the Lebesgue spaces L^p the Zippin indices coincide and are equal to $1/p$, but there are Orlicz spaces L^Φ for which $0 < p_{L^\Phi} < q_{L^\Phi} < 1$ (see, e.g., [25]).

We refer to the monographs [14, Chap. 13] and [21, Chap. 4] and the survey paper [22] for a more detailed account of the history of the problem of calculation (or estimation) of the norms and the essential norms of the operators $aI + bH$ with $a, b \in \mathbb{C}$.

Our first main result is the following extension of [23, Theorem 1] (see (1.5)).

Theorem 1.1. *Let X be a rearrangement-invariant Banach function space on the unit circle $\mathbb{T}$. If $H \in \mathcal{B}(X)$ and $a, b \in \mathbb{C}$, then*

$$\|aI + bH\|_{\mathcal{B}(X),\mathcal{X}} = \|aI + bH\|_{\mathcal{B}(X),\mathrm{e}} = \|aI + bH\|_{\mathcal{B}(X)}.$$

The first equality appears in this work for the first time. We also would like to underline that, in contrast to [23], we do not require that the space X is reflexive. It is instructive to analyze the case of the Lorentz spaces $L^{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$, which are rearrangement-invariant Banach function spaces (see, e.g., [4, Chap. 4, Theorem 4.6]). It is well known that the operator H is bounded on all such spaces $L^{p,q}$. This follows from the boundedness of H on L^p for $1 < p < \infty$ (see, e.g., [13, Chap. 1, Lemma 1]), Boyd's interpolation theorem (see [7] and also [4, Chap. 3, Theorem 5.16]), and [4, Chap. 4, Theorem 4.6]. Note that if $1 < p < \infty$, then the spaces $L^{p,q}$ are separable and reflexive for $1 < q < \infty$; the spaces $L^{p,1}$ are separable and non-reflexive; and the spaces $L^{p,\infty}$ are non-separable and non-reflexive. So, all equalities in Theorem 1.1 are new for $L^{p,1}$ and $L^{p,\infty}$.

Now we pass to the backward shift operator. For $f \in L^1$, let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi}\, d\varphi, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. For $1 \le p \le \infty$, the classical Hardy spaces H^p are defined by

$$H^p := \left\{ f \in L^p \;:\; \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

Consider the functions

$$\mathbf{e}_n(z) := z^n, \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}.$$

The backward shift operator S is defined on H^p, $1 \le p \le \infty$, by

$$(Sf)(t) := \mathbf{e}_{-1}(t) \left(f(t) - \widehat{f}(0) \right), \quad t \in \mathbb{T}. \tag{1.7}$$

We refer to the monograph by Cima and Ross [9] for a systematic study of this operator on the classical Hardy spaces $H^p(\mathbb{D})$ over the unit disk $\mathbb{D}$ for $0 < p \le \infty$. The operator S is one of the simplest Toeplitz operators (see, e.g., [6]). It plays an important role in the study of more general Toeplitz operators with continuous symbols on the classical Hardy spaces H^p with $1 < p < \infty$ (see [5] and [28]). The second author proved in [28, Theorems 3.1 and 5.1] that for the Hardy spaces H^p, $1 < p < \infty$, one has

$$\|S\|_{\mathcal{B}(H^p),\mathcal{X}} = \|S\|_{\mathcal{B}(H^p),\mathrm{e}} = \|S\|_{\mathcal{B}(H^p)} \le 2^{|1-2/p|}. \tag{1.8}$$

Note that the exact value of the norm of the operator S is unknown even in the case of the Hardy space H^1 (see [11]).

Let X be a rearrangement-invariant Banach function space (see Section 2.2). It is continuously embedded into L^1. Following [29, p. 877], we consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \left\{ f \in X : \; \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

This is a Banach space with respect to the norm

$$\|f\|_{H[X]} := \|f\|_X.$$

It is clear that if $1 \le p \le \infty$, then $H[L^p]$ is the classical Hardy space H^p.

Our second main result is the following extension of the equalities in (1.8) from the setting of classical Hardy spaces $H^p = H[L^p]$ with $1 < p < \infty$ to the case of abstract Hardy spaces $H[X]$ built upon arbitrary rearrangement-invariant Banach function spaces X.

Theorem 1.2. *Let $H[X]$ be the abstract Hardy space built upon a rearrangement-invariant Banach function space X on the unit circle $\mathbb{T}$. Then the backward shift operator S is bounded on $H[X]$ and*

$$\|S\|_{\mathcal{B}(H[X]),\mathcal{X}} = \|S\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|S\|_{\mathcal{B}(H[X])}.$$

Once again, since we do not require that X is reflexive, our result covers the case of classical Hardy spaces $H^1 = H[L^1]$ and $H^\infty = H[L^\infty]$, which were not considered in [28, Theorem 5.1].

The paper is organized as follows. In Section 2, we collect definitions of a rearrangement-invariant Banach function space and its associate space, as well as auxiliary facts on measure preserving transformations of the unit circle onto itself generated by inner functions vanishing at the origin. In Section 3, we show that the operators $B_n f = \mathbf{e}_n(f \circ \mathbf{e}_{2n})$ commute with $A = aI + bH$ for $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Further, we prove Theorem 1.1 following the scheme of the proof of [28, Theorem 5.1]. In Section 4, we show that the backward shift operators S is bounded on the abstract Hardy space built upon an arbitrary Banach function space (not necessarily rearrangement-invariant). Finally, we prove Theorem 1.2.

2. Rearrangement-invariant Banach function spaces

2.1. **Banach function spaces and their associate spaces.** Let $\mathcal{M}$ be the set of all measurable complex-valued functions on $\mathbb{T}$ equipped with the normalized measure $dm(t) = |dt|/(2\pi)$ and let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$. The characteristic (indicator) function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E.

Following [4, Chap. 1, Definition 1.1], a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in \mathcal{M}^+$ with $n \in \mathbb{N}$, and for all constants $a \geq 0$, the following properties hold:

(A1) $\quad \rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,

(A2) $\quad 0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),

(A3) $\quad 0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\quad \rho(1) < \infty$,

(A5) $\quad \displaystyle\int_{\mathbb{T}} f(t)\, dm(t) \leq C\rho(f)$

with a constant $C \in (0, \infty)$ that may depend on ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X equipped with the natural linear space operations and this norm becomes a Banach space (see [4, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(t)g(t)\, dm(t) \ : \ f \in \mathcal{M}^+,\ \rho(f) \leq 1\right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [4, Chap. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X. The associate space X' can be viewed as a subspace of the Banach dual space X^*.

For $f \in X$ and $g \in X'$, put

$$\langle f, g\rangle := \int_{\mathbb{T}} f(t)\overline{g(t)}\, dm(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\, d\theta.$$

The following statement is a consequence of the Lorentz-Luxemburg theorem (see [4, Chap. 1, Theorem 2.7]). It can be proved in exactly the same way as [19, Lemma 2.10].

Lemma 2.1. *Let S_0 be the set of all simple functions on $\mathbb{T}$ and let X be a Banach function space on $\mathbb{T}$. If $f \in X$, then*

$$\|f\|_X = \sup\left\{|\langle f, s\rangle| \ : \ s \in S_0,\ \|s\|_{X'} \le 1\right\}.$$

2.2. **Rearrangement-invariant Banach function spaces and their associate spaces.** Let $\mathcal{M}_0$ (resp. $\mathcal{M}_0^+$) denote the set of all a.e. finite functions in $\mathcal{M}$ (resp. in $\mathcal{M}^+$). Following [4, Chap. 2, Definitions 1.1 and 1.2], the distribution function m_f of a function $f \in \mathcal{M}_0$ is given by

$$m_f(\lambda) := m\left\{t \in \mathbb{T} \ : \ |f(t)| > \lambda\right\}, \quad \lambda \ge 0.$$

Two functions $f, g \in \mathcal{M}_0$ are said to be equimeasurable if $m_f(\lambda) = m_g(\lambda)$ for all $\lambda \ge 0$. A Banach function norm $\rho : \mathcal{M} \to [0, \infty]$ is said to be rearrangement-invariant if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$. In that case, the Banach function space X generated by ρ is said to be a rearrangement-invariant Banach function space (see [4, Chap. 2, Definition 4.1]). It follows from [4, Chap. 2, Proposition 4.2] that if a Banach function space X is rearrangement-invariant, then its associate space X' is also a rearrangement-invariant Banach function space.

2.3. **Measure preserving transformations defined by inner functions.** Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^\infty(\mathbb{D})$ if

$$\|F\|_{H^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} |F(z)| < \infty.$$

Recall that an inner function is a function $u \in H^\infty(\mathbb{D})$ such that $|u(e^{i\theta})| = 1$ for a.e. $\theta \in [-\pi, \pi]$.

Lemma 2.2. *If u is an inner function such that $u(0) = 0$, then u is a measure preserving transformation from $\mathbb{T}$ onto itself.*

This lemma goes back to Nordgren (see corollary to [26, Lemma 1] and also [8, Remark 9.4.6], [19, Lemma 2.5], [10, Theorem 5.5]).

Lemma 2.3. *Let X be a rearrangement-invariant Banach function spaces on the unit circle $\mathbb{T}$ an let $H[X]$ be the abstract Hardy space built upon X.*

(a) *If $f \in X$ and $n \in \mathbb{N}$, then $f \circ \mathbf{e}_n \in X$ and $\|f \circ \mathbf{e}_n\|_X = \|f\|_X$.*
(b) *If $f \in H[X]$ and $n \in \mathbb{N}$, then $f \circ \mathbf{e}_n \in H[X]$ and $\|f \circ \mathbf{e}_n\|_{H[X]} = \|f\|_{H[X]}$.*

Proof.

(a) It is clear that $\mathbf{e}_n$ is an inner function and $\mathbf{e}_n(0) = 0$ for every $n \in \mathbb{N}$. It follows from Lemma 2.2 that $m_f = m_{f\circ \mathbf{e}_n}$ for every $f \in X$ and every $n \in \mathbb{N}$. Then $f \circ \mathbf{e}_n \in X$ and $\|f \circ \mathbf{e}_n\|_X = \|f\|_X$ because X is rearrangement-invariant.

(b) If $f \in H^1$, then it follows from [10, Theorem 5.5] that $f \circ \mathbf{e}_n \in H^1$ for every $n \in \mathbb{N}$. Combining this observation with part (a), we see that if $f \in H[X] = X \cap H^1$, then $f \circ \mathbf{e}_n \in X \cap H^1 = H[X]$ and $\|f \circ \mathbf{e}_n\|_{H[X]} = \|f\|_{H[X]}$.

□

3. Singular integral operators with constant coefficients

3.1. Operators B_n. For $f \in L^1$ and $n \in \mathbb{N}$, let

$$B_n f := \mathbf{e}_n(f \circ \mathbf{e}_{2n}). \tag{3.1}$$

Lemma 3.1. *If $a, b \in \mathbb{C}$ and $A = aI + bH$, then $AB_n\varphi = B_nA\varphi$ for all $\varphi \in L^1$ and $n \in \mathbb{N}$.*

Proof. Let $\mathcal{P}$ denote the set of all trigonometric polynomials. If $f \in \mathcal{P}$, then it is easy to see that for every $k, n \in \mathbb{N}$,

$$\widehat{f \circ \mathbf{e}_n}(k) = \begin{cases} \widehat{f}(m), & \text{if } k = nm,\ m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Since $\mathcal{P}$ is dense in L^1 (see, e.g., [20, Chap. 1, Theorem 2.12]), identity (3.2) remains true for every $f \in L^1$.

By Lemma 2.3(a), $B_n\varphi \in L^1$. For any $z \in \mathbb{D}$, we get using (3.2)

$$\begin{aligned}
F(z) &:= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(B_n\varphi)(\tau)}{\tau - z}\,d\tau = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\tau^n \varphi\left(\tau^{2n}\right)}{\tau - z}\,d\tau \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{in\theta}\varphi\left(e^{2in\theta}\right)}{e^{i\theta} - z}\,e^{i\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{in\theta}\varphi\left(e^{2in\theta}\right)}{1 - ze^{-i\theta}}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\varphi\left(e^{2in\theta}\right)\left(\sum_{k=0}^{\infty} z^k e^{-ik\theta}\right)d\theta \\
&= \sum_{k=0}^{\infty} z^k \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi\left(e^{2in\theta}\right)e^{-i(k-n)\theta}\,d\theta = \sum_{k=0}^{\infty} z^k \widehat{\varphi \circ \mathbf{e}_{2n}}(k-n) \\
&= \sum_{m=0}^{\infty} z^{2nm+n}\widehat{\varphi}(m) = z^n \sum_{m=0}^{\infty} \left(z^{2n}\right)^m \widehat{\varphi}(m) \\
&= \frac{z^n}{2\pi}\int_{-\pi}^{\pi} \varphi\left(e^{i\theta}\right)\left(\sum_{m=0}^{\infty} z^{2nm}e^{-im\theta}\right)d\theta \\
&= \frac{z^n}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi\left(e^{i\theta}\right)}{1 - z^{2n}e^{-i\theta}}\,d\theta = \frac{z^n}{2\pi i}\int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - z^{2n}}\,d\tau =: G(z).
\end{aligned}$$

Since $\varphi, B_n\varphi \in L^1$, it follows from Privalov's theorem (see, e.g., [16, Chap. X, §3, Theorem 1]) that the nontangential limit of $F(z)$ as $z \to e^{i\vartheta}$ coincides with $\big(P(B_n\varphi)\big)(e^{i\vartheta})$ for a.e. $\vartheta \in [-\pi, \pi]$, while the nontangential limit of $G(z)$ coincides with $\big(B_n(P\varphi)\big)(e^{i\vartheta})$. Hence $PB_n\varphi = B_nP\varphi$, which implies $AB_n\varphi = B_nA\varphi$ for all $\varphi \in L^1$, since $H = 2P - I$. □

3.2. **Proof of Theorem 1.1.** The proof is similar to that of [28, Theorem 5.1]. In view of inequality (1.1), it is sufficient to prove that

$$\|A\|_{\mathcal{B}(X),\mathcal{X}} \geq \|A\|_{\mathcal{B}(X)}. \tag{3.3}$$

For any $\varepsilon > 0$, there exists $q \in X$, such that $\|q\|_X = 1$ and

$$\|Aq\|_X \geq \|A\|_{\mathcal{B}(X)} - \varepsilon. \tag{3.4}$$

By Lemma 2.1, there exists a simple function $h \in X' \cap L^\infty$ such that $\|h\|_{X'} \leq 1$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(Aq)\left(e^{i\theta}\right)h\left(e^{i\theta}\right)\,d\theta \geq \|Aq\|_X - \varepsilon. \tag{3.5}$$

For $n \in \mathbb{N}$, set $h_n := \mathbf{e}_{-n}(h \circ \mathbf{e}_{2n})$. Since X' is rearrangement-invariant (see [4, Chap. 2, Proposition 4.2]), it follows from Lemma 2.3(a) that

$$\|h_n\|_{X'} = \|h\|_{X'} \leq 1, \quad n \in \mathbb{N}. \tag{3.6}$$

On the other hand, taking into account that $\mathbf{e}_{2n}$ is a measure preserving transformation of $\mathbb{T}$ onto itself (see Lemma 2.2), we see that for all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(B_nAq)\left(e^{i\theta}\right)h_n\left(e^{i\theta}\right)\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}((Aq)\circ\mathbf{e}_{2n})\left(e^{i\theta}\right)(h\circ\mathbf{e}_{2n})\left(e^{i\theta}\right)\,d\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(Aq)\left(e^{i\theta}\right)h\left(e^{i\theta}\right)\,d\theta.\end{aligned} \tag{3.7}$$

Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset X \subseteq L^1$. It is clear that $h_n = h_1 \circ \mathbf{e}_n$, and it follows from (3.2) that $\widehat{h_1}(0) = 0$. By Fejér's lemma (see [30, Chap. II, Theorem 4.15]), we have for every $j \in \{1, \dots, m\}$,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_j\left(e^{i\theta}\right)h_n\left(e^{i\theta}\right)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_j\left(e^{i\theta}\right)\,d\theta\,\frac{1}{2\pi}\int_{-\pi}^{\pi}h_1\left(e^{i\theta}\right)\,d\theta = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_j\left(e^{i\theta}\right)h_N\left(e^{i\theta}\right)\,d\theta\right| < \varepsilon, \quad j \in \{1, \dots, m\}. \tag{3.8}$$

It follows from (3.5), (3.7), (3.8) that for $j \in \{1, \dots, m\}$,

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(B_NAq - \varphi_j)\left(e^{i\theta}\right)h_N\left(e^{i\theta}\right)\,d\theta\right|\\ &\quad\geq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(B_NAq)\left(e^{i\theta}\right)h_N\left(e^{i\theta}\right)\,d\theta\right| - \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_j\left(e^{i\theta}\right)h_N\left(e^{i\theta}\right)\,d\theta\right|\\ &\quad> \|Aq\|_X - 2\varepsilon.\end{aligned} \tag{3.9}$$

On the other hand, Hölder's inequality for X (see [4, Chap. 1, Theorem 2.4]) and (3.6) imply that for $j \in \{1, \dots, m\}$,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(B_NAq - \varphi_j)\left(e^{i\theta}\right)h_N\left(e^{i\theta}\right)\,d\theta\right| &\leq \|B_NAq - \varphi_j\|_X\|h_N\|_{X'}\\ &\leq \|B_NAq - \varphi_j\|_X.\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we see that for $j \in \{1, \dots, m\}$,

$$\|B_NAq - \varphi_j\|_X > \|Aq\|_X - 2\varepsilon. \tag{3.11}$$

It follows from $X \subset L^1$, Lemma 3.1 and inequalities (3.4) and (3.11) that for all $j \in \{1, \dots, m\}$,

$$\|AB_N q - \varphi_j\|_X = \|B_N A q - \varphi_j\|_X > \|Aq\|_X - 2\varepsilon \geq \|A\|_{\mathcal{B}(X)} - 3\varepsilon.$$

By Lemma 2.3(a), $\|B_N q\|_X = \|q\|_X = 1$. So, for every finite set $\{\varphi_1, \dots, \varphi_m\} \subset X$, there exists an element of the image of the unit ball $A(B_X)$ that lies at a distance greater than $\|A\|_{\mathcal{B}(X)} - 3\varepsilon$ from every element of $\{\varphi_1, \dots, \varphi_m\}$. This means that $A(B_X)$ cannot be covered by a finite family of open balls of radius $\|A\|_{\mathcal{B}(X)} - 3\varepsilon$. Hence

$$\|A\|_{\mathcal{B}(X),\chi} \geq \|A\|_{\mathcal{B}(X)} - 3\varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Passing to the limit as $\varepsilon \to 0+$, we arrive at (3.3), which completes the proof. □

4. Backward shift operator

4.1. **Boundedness of the backward shift operator.** First, we observe that in order to guarantee the boundedness of the backward shift operator on the abstract Hardy space $H[X]$, we do not need to require that the underlying Banach function space X is rearrangement-invariant or reflexive.

Lemma 4.1. *Let $H[X]$ be the abstract Hardy space built upon an arbitrary Banach function space X on the unit circle $\mathbb{T}$. Then the backward shift operator S defined by* (1.7) *is bounded on the space $H[X]$.*

Proof. For every $f \in H[X]$, one has

$$\begin{aligned}\|Sf\|_{H[X]} &= \left\|\mathbf{e}_{-1}\left(f - \widehat{f}(0)\right)\right\|_{H[X]} = \left\|\mathbf{e}_{-1}\left(f - \widehat{f}(0)\right)\right\|_{X} = \left\|f - \widehat{f}(0)\right\|_{H[X]} \\ &\leq \|f\|_{H[X]} + \left|\widehat{f}(0)\right| \|1\|_{H[X]} \leq \|f\|_{H[X]} + \|f\|_{L^1}\|1\|_X.\end{aligned}$$

By Axiom (A4), $1 \in X$. On the other hand, in view of Axiom (A5), $\|f\|_{L^1} \leq C\|f\|_X$ for some constant $C > 0$ independent of $f \in X$. Hence

$$\|Sf\|_{H[X]} \leq \|f\|_{H[X]} + C\|f\|_X\|1\|_{H[X]} = (1 + C\|1\|_X)\,\|f\|_{H[X]}.$$

Therefore $S \in \mathcal{B}(H[X])$ and

$$\|S\|_{\mathcal{B}(H[X])} \leq 1 + C\|1\|_X, \tag{4.1}$$

which completes the proof. □

Note that estimate (4.1) is quite crude. If $X = L^p$ with $1 < p < \infty$, then $\|1\|_{L^p} = \|1\|_{L^{p'}} = 1$, where $1/p + 1/p' = 1$. By Hölder's inequality, $\|f\|_{L^1} \leq \|f\|_{L^p}$ and the constant $C = 1$ on the right-hand side of this inequality is best possible. Thus, it follows from (4.1) that $\|S\|_{\mathcal{B}(H^p)} \leq 2$ for all $1 < p < \infty$. On the other hand, the inequality in (1.8) gives a better estimate for the norm of the backward shift operator on the classical Hardy spaces: $\|S\|_{\mathcal{B}(H^p)} \leq 2^{|1-2/p|}$.

4.2. **Proof of Theorem 1.2.** The proof is similar to that of Theorem 1.1 (and of [28, Theorem 5.1]). In view of inequality (1.1), it is sufficient to prove that

$$\|S\|_{\mathcal{B}(H[X]),\chi} \geq \|S\|_{\mathcal{B}(H[X])}. \tag{4.2}$$

Fix $\varepsilon > 0$. Then there exists $q \in H[X]$, such that $\|q\|_{H[X]} = 1$ and

$$\|Sq\|_{H[X]} \geq \|S\|_{\mathcal{B}(H[X])} - \varepsilon. \tag{4.3}$$

Let $q_0 := q - \widehat{q}(0) \in H[X]$. Then

$$\|q_0\|_{H[X]} = \|\mathbf{e}_{-1} q_0\|_{H[X]} = \|Sq\|_{H[X]}. \tag{4.4}$$

By Lemma 2.1, there exists a simple function $h \in X' \cap L^\infty$ such that $\|h\|_{X'} \le 1$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} q_0\left(e^{i\theta}\right) h\left(e^{i\theta}\right)\, d\theta \ge \|q_0\|_X - \varepsilon. \tag{4.5}$$

For $n \in \mathbb{N}$, set $h_n := h \circ \mathbf{e}_n$. Since X' is rearrangement-invariant (see [4, Chap. 2, Proposition 4.2]), it follows from Lemma 2.3(a) that

$$\|h_n\|_{X'} = \|h\|_{X'} \le 1, \quad n \in \mathbb{N}. \tag{4.6}$$

On the other hand, taking into account that $\mathbf{e}_n$ is a measure preserving transformation of $\mathbb{T}$ onto itself (see Lemma 2.2), we see that for all $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_n)\left(e^{i\theta}\right) h_n\left(e^{i\theta}\right)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} q_0\left(e^{i\theta}\right) h\left(e^{i\theta}\right)\, d\theta. \tag{4.7}$$

Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset H[X] \subseteq H^1$. Let $\psi_j := \mathbf{e}_1 \varphi_j$ for $j \in \{1, \dots, m\}$. It is clear that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_j\left(e^{i\theta}\right) d\theta = \widehat{\psi_j}(0) = 0, \quad j \in \{1, \dots, m\}.$$

By Fejér's lemma (see [30, Chap. II, Theorem 4.15]), for every $j \in \{1, \dots, m\}$,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_j\left(e^{i\theta}\right) h_n\left(e^{i\theta}\right)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_j\left(e^{i\theta}\right) d\theta \, \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(e^{i\theta}\right) d\theta = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_j\left(e^{i\theta}\right) h_N\left(e^{i\theta}\right)\, d\theta\right| < \varepsilon, \quad j \in \{1, \dots, m\}. \tag{4.8}$$

It follows from (4.5), (4.7), (4.8) that for $j \in \{1, \dots, m\}$,

$$\begin{aligned} &\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j)\left(e^{i\theta}\right) h_N\left(e^{i\theta}\right)\, d\theta\right| \\ &\quad \ge \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N)\left(e^{i\theta}\right) h_N\left(e^{i\theta}\right)\, d\theta\right| - \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_j\left(e^{i\theta}\right) h_N\left(e^{i\theta}\right)\, d\theta\right| \\ &\quad > \|q_0\|_X - 2\varepsilon. \end{aligned} \tag{4.9}$$

On the other hand, Hölder's inequality for X (see [4, Chap. 1, Theorem 2.4]) and (4.6) imply that for $j \in \{1, \dots, m\}$,

$$\begin{aligned} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j)\left(e^{i\theta}\right) h_N\left(e^{i\theta}\right)\, d\theta\right| &\le \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j\|_X \|h_N\|_{X'} \\ &\le \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j\|_X. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we see that for $j \in \{1, \dots, m\}$,

$$\|q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j\|_X > \|q_0\|_X - 2\varepsilon = \|q_0\|_{H[X]} - 2\varepsilon. \tag{4.11}$$

Inequality (4.11), equality (4.4) and inequality (4.3) imply that for $j \in \{1, \dots, m\}$,

$$\begin{aligned} \|S(q \circ \mathbf{e}_N) - \varphi_j\|_{H[X]} &= \|\mathbf{e}_{-1}(q_0 \circ \mathbf{e}_N) - \mathbf{e}_{-1}\mathbf{e}_1\varphi_j\|_X = \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1\varphi_j\|_X \\ &> \|q_0\|_{H[X]} - 2\varepsilon \ge \|S\|_{\mathcal{B}(H[X])} - 3\varepsilon. \end{aligned}$$

So, for every finite set $\{\varphi_1, \ldots, \varphi_m\} \subset H[X]$, there exist a function $S(q \circ \mathbf{e}_N)$ that lies at a distance greater than $\|S\|_{\mathcal{B}(H[X])} - 3\varepsilon$ from every element of $\{\varphi_1, \ldots, \varphi_m\}$. It follows from Lemma 2.3(b) that $\|q \circ \mathbf{e}_N\|_{H[X]} = \|q\|_{H[X]} = 1$, that is, the function $S(q \circ \mathbf{e}_N)$ lies in the image $S\left(B_{H[X]}\right)$ of the unit ball of the space $H[X]$ by the operator S. This means that $S\left(B_{H[X]}\right)$ cannot be covered by a finite family of open balls of radius $\|S\|_{\mathcal{B}(H[X])} - 3\varepsilon$. Hence

$$\|S\|_{\mathcal{B}(H[X]),\mathcal{X}} \geq \|S\|_{\mathcal{B}(H[X])} - 3\varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Passing to the limit as $\varepsilon \to 0+$, we arrive at (4.2), which completes the proof. $\square$

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