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Representations and identities of plactic-like monoids

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ABSTRACT

We exhibit faithful representations of the hypoplactic, stalactic, taiga, sylvester, Baxter and right patience sorting monoids of each finite rank as monoids of upper triangular matrices over any semiring from a large class including the tropical semiring and fields of characteristic 0. By analysing the image of these representations, we show that the variety generated by a single hypoplactic (respectively, stalactic or taiga) monoid of rank at least 2 coincides with the variety generated by the natural numbers together with a fixed finite monoid $\mathcal{H}$ (respectively, $\mathcal{F}$) and forms a proper subvariety of the variety generated by the plactic monoid of rank 2.

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Tropical semiring
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1. Introduction

The plactic monoid (the monoid of Young tableaux) is famous for its connections to such diverse areas as symmetric functions [30], representation theory and algebraic combinatorics [13,27], Kostka–Foulkes polynomials [28,29], and musical theory [21]. Its finite-rank versions were shown to have faithful tropical representations by the second and third authors [22, Theorem 2.8]. An important consequence of these representations, which are specifically representations using upper triangular tropical matrices, is that each finite-rank plactic monoid satisfies a non-trivial semigroup identity [22, Theorem 3.1], which had been an actively-studied question [24]. The dimension of the representation and thus the lengths of the resulting identities are dependent on the rank of the monoid. The first and fourth authors, together with Kubat, Klein, and Okniński, showed that the rank-n plactic monoid does not satisfy any non-trivial identity of length less than or equal to n, which implies that there is no single non-trivial identity satisfied by all finite-rank plactic monoids, and, moreover, that the infinite-rank plactic monoid does not satisfy any non-trivial semigroup identity [5]. Kubat and Okniński [23], and Cédo and the same authors [6] also studied representations over a field of the plactic algebra of ranks 3 and 4 (that is, the monoid ring of the plactic monoid of ranks 3 or 4 over the same field).

Plactic monoids belong to a family of 'plactic-like' monoids which are connected with combinatorics and whose elements can be identified with combinatorial objects. Others in the family include the hypoplactic monoids, whose elements are quasi-ribbon tableaux and which play a role in the theory of quasi-symmetric functions analogous to that of the plactic monoid for symmetric functions [25,26,31]; the sylvester and #-sylvester monoids, whose elements are respectively right strict and left strict binary search trees [17]; the taiga monoids, whose elements are binary search trees with multiplicities [34]; the stalactic monoids, whose elements are stalactic tableaux [18,34]; the Baxter monoids, whose elements are pairs of twin binary search trees [14,15] and which are linked to the theory of Baxter permutations; and the left and right patience sorting monoids [10,35], whose elements are patience sorting tableaux.

For each species of plactic-like monoid, with the exception of the patience sorting monoids, there exist fixed identities satisfied by all finite- and infinite-rank monoids of that species [7]. For the left patience sorting monoid and its finite-rank versions, only the rank-1 monoid satisfies a non-trivial identity (since it is commutative); for higher ranks, it contains a free submonoid of rank 2 [10, Corollary 4.4]. For the right patience sorting monoid, the situation is similar to the plactic monoid: the rank-n monoid satisfies no identity of length less than of equal to n, and consequently the infinite-rank monoid satisfies no identity, but there are identities, dependent on rank, satisfied by the finite-rank right patience sorting monoids.

This state of knowledge naturally raises the question of whether these plactic-like monoids admit faithful tropical representations. Left patience sorting monoids of rank greater than 2 are immediately excluded by their free submonoids of rank 2: it is known that finitely generated semigroups of tropical matrices have polynomial growth [12] and therefore cannot contain free submonoids of rank 2 or more. In this paper, we show that, with this exception, faithful tropical representations exist for all the plactic-like monoids mentioned above, namely the hypoplactic, sylvester, #-sylvester, Baxter, stalactic, taiga, and right patience sorting monoids. In fact we show that each of these monoids can be faithfully represented by upper triangular matrices over any unital semiring with zero containing an element of infinite multiplicative order.

The paper is structured as follows. In Section 2 we outline some preliminary material on words; the main combinatorial objects, insertion algorithms and monoids studied in this paper (the 'plactic-like' monoids); representations; and identities. Each of Sections 3–7 studies a family of plactic-like monoids. Each monoid in a given family is associated to a class of combinatorial object, and arises from an algorithm that inserts a symbol into such an object. Starting from an empty object, it is therefore possible to compute a combinatorial object from a word, and the elements of the monoid are equivalence classes of words that correspond to the same object. We shall show that these monoids admit faithful representations by upper triangular matrices over certain semirings. Section 3 concerns representations of the hypoplactic monoid; Section 4, representations of the stalactic monoid; Section 5, representations of the taiga monoid; Section 6, representations of the sylvester and Baxter monoids; and Section 7, representations of the right patience sorting monoid. We use our representations to prove results about the variety of monoids generated by a single plactic-like monoid.

2. Preliminaries

2.1. Words

We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. For $n \in \mathbb{N}$ we write $[n]$ to denote the set $\{1, \dots n\}$. For $i, j \in \mathbb{N}$ we write $[[i, j]]$ to denote the set $\{ k \in \mathbb{N} : \min(i, j) \leq k \leq \max(i, j) \}$, or simply $[i, j]$ in the case where $i < j$, and refer to such subsets as *intervals*. For a non-empty subset $X \subseteq \mathbb{N}$ we write X^* to denote the free monoid generated by the set X, that is the set of all words on the (possibly ordered) alphabet X, where ε denotes the empty word. We write X^+ to denote the set of all non-empty words over X. For $w \in \mathbb{N}^*$ we write $|w|$ to denote the length of the word w, and for each $i \in \mathbb{N}$ we write $|w|_i$ to denote the number of occurrences of the letter i in w.

Each word $w \in \mathbb{N}^*$ determines a function $\mathbb{N} \to \mathbb{N}_0$ via $x \mapsto |w|_x$ called the *content* or *evaluation* of w, denoted $\mathrm{ev}(w)$ and a subset $\mathrm{supp}(w) = \{\, x \in \mathbb{N} : |w|_x \neq 0 \,\} \subset \mathbb{N}$, called the *support* of w.

2.2. Combinatorial objects and insertion algorithms

2.2.1. Quasi-ribbon tableaux

A *quasi-ribbon tableau* is a planar diagram consisting of a finite array of adjacent symbols from $\mathbb{N}$, with each symbol lying either to the right or below the previous symbol, and with the property that symbols lying in the same 'row' form a non-decreasing sequence when read from left to right and symbols lying in the same 'column' are strictly increasing when read from top to bottom. An example of a quasi-ribbon tableau is:

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & & & \\ \hline & & 3 & 4 & 4 & \\ \hline & & & & 5 & \\ \hline & & & & 6 & 6 \\ \hline \end{array} \, . \tag{2.1}$$

Notice that the same symbol cannot appear in two different rows of a quasi-ribbon tableau.

The insertion algorithm is as follows:

Algorithm 2.1 *([25, § 7.2]).*

Input: A quasi-ribbon tableau T and a symbol $a \in \mathbb{N}$.

Output: A quasi-ribbon tableau $T \leftarrow a$.

Method: If there is no entry in T that is less than or equal to a, output the quasi-ribbon tableau obtained by creating a new entry a and attaching (by its top-left-most entry) the quasi-ribbon tableau T to the bottom of a.

If there is no entry in T that is greater than a, output the quasi-ribbon tableau obtained by creating a new entry a and attaching (by its bottom-right-most entry) the quasi-ribbon tableau T to the left of a.

Otherwise, let x be the right-most and bottom-most entry of T that is less than or equal to a. Put a new entry a to the right of x and glue the remaining part of T (below and to the right of x) onto the bottom of the new entry a. Output the resulting tableau.

2.2.2. Stalactic tableaux

A *stalactic tableau* is a finite array of symbols from $\mathbb{N}$ in which columns are top-aligned, and two symbols appear in the same column if and only if they are equal. For example,

3	1	2	6	5
3	1		6	5
	1			5
	1			

(2.2)

is a stalactic tableau. The insertion algorithm is very straightforward:

Algorithm 2.2.

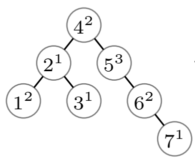
Input: A stalactic tableau T and a symbol $a \in \mathbb{N}$.

Output: A stalactic tableau $a \to T$.

Method: If a does not appear in T, add a to the left of the top row of T. If a does appear in T, add a to the bottom of the (by definition, unique) column in which a appears. Output the new tableau.

2.2.3. Binary search trees with multiplicities

A *binary search tree with multiplicities* is an ordered (children of each vertex being designated *left* and *right*), rooted (unless empty) binary tree in which: each vertex is labelled by a positive integer; distinct vertices have distinct labels; the label of each vertex is greater than the label of every vertex in its left subtree, and less than the label of every vertex in its right subtree; each vertex label is assigned a positive integer called its *multiplicity*. An example of a binary search tree with multiplicities is:



. (2.3)

The superscript on the label in a vertex denotes its multiplicity.

Algorithm 2.3.

Input: A binary search tree with multiplicities T and a symbol $a \in \mathbb{N}$.

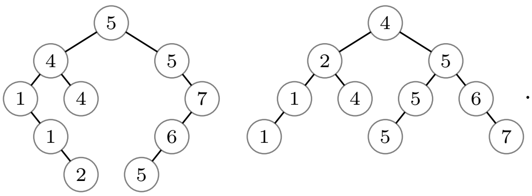
Output: A binary search tree with multiplicities $a \to T$.

Method: If T is empty, create a vertex, label it by a, and assign it multiplicity 1. If T is non-empty, examine the label x of the root vertex; if $a < x$, recursively insert a into the left subtree of the root; if $a > x$, recursively insert a into the right subtree of the root; if $a = x$, increment by 1 the multiplicity of the vertex label x.

2.2.4. Right strict and left strict binary search trees

A *right (respectively, left) strict binary search tree* is an ordered (children of each vertex being designated *left* and *right*), rooted (unless empty) binary tree in which each vertex is labelled by a positive integer and the label of each vertex is greater than or equal to the label of every vertex in its left subtree, and strictly less than the label of every vertex in its right subtree (respectively, the label of each vertex is strictly greater than or equal to the label of every vertex in its left subtree, and less than or equal to the label of every vertex in its right subtree).

The following are examples of, respectively, left strict and right strict binary search trees:



(2.4)

The insertion algorithm for right (respectively, left) strict binary search trees adds the new symbol as a leaf vertex in the unique place that maintains the property of being a right (respectively, left) strict binary search tree.

Algorithm 2.4 *(Right strict leaf insertion).*

Input: A right strict binary search tree T and a symbol $a \in \mathbb{N}$.

Output: A right strict binary search tree $a \to T$.

Method: If T is empty, create a vertex and label it a. If T is non-empty, examine the label x of the root; if $a \leq x$, recursively insert a into the left subtree of the root; otherwise recursively insert a into the right subtree of the root. Output the resulting tree.

Algorithm 2.5 *(Left strict leaf insertion).*

Input: A left strict binary search tree T and a symbol $a \in \mathbb{N}$.

Output: A left strict binary search tree $T \leftarrow a$.

Method: If T is empty, create a vertex and label it a. If T is non-empty, examine the label x of the root; if $a \geq x$, recursively insert a into the right subtree of the root; otherwise recursively insert a into the left subtree of the root. Output the resulting tree.

2.2.5. Right patience-sorting tableaux

An *rPS-tableau* is a finite array of symbols of $\mathbb{N}$ in which columns are bottom-aligned, the entries in the bottom row are strictly increasing from left to right, and the entries in each column are non-increasing from top to bottom. For example,

3			7	
2		6	6	
2	4	6	5	
1	3	4	5	7

is an rPS-tableau. The insertion algorithm is as follows:

Algorithm 2.6.

Input: An rPS-tableau T and a symbol $a \in \mathbb{N}$.

Table 1
Insertion algorithms used to compute combinatorial objects, their associated directions, monoids, and symbols.

Algorithm (for combinatorial object)	Direction	Monoid	Symbol
2.1 (Quasi-ribbon tableau)	$\rightarrow$	Hypoplactic	hypo
2.2 (Stalactic tableau)	$\leftarrow$	Stalactic	stal
2.3 (Binary search tree with multiplicities)	$\leftarrow$	Taiga	taig
2.4 (Right strict binary search tree)	$\leftarrow$	Sylvester	sylv
2.5 (Left strict binary search tree)	$\rightarrow$	#-sylvester	$\mathsf{sylv}^{\#}$
2.6 (rPS tableau)	$\rightarrow$	Right patience sorting	rPS

Output: An rPS-tableau $T \leftarrow a$.

Method: If a is greater than every symbol that appears in the bottom row of T, add a to the right of the bottom row of T. Otherwise, let C be the leftmost column whose bottom-most symbol is greater than or equal to a. Slide column C up by one space and add a as a new entry of C. Output the new tableau.

2.3. Monoids from insertion

For $u \in \mathbb{N}^*$ and each insertion algorithm described above, one can compute from u a combinatorial object (of the type associated to the algorithm) by starting with the empty combinatorial object (of the same type) and inserting the symbols of u one-by-one using the appropriate insertion algorithm and proceeding through the word u either left-to-right or right-to-left. In Table 1 we associate to each algorithm a symbol and a direction (either $\rightarrow$ or $\leftarrow$). For each of the symbols $\mathsf{M} \in \{\mathsf{hypo}, \mathsf{stal}, \mathsf{taig}, \mathsf{sylv}, \mathsf{sylv}^{\#}, \mathsf{rPS}\}$ we may then write $\mathrm{P}_{\mathsf{M}}(u)$ to denote the combinatorial object obtained from u by applying the algorithm associated to M in the direction specified (i.e. where $\rightarrow$ denotes that words are read from left-to-right and $\leftarrow$ denotes that words are read from right-to-left.) We also define $\mathrm{P}_{\mathsf{baxt}}(u) = \big(\mathrm{P}_{\mathsf{sylv}^{\#}}(u), \mathrm{P}_{\mathsf{sylv}}(u)\big)$.

Now, for each $\mathsf{M} \in \{\mathsf{hypo}, \mathsf{stal}, \mathsf{taig}, \mathsf{sylv}, \mathsf{sylv}^{\#}, \mathsf{baxt}, \mathsf{rPS}\}$, define the relation $\equiv_{\mathsf{M}}$ on $\mathbb{N}^*$ by

$$u \equiv_{\mathsf{M}} v \iff \mathrm{P}_{\mathsf{M}}(u) = \mathrm{P}_{\mathsf{M}}(v).$$

In each case, it turns out that the relation $\equiv_{\mathsf{M}}$ is a congruence on $\mathbb{N}^*$, and so the factor monoid $\mathsf{M} = \mathbb{N}^*/{\equiv_{\mathsf{M}}}$ can be formed, and is named as in Table 1. The rank-n analogue is the factor monoid $\mathsf{M}_n = [n]^*/{\equiv_{\mathsf{M}}}$, where the relation $\equiv_{\mathsf{M}}$ is naturally restricted to $[n]^* \times [n]^*$.

It follows from the definition of $\equiv_{\mathsf{M}}$ that each element $[u]_{\equiv_{\mathsf{M}}}$ of the factor monoid M can be identified with the combinatorial object $\mathrm{P}_{\mathrm{M}}(u)$. We refer to these monoids as *plactic-like* monoids. Each of the plactic-like monoids considered in this paper has a canonical generating set with the property that all words representing a given element have the same content (and thus the same support). Hence it makes sense to define, for an element m of the monoid, $\mathrm{ev}(m)$ and $\mathrm{supp}(m)$ to be the content and support of the words representing m with respect to the canonical generators.

To help the reader to keep the established left-to-right or right-to-left reading conventions in mind, from now on for each symbol $\mathsf{M} \in \{\mathsf{hypo}, \mathsf{sylv}^{\#}, \mathsf{rPS}\}$ we shall write $\mathrm{P}^{\rightarrow}_{\mathsf{M}}(u)$ in place of $\mathrm{P}_{\mathrm{M}}(u)$, and likewise, for each symbol $\mathsf{M} \in \{\mathsf{stal}, \mathsf{taig}, \mathsf{sylv}\}$ we write $\mathrm{P}^{\leftarrow}_{\mathsf{M}}(u)$.

2.4. Matrix representations over semirings

Throughout this paper S will be a commutative unital semiring (with zero denoted by 0_S and unit denoted by 1_S) containing an element of infinite multiplicative order. Of particular interest is the *tropical semiring* $\mathbb{T}$, which is the set $\mathbb{R} \cup \{-\infty\}$ under the operations $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for all $a, b \in \mathbb{T}$, where we define $\max(a, -\infty) = a = \max(-\infty, a)$ and $-\infty + a = a + -\infty = -\infty$ for all $a \in \mathbb{T}$. Notice that $0_{\mathbb{T}} = -\infty$, $1_{\mathbb{T}} = 0$ and all other elements have infinite multiplicative order.

We write $\mathrm{M}_n(S)$ to denote the monoid of all $n \times n$ matrices with entries from S under the matrix multiplication induced from operations of S in the obvious way. The $n \times n$ *identity matrix* (with all diagonal entries equal to 1_S and all other entries equal to 0_S) and *zero matrix* (with all entries equal to 0_S) are respectively the identity element and zero element in $\mathrm{M}_n(S)$. We say that $A \in \mathrm{M}_n(S)$ is *upper triangular* if $A_{i,j} = 0_S$ for all $i > j$, and write $\mathrm{UT}_n(S)$ for the submonoid of $n \times n$ upper triangular matrices over S. If T is a finite set we write $\mathrm{M}_T(S)$ for the semigroup of matrices with rows and columns indexed by elements of T; this is of course isomorphic to $\mathrm{M}_{|T|}(S)$ but it is often convenient to index entries by elements of a particular finite set.

From now on fix an element of infinite multiplicative order $\alpha \in S$ and let $c_n : [n]^* \to \mathrm{UT}_n(S)$ be the homomorphism extending the map defined for $x \in [n]$ by

$$c_n(x)_{p,q} = \begin{cases} \alpha & \text{if } p = q = x, \\ 1_S & \text{if } p = q \neq x, \\ 0_S & \text{otherwise.} \end{cases}$$

The image of this morphism is the (commutative) semigroup of diagonal matrices, with entries from $\{\, \alpha^i : i \in \mathbb{N}_0 \,\}$ on the diagonal. Since α is an element of infinite multiplicative order, this image is isomorphic to n copies of the monoid $(\mathbb{N}_0, +)$. Two words $w, v \in [n]^*$ have the same image under c_n if and only if they have the same content. (Of course, if S has the stronger property that its multiplicative monoid contains a free commutative monoid of each finite rank n (as is the case for the tropical semiring, for example), then we can instead construct a 1-dimensional representation of $[n]^*$ which records the content of a word.) For each $\mathsf{M} \in \{\mathsf{hypo}, \mathsf{stal}, \mathsf{taig}, \mathsf{sylv}, \mathsf{sylv}^{\#}, \mathsf{baxt}, \mathsf{rPS}\}$, words in the same $\equiv_{\mathsf{M}}$ class have the same evaluation, and therefore $\equiv_{\mathsf{M}}$-equivalent elements of $[n]^*$ have the same image under the map c_n. By a slight abuse of notation, we shall therefore consider c_n to be a homomorphism from the rank n plactic-like monoid $\mathsf{M}_n = [n]^*/\equiv_{\mathsf{M}}$ to $\mathrm{UT}_n(S)$, with image isomorphic to n copies of the natural numbers.

2.5. Identities

A *semigroup identity* is a formal equality $u = v$ where u and v are non-empty words over some alphabet of variables X. An identity is *non-trivial* if u and v are not equal as words. Such a semigroup identity $u = v$ is *satisfied* by a semigroup S if, for every homomorphism $\phi : X^+ \to S$, the equality $\phi(u) = \phi(v)$ holds in S. The identity is *balanced* (or *multihomogeneous*) if $\mathrm{ev}(u) = \mathrm{ev}(v)$.

It is easy to see that any identity satisfied by a semigroup containing a free submonoid of rank 1 must be balanced: to see that u and v must contain the same number of the variable x, consider the homomorphism sending x to the generator of the free submonoid and all other variables to the identity element of the monoid. All of the plactic-like monoids considered in this paper contain free submonoids of rank 1.

On the other hand, the monoid variety defined by all balanced identities is **Comm**, the class of commutative monoids.

3. Hypoplactic monoids

For $u \in [n]^*$ and $1 \leq i < j \leq n$, let $H_{i,j}(u)$ denote the statement 'u contains i and j, no symbol k with $i < k < j$, and no scattered subword ji'. The following characterisation of the hypoplactic monoid is a consequence of [31, Theorem 4.18 and Note 4.10].

Proposition 3.1. *Let n be a fixed positive integer and $u, v \in [n]^*$. The quasi-ribbon tableaux $\mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(u)$ and $\mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(v)$ are equal if and only if:*

(1) *u and v have the same content; and*
(2) *for $1 \leq i < j \leq n$, $H_{i,j}(u) \iff H_{i,j}(v)$.*

Let

$$I = \begin{bmatrix} 1_S & 1_S \\ 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 0_S & 0_S \\ 0_S & 1_S \end{bmatrix}.$$

Let $K = JI$, $L = IJ$, and let E be the 2×2 identity matrix. Note that K is the 2×2 zero matrix. It is easy to see that $\mathcal{H} = \{E, I, J, K, L\}$ is a submonoid of $\mathrm{UT}_2(S)$, with presentation $\langle I, J \mid I^2 = I, J^2 = J, IJI = JI = JIJ \rangle$. It can also be verified that $\mathcal{H}$ is isomorphic to the monoid $\mathcal{C}_3$ of order-preserving and extensive transformations of the 3-element chain, as studied in [39].

For all $i, j \in \mathbb{N}$ with $i < j$, consider the monoid homomorphism $h_{i,j} : \mathbb{N}^* \to \mathrm{UT}_2(S)$ defined by $i \mapsto I$, $j \mapsto J$, each k with $i < k < j$ maps to K, and all other letters map to E. Note that the image of $h_{i,j}$ is $\mathcal{H}$. Straightforward calculation shows that for $w \in \mathbb{N}^*$,

$$h_{i,j}(w) = \begin{cases} E & \text{if } w \text{ contains no symbols in the interval } [i,j]; \\ I & \text{if } w \text{ contains } i \text{ and no other symbols from } [i,j]; \\ J & \text{if } w \text{ contains } j \text{ and no other symbols from } [i,j]; \\ L & \text{if } w \text{ contains } i \text{ and } j, \text{ no other symbols from } [i,j], \\ & \text{and no scattered subword } ji; \text{ and} \\ K & \text{otherwise.} \end{cases} \tag{3.1}$$

Lemma 3.2. *Let $n \geq 2$ be a fixed positive integer and let $1 \leq i < j \leq n$. The homomorphism $h_{i,j} : \mathbb{N}^* \to \mathrm{UT}_2(S)$ factors to give a homomorphism from the hypoplactic monoid of rank n to the monoid $\mathcal{H}$.*

Proof. Suppose that $u, v \in [n]^*$ are in the same hypoplactic class. Since u and v have the same content, it is immediate that either both or neither u and v have image X where $X \in \{E, I, J\}$. Moreover, by Proposition 3.1, u has image L if and only if v has image L. Since the only other possible image is K, the result now follows. $\square$

Theorem 3.3. *The hypoplactic monoid hypo_n (respectively, hypo) embeds into a direct sum of n copies (respectively, countably infinite copies) of $(\mathbb{N}_0, +)$ with $\binom{n}{2}$ copies (respectively, countably infinite copies) of the finite monoid $\mathcal{H}$.*

Proof. The direct product of the morphisms c_i for $i \in [n]$ (respectively, $i \in \mathbb{N}$) and $h_{i,j}$ for $i, j \in [n]$ (respectively, $i, j \in \mathbb{N}$) with $i < j$ give a morphism to the required monoid, and it is clear that for each fixed word w, $c_i(w) = 1_S$ for all but finitely many i, and $h_{i,j}(w) = E$ for all but finitely many pairs i, j. The fact that this is an embedding now follows from Proposition 3.1 and from observing from (3.1) that the image of an element w under $h_{i,j}$ is equal to L if and only if w contains i and j, no other symbols from $[i, j]$, and no scattered subword ji. $\square$

Theorem 3.4. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The hypoplactic monoid of rank n admits a faithful representation by upper triangular matrices of size n^2 over S having block-diagonal structure with largest block of size 2 (or size 1 if $n = 1$).*

Proof. Since $(\mathbb{N}_0, +)$ embeds in $\mathrm{UT}_1(S)$ and $\mathcal{H}$ by definition embeds in $\mathrm{UT}_2(S)$, this is immediate from Theorem 3.3. $\square$

3.1. Variety generated by the hypoplactic monoid

The identities satisfied by the hypoplactic monoid have been completely characterised by the first and fourth authors and Ribeiro [8, Theorem 4.1], but we deduce here some more information about the corresponding variety.

Let J_k denote the set of identities $u = v$ with the property that u and v admit the same set of scattered subwords of length at most k. Let **Comm** denote the variety of commutative monoids (which is the variety with equational theory given by the balanced identities), $\mathbf{B}$ denote the variety of monoids generated by the bicyclic monoid, and $\mathbf{J}_k$ the variety of monoids determined by the set J_k.

The monoid varieties $\mathbf{J}_k$ have been studied extensively [3,4,33,37,39]. By a result of Volkov [39, Theorem 2], $\mathbf{J}_k$ is generated by any one of: the monoid of unitriangular Boolean matrices of rank $k+1$; the monoid of reflexive binary relations on a set of size $k+1$; and most importantly for our purposes, the monoid of order-preserving and extensive transformations of a chain with $k+1$ elements. In particular, the latter means that the five-element monoid $\mathcal{H}$ generates the variety $\mathbf{J}_2$. By a result of Tischenko [38] any five element monoid generates a finitely based variety, and so certainly $\mathbf{J}_2$ is finitely based. More generally, Blanchet-Sadri [3,4] has shown that the monoid variety $\mathbf{J}_k$ is finitely based if and only if $k \leq 3$, providing a basis of identities in those cases: a finite basis of identities for $\mathbf{J}_2$ is $xyxzx = xyzx$, and $xyxy = yxyx$.

Corollary 3.5. *Let $n \geq 2$ be a fixed positive integer. The variety of monoids generated by the hypoplactic monoid of rank n is:*

(1) *a proper subvariety of* $\mathbf{B}$*;*
(2) *the join of* $\mathbf{Comm}$ *and* $\mathbf{J}_2$*;*
(3) *equal to the variety generated by the (infinite-rank) hypoplactic monoid.*

Proof. (1) We have seen that the hypoplactic monoid of rank n embeds in the direct product of n copies of $(\mathbb{N}_0, +)$ (which embeds in $\mathrm{UT}_1(\mathbb{T})$) and $\binom{n}{2}$ copies of $\mathrm{UT}_2(\mathbb{T})$. Thus hypo_n is contained in the variety generated by $\mathrm{UT}_2(\mathbb{T})$. By [11], the latter is equal to the variety generated by the bicyclic monoid. That these varieties are distinct follows from the fact that the shortest identity satisfied by $\mathrm{UT}_2(\mathbb{T})$ has length 10, whilst hypo_n satisfies the identity $xyxy = yxyx$ [7, Proposition 12].

(2) We begin by showing that the identities satisfied by the hypoplactic monoid of rank n are precisely the balanced identities satisfied by the monoid $\mathcal{H}$. By Theorem 3.3, it is clear that the hypoplactic monoid of rank n embeds in a direct product of copies of $(\mathbb{N}_0, +)$ with copies of $\mathcal{H}$. Thus hypo_n satisfies every identity satisfied by both $(\mathbb{N}_0, +)$ and $\mathcal{H}$. The identities satisfied by $(\mathbb{N}_0, +)$ are precisely identities of the form $u = v$ where u and v have the same content, thus hypo_n satisfies all balanced identities satisfied by $\mathcal{H}$. On the other hand, $\mathcal{H}$ is an image of hypo_n under any of the homomorphisms $h_{i,j}$, and so $\mathcal{H}$ satisfies every identity satisfied by hypo_n. Finally, note that all identities satisfied by hypo_n are balanced.

It now follows from the fact that $\mathcal{H}$ generates the variety $\mathbf{J}_2$ [39, Theorem 2] that the variety generated by hypo_n is the join of $\mathbf{Comm}$ and $\mathbf{J}_2$.

(3) It is clear that each hypo_n embeds in hypo, while it is known [8, Proposition 3.6] that hypo can be embedded into a direct product of copies of hypo_2. Thus, the varieties generated by all of these monoids coincide. $\square$

4. Stalactic monoid

Let $u \in [n]^*$. Notice that the order in which the symbols appear along the first row in $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u)$ is the same as the order of the rightmost instances of the symbols that appear in u. Moreover, if T is a stalactic tableau consisting of a single row (that is, with all columns having height 1), then there is a unique word $u \in \mathbb{N}^*$, formed by reading the entries of T left-to-right, such that $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u) = T$.

For $u \in [n]^*$ and $i, j \in [n]$ with $i \neq j$, let $S_{i,j}(u)$ denote the statement 'u factors as $u = u'iu''$ where u'' contains j but not i'. Notice that if i and j are in the support of u, then exactly one of $S_{i,j}(u)$ and $S_{j,i}(u)$ is true. Otherwise both statements are false.

Proposition 4.1. *Let n be a fixed positive integer and $u, v \in [n]^*$. The stalactic tableaux $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(v)$ are equal if and only if:*

(1) *u and v have the same content; and*
(2) *for $1 \leq i < j \leq n$, $S_{i,j}(u) \iff S_{i,j}(v)$.*

Proof. It follows from the insertion algorithm that $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{stal}}(v)$ if and only if u and v have the same content and the order of the rightmost instances of the symbols that appear in u is equal to the order of the rightmost instances of the symbols that appear in v. For each pair of elements i, j in the support of u, the statement $S_{i,j}(u)$ (if $i < j$) or $S_{j,i}(u)$ (if $j < i$) can be used to determine whether j occurs to the right of the right-most i, and hence (1) and (2) together imply $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{stal}}(v)$.

Conversely, suppose that $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{stal}}(v)$. We immediately have (1) (and hence in particular, the support of u is equal to the support of v) and so it remains to show that (2) holds. For $i \in \mathrm{supp}(u)$ it follows from the insertion algorithm that if $u = u'iu''$ where $i \notin \mathrm{supp}(u'')$, then i occurs in the $(|\mathrm{supp}(u'')| + 1)$-th column from the right of the tableau (recalling here that the insertion algorithm reads the word from right to left) and the support of u'' is the set of entries in the first row preceding the symbol i. Thus $S_{i,j}(u)$ holds if and only if in the first row of $\mathrm{P}^{\leftarrow}_{\mathsf{stal}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{stal}}(v)$ symbol j precedes symbol i if and only if $S_{i,j}(v)$ holds. $\square$

Let

$$I = \begin{bmatrix} 1_S & 1_S \\ 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 1_S & 0_S \\ 0_S & 0_S \end{bmatrix},$$

and let E be the 2×2 identity matrix. Then $\mathcal{F} = \{E, I, J\}$ is isomorphic to the 'flip-flop monoid' (a two element right zero semigroup with an identity adjoined), presented by $\langle I, J \mid JI = I = I^2, IJ = J = J^2 \rangle$.

For all $i, j \in \mathbb{N}$ with $i < j$, consider the monoid homomorphism $s_{i,j} : \mathbb{N}^* \to \mathrm{UT}_2(S)$ defined by $i \mapsto I$, $j \mapsto J$, and all other letters map to E. Note that the image of $s_{i,j}$ is $\mathcal{F}$. It is easy to see that for $w \in \mathbb{N}^*$

$$s_{i,j}(w) = \begin{cases} E & \text{if } i, j \notin \operatorname{supp}(w); \\ I & \text{if } i \in \operatorname{supp}(w) \text{ and } w \text{ factors as } w = w'iw'' \text{ with } j \notin \operatorname{supp}(w''); \\ J & \text{if } j \in \operatorname{supp}(w) \text{ and } w \text{ factors as } w = w'jw'' \text{ with } i \notin \operatorname{supp}(w''). \end{cases} \tag{4.1}$$

Lemma 4.2. *Let $n \geq 2$ be a fixed positive integer and let $1 \leq i < j \leq n$. The homomorphism $s_{i,j} : \mathbb{N}^* \to \mathrm{UT}_2(S)$ factors to give a homomorphism from the stalactic monoid of rank n to the monoid $\mathcal{F}$.*

Proof. Let $u, v \in \mathbb{N}^*$ be in the same stalactic class. Since u and v have the same content, it is immediate that either both or neither have image E. Moreover, by Proposition 4.1, u has image I if and only if v has image I. Since J is the only other possible image, this completes the proof. □

Theorem 4.3. *The stalactic monoid stal_n (respectively, stal) embeds into a direct product of n copies (respectively, countably infinite copies) of $(\mathbb{N}_0, +)$ and $\binom{n}{2}$ copies (respectively, countably infinite copies) of the finite monoid $\mathcal{F}$.*

Proof. The direct product of the morphisms c_i for $i \in [n]$ (respectively, $i \in \mathbb{N}$) and $s_{i,j}$ for $i, j \in [n]$ (respectively, $i, j \in \mathbb{N}$) with $i < j$ give a morphism to the appropriate monoid, and it is clear that for each fixed word w, $c_i(w) = 1_S$ for all but finitely many i, and $s_{i,j}(w) = E$ for all but finitely many pairs i, j. The fact that this is an embedding follows from Proposition 4.1 and observing from (4.1) that in the image of an element w, for $i < j$ (respectively $j < i$) the block corresponding to $s_{i,j}$ is equal to I (respectively J) if and only if w contains i and factors as $w = w'iw''$ with $j \notin \operatorname{supp}(w'')$. □

Theorem 4.4. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The stalactic monoid of rank n admits a faithful representation by upper triangular matrices of size n^2 over S having block-diagonal structure with largest block of size 2 (or size 1 if $n = 1$).*

Proof. Since $(\mathbb{N}_0, +)$ embeds in $\mathrm{UT}_1(S)$ and $\mathcal{F}$ by definition embeds in $\mathrm{UT}_2(S)$, this is immediate from Theorem 4.3. □

4.1. Variety generated by the stalactic monoid

The finite basis problem for the stalactic monoid has been studied by Han and Zhang [19, Theorem 4.2], but we deduce here some more information about the corresponding variety.

For each word u over a finite alphabet, define $\sigma_u : \operatorname{supp}(u) \to [|\operatorname{supp}(u)|]$ to be the bijection taking each symbol x to the number of distinct symbols appearing in the shortest suffix of u containing x, that is, the position of x in an ordering of $\operatorname{supp}(u)$ according to first occurrence when reading u from right-to-left. For example, if $u = dfeebdbf$ is a word over alphabet $\{a, b, c, d, e, f\}$, then $\sigma_u = \left(\begin{smallmatrix} b & d & e & f \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$.

Remark 4.5. Let F denote the set of identities $u = v$ with the property that $\sigma_u = \sigma_v$ (and so in particular, the two words must have the same support). It is straightforward to verify that the identities $x^2 = x, xyx = yx$ form a basis for those in F and hence F defines the variety of right regular bands **RRB** (see for example [16]). Moreover, the variety **RRB** is generated by the flip-flop monoid $\mathcal{F}$ (see [36, Proposition 7.3.2]).

Recall that **Comm** denotes the variety of commutative monoids, and **B** the variety of monoids generated by the bicyclic monoid.

Corollary 4.6. *Let $n \geq 2$ be a fixed positive integer. The variety of monoids generated by the stalactic monoid of rank n is:*

(1) *a proper subvariety of* **B***;*
(2) *the join of* **Comm** *and* **RRB***;*
(3) *defined by the single identity $xyx = yxx$;*
(4) *equal to the variety generated by the (infinite-rank) stalactic monoid.*

Proof. (1) By Theorem 4.3, the stalactic monoid of rank n embeds in the direct product of copies of $(\mathbb{N}_0, +)$ (which embeds in $\mathrm{UT}_1(\mathbb{T})$) and copies of $\mathrm{UT}_2(\mathbb{T})$. Thus stal_n is contained in the variety generated by $\mathrm{UT}_2(\mathbb{T})$. By [11], the latter is equal to the variety generated by the bicyclic monoid. The fact that stal_n generates a proper variety of the bicyclic variety now follows from the fact that Adjan's identity is a minimal length identity for the bicyclic monoid $\mathcal{B}$ [1], whilst stal_n satisfies the identity $xyx = yxx$ [7, Proposition 15].

(2) We first show that the identities satisfied by the stalactic monoid of rank n are precisely the balanced identities satisfied by the flip-flop monoid $\mathcal{F}$. By Theorem 4.3, it is clear that the stalactic monoid of rank n embeds in the direct product of copies of $(\mathbb{N}_0, +)$ and copies of $\mathcal{F}$. Thus stal_n satisfies every identity satisfied by both $(\mathbb{N}_0, +)$ and $\mathcal{F}$. The identities satisfied by $(\mathbb{N}_0, +)$ are precisely identities of the form $u = v$ where u and v have the same content, thus stal_n satisfies all balanced identities satisfied by $\mathcal{F}$. On the other hand, $\mathcal{F}$ is an image of stal_n under any of the homomorphisms $s_{i,j}$, and so $\mathcal{F}$ satisfies every identity satisfied by stal_n. Finally, note that the identities satisfied by stal_n are balanced. It thus follows that the identities satisfied by the stalactic monoid of rank n are precisely the balanced identities satisfied by the flip-flop monoid $\mathcal{F}$. Hence the variety generated by stal_n is the join of **Comm** and the variety generated by $\mathcal{F}$. The result now follows from Remark 4.5.

(3) This has been established in [19, Theorem 4.2], but we give an alternative proof here. We show that each balanced identity in F can be deduced from $xyx = yxx$. Let $u = v$ be such an identity, and suppose that x is the rightmost letter of u. By repeatedly applying the identity $xyx = yxx$ where y is a factor of u lying between two symbols x, one can move all symbols x to the right. Inductively it follows that for any word u we can deduce from the identity $xyx = yxx$ an identity of the

form $u = (\sigma_u^{-1}(\ell))^{\alpha_\ell} \cdots (\sigma_u^{-1}(1))^{\alpha_1}$, where $\ell = |\mathrm{supp}(u)|$ and the exponents α_i are determined by the content of u. Since $\sigma_u = \sigma_v$ and u and v have the same content, we may therefore also deduce $u = v$. This shows that each stalactic monoid of rank at least 2 generates the same variety, namely the variety defined by $xyx = yxx$.

(4) Finally, it is clear that stal_n is contained in the variety generated by stal. Since stal satisfies the identity $xyx = yxx$, this completes the proof. □

5. Taiga monoid

For a word $u \in \mathbb{N}^*$ and a symbol $k \in \mathrm{supp}(u)$ there is a unique node of $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(u)$ containing the symbol k with multiplicity $|u|_k$. The number of vertices of $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(u)$ is therefore equal to $|\mathrm{supp}(u)|$. In a similar way to the stalactic monoid (where the structure of $\mathrm{P}_{\mathsf{stal}}^{\leftarrow}(u)$ is determined by the content of the word and the order of the rightmost instances of the symbols that appear in u), it is clear that the structure of the tree $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(u)$ is determined by the content of the word and the order of the rightmost instances of the symbols that appear in u. The difference here is that the construction of the tree also takes into account the ordering of these symbols in the underlying alphabet $\mathbb{N}$.

Recall that for each $u \in \mathbb{N}^*$ the bijection $\sigma_u : \mathrm{supp}(u) \to [|\mathrm{supp}(u)|]$ is defined to take each symbol to the number of distinct symbols appearing in the shortest suffix of u containing that letter.

Lemma 5.1. *Let $u \in \mathbb{N}^*$ and $i, j \in \mathrm{supp}(u)$ with $i \neq j$. The following are equivalent:*

(1) *i occurs in a subtree of j in $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(u)$;*
(2) *$\sigma_u(j) < \sigma_u(i)$ and there does not exist $k \in [[i, j]]$ with $\sigma_u(k) < \sigma_u(j)$;*
(3) *u factorises as $u = u'iu''ju'''$ where $u', u'', u''' \in \mathbb{N}^*$, $\mathrm{supp}(u''') \cap [[i, j]] = \emptyset$.*

Proof. By the insertion algorithm, i is in a subtree of j if and only if i is inserted after j and no symbol k with $i < k < j$ was inserted before j. The statement (1) is the left-hand side of this equivalence; (2) and (3) are different formulations of the right-hand side. □

For $u \in [n]^*$ and $1 \leq i \neq j \leq n$, let $T_{i,j}(u)$ denote the statement 'u factorises as $u = u'iu''ju'''$ where $u', u'', u''' \in \mathbb{N}^*$, $\mathrm{supp}(u''') \cap [[i, j]] = \emptyset$'. Notice that if i and j are in the support of u, then at most one of $T_{i,j}(u)$ and $T_{j,i}(u)$ is true. Otherwise both statements are false.

Proposition 5.2. *Let n be a fixed positive integer and $u, v \in [n]^*$. The binary search trees with multiplicities $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(u)$ and $\mathrm{P}_{\mathsf{taig}}^{\leftarrow}(v)$ are equal if and only if:*

(1) *u and v have the same content; and*
(2) *for $1 \leq i \neq j \leq n$, $T_{i,j}(u) \iff T_{i,j}(v)$.*

Table 2
The multiplication table of the monoid $\mathcal{T}$.

	E	K	J	I	L
E	E	K	J	I	L
K	K	K	J	I	L
J	J	K	J	I	L
I	I	I	L	I	L
L	L	I	L	I	L

Proof. If $\mathrm{P}^{\leftarrow}_{\mathsf{taig}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{taig}}(v)$, then in particular both trees contain the same number of each symbol (hence (1) holds) and both trees have the same parent-child structure (and hence by Lemma 5.1 (2) also holds).

Conversely, suppose first that conditions (1) and (2) hold. By Lemma 5.1 we have that i occurs in a subtree of j in $\mathrm{P}^{\leftarrow}_{\mathsf{taig}}(u)$ if and only if i occurs as a subtree of j in $\mathrm{P}^{\leftarrow}_{\mathsf{taig}}(v)$. (Note, if $i < j$, it will be the left subtree; if $i > j$, it will be the right subtree.) Thus the statements in (2) determine the same parent-child structure of the two trees (the ordering of the children of a given vertex is determined by the order on $\operatorname{supp}(u) \subseteq [n]$), while the content of the two words determines the multiplicities of the symbols in each tree. Thus (1) and (2) together imply that $\mathrm{P}^{\leftarrow}_{\mathsf{taig}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{taig}}(v)$. $\square$

Let

$$I = \begin{bmatrix} 1_S & 1_S & 0_S \\ 0_S & 0_S & 0_S \\ 0_S & 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 1_S & 0_S & 0_S \\ 0_S & 1_S & 1_S \\ 0_S & 0_S & 0_S \end{bmatrix},$$

$$K = \begin{bmatrix} 1_S & 0_S & 0_S \\ 0_S & 1_S & 0_S \\ 0_S & 0_S & 0_S \end{bmatrix}, \quad L = \begin{bmatrix} 1_S & 1_S & 1_S \\ 0_S & 0_S & 0_S \\ 0_S & 0_S & 0_S \end{bmatrix},$$

and let E be the 3×3 identity matrix. Straightforward calculation shows that $\mathcal{T} = \{E, K, J, I, L\}$ is a submonoid of $\mathrm{UT}_3(S)$ with multiplication table as given in Table 2.

For all $i, j \in \mathbb{N}$ with $i \neq j$, consider the homomorphism $t_{i,j} : \mathbb{N}^* \to \mathcal{T}$ defined by $i \mapsto I$, $j \mapsto J$, all other letters $k \in [[i,j]]$ map to K, and all remaining letters map to E.

Lemma 5.3. *Let $w \in \mathbb{N}^*$. Then*

(1) $t_{i,j}(w) = E$ *if and only if* $\operatorname{supp}(w) \cap [[i,j]] = \emptyset$;
(2) $t_{i,j}(w) = J$ *if and only if* $w = w'jw''$ *where* $i \notin \operatorname{supp}(w')$ *and* $\operatorname{supp}(w'') \cap [[i,j]] = \emptyset$;
(3) $t_{i,j}(w) = L$ *if and only if w can be factored as* $w = w'iw''jw'''$ *where* $\operatorname{supp}(w''') \cap [[i,j]] = \emptyset$;
(4) $t_{i,j}(w) = I$ *if and only if i is in the support of w and w cannot be factored as* $w = w'iw''jw'''$ *where* $\operatorname{supp}(w''') \cap [[i,j]] = \emptyset$; *and*
(5) $t_{i,j}(w) = K$ *otherwise.*

Proof. Let $W = t_{i,j}(w)$. It is easy to see that $W_{3,3} = 1_S$ if and only if $\mathrm{supp}(w) \cap [[i,j]] = \emptyset$, and E is the only element of $\mathcal{T}$ in which this entry is 1_S. It is also easy to see that $W_{2,3} = 1_S$ if and only if $w = w'jw''$ where $i \notin \mathrm{supp}(w')$ and $\mathrm{supp}(w'') \cap [[i,j]] = \emptyset$, and J is the only element of $\mathcal{T}$ in which this entry is 1_S. A similar computation shows that $W_{1,3} = 1_S$ if and only if $w = w'iw''jw'''$ where $i \notin \mathrm{supp}(w'')$ and $\mathrm{supp}(w''') \cap [[i,j]] = \emptyset$, and L is the only element of $\mathcal{T}$ in which this entry is 1_S. From the multiplication table above it is easy to see that if $i \in \mathrm{supp}(w)$ then either $W = L$ or $W = I$. Thus $W = I$ if and only if i is contained in the support of w, but w cannot be factored as $w = w'iw''jw'''$ where $i \notin \mathrm{supp}(w'')$ and $\mathrm{supp}(w''') \cap [[i,j]] = \emptyset$. □

Lemma 5.4. *Let $n \geq 2$ be a fixed positive integer and let $1 \leq i \neq j \leq n$. Each of the maps $t_{i,j} : \mathbb{N}^* \to \mathcal{T}$ factors to give a homomorphism from the taiga monoid of rank n to $\mathcal{T}$.*

Proof. Suppose that $u, v \in [n]^*$ are in the same taiga class of rank n. Since u and v have the same content, by Lemma 5.3 either both or neither have image E. Moreover, it follows from Proposition 5.2 and Lemma 5.3 that u has image L if and only if v has image L. Hence $i \in \mathrm{supp}(w)$ and $t_{i,j}(u) \neq L$ if and only if $i \in \mathrm{supp}(v)$ and $t_{i,j}(v) \neq L$, showing that u has image I if and only if v has image I.

Thus it suffices to show that $t_{i,j}(u) = J$ if and only if $t_{i,j}(v) = J$, or equivalently, that either both words admit a factorisation of the form in case (2) of Lemma 5.3, or neither do. To this end note that a word admitting a factorisation of this form is equivalent to j being the first symbol inserted from the interval $[[i,j]]$ and the symbol i is never inserted. Given two words u and v in the same taiga class, it is clear that symbol i is either in the support of both or neither. Suppose then that i is in the support of neither but, with the aim of obtaining a contradiction, that j is the first symbol to be inserted from the interval $[[i,j]]$ when reading u, whilst some symbol k with $i \neq k \neq j$ is the first symbol from $[[i,j]]$ to be inserted when reading v. Since u and v have the same support, it follows that we may write $u = u'ku''ju'''$ and $v = v'jv''kv'''$, where $k \notin \mathrm{supp}(u'')$, $j \notin \mathrm{supp}(v'')$, and $\mathrm{supp}(u''') \cap [[k,j]] = \emptyset = \mathrm{supp}(v''') \cap [[k,j]]$. But then Lemma 5.1 shows that in the tree associated to w symbol k occurs in a subtree of symbol j, whilst in the tree associated to v symbol j occurs in a subtree of symbol k. This gives the desired contradiction. □

Theorem 5.5. *The taiga monoid* taig_n *(respectively,* taig*) embeds into a direct product of n copies (respectively, countably infinite copies) of $(\mathbb{N}_0, +)$ and n^2-n copies (respectively, countably infinite copies) of the finite monoid $\mathcal{T}$.*

Proof. The direct product of the morphisms c_i for $i \in [n]$ (respectively, $i \in \mathbb{N}$) and $t_{i,j}$ for $i, j \in [n]$ (respectively, $i, j \in \mathbb{N}$) with $i \neq j$ gives a morphism to the appropriate monoid, and it is clear that for each fixed word w, $c_i(w) = 1_S$ for all but finitely many i, and $t_{i,j}(w) = E$ for all but finitely many pairs i, j. The fact that this is an embedding follows from Proposition 5.2 together with the observation from Lemma 5.3 that in the image of an element w, the block corresponding to the homomorphism $t_{i,j}$ equals L if and only if the statement $T_{i,j}(w)$ is true. □

Theorem 5.6. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The taiga monoid of rank n admits a faithful representation by upper triangular matrices of size $3n^2 - 2n$ over S having block-diagonal structure with largest block of size 3 (or size 1 if $n = 1$).*

Proof. Since $(\mathbb{N}_0, +)$ embeds in $\mathrm{UT}_1(\mathbb{T})$ and $\mathcal{T}$ by definition embeds in $\mathrm{UT}_3(\mathbb{T})$, this is immediate from Theorem 5.5. □

5.1. Variety generated by the taiga monoid

The finite basis problem for the taiga monoid has been studied by Han and Zhang; in the next corollary we show how our results can be used to give an alternative proof of [19, Theorem 4.2]. Recall that **Comm** denotes the variety of commutative monoids and that (from Subsection 4.1) **RRB** denotes the variety of right regular bands.

Remark 5.7. The variety **RRB** is generated by the monoid $\mathcal{T}$. Straightforward calculation shows that $\mathcal{T}$ satisfies the identities $x^2 = x$ and $xyx = yx$. On the other hand, as can be seen from Table 2, the monoid $\mathcal{T}$ contains the submonoid $\{E, I, L\}$, which is a two element right zero semigroup with an identity adjoined and so isomorphic to $\mathcal{F}$. Thus $\mathcal{F}$ lies in the variety generated by $\mathcal{T}$, and so the monoids $\mathcal{F}$ and $\mathcal{T}$ generate the same variety **RRB**.

Corollary 5.8. *Let $n \geq 2$ be a fixed positive integer. The monoids stal, stal_n, taig_n, and taig each generate the same variety.*

Proof. First, taig is a homomorphic image of stal [34, § 5] and thus lies in the variety generated by stal. Since taig_n is a submonoid of taig, it also lies in the variety generated by stal.

On the other hand, the variety generated by taig_n contains all commutative monoids and the monoid $\mathcal{T}$. By Remark 5.7, this variety also contains the flip-flop monoid $\mathcal{F}$, and so by Theorem 4.3 it contains stal.

Hence stal, taig, and taig_n all generate the same variety. Finally, by Corollary 4.6, stal_n generates the same variety. □

6. Sylvester and Baxter monoids

6.1. The sylvester monoid

The first and fourth authors and Ribeiro proved that the sylvester monoid of rank n embeds into $\binom{n}{2}$ copies of the sylvester monoid of rank 2, via a map we now define. For $1 \leq i < j \leq n$, define a homomorphism $\phi_{i,j} : [n]^* \to \mathsf{sylv}_2$ by

$$a \mapsto \begin{cases} [1]_{\mathsf{sylv}} & \text{if } a = i, \\ [21]_{\mathsf{sylv}} & \text{if } i < a < j, \\ [2]_{\mathsf{sylv}} & \text{if } a = j, \\ [\varepsilon]_{\mathsf{sylv}} & \text{otherwise.} \end{cases}$$

Each of these maps factors to give a homomorphism $\phi_{i,j} : \mathsf{sylv}_n \to \mathsf{sylv}_2$. Define a homomorphism $\Phi : \mathsf{sylv}_n \to \prod_{1 \leq i < j \leq n} \mathsf{sylv}_2$, where the (i,j)-th component of the image of $u \in [n]^*$ is $\phi_{i,j}(u)$. The map Φ is injective, as the following result shows:

Proposition 6.1 *([9, Lemma 3.6]). Let n be a fixed positive integer and $u, v \in [n]^*$. The binary search trees $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ are equal if and only if $\phi_{i,j}(u) = \phi_{i,j}(v)$ for $i, j \in [n]$ with $i \leq j$).*

Let

$$I = \begin{bmatrix} 1_S & 1_S \\ 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 1_S & 0_S \\ 0_S & \alpha \end{bmatrix},$$

where α is an element of infinite multiplicative order in the commutative semiring S. Then let $\mathcal{M}$ be the submonoid of $\mathrm{UT}_2(S)$ generated by $\{I, J\}$. Straightforward calculation shows that I is idempotent, $JI = I$, and that for any $k \in \mathbb{N}_0$,

$$J^k = \begin{bmatrix} 1_S & 0_S \\ 0_S & \alpha^k \end{bmatrix} \text{ and } IJ^k = \begin{bmatrix} 1_S & \alpha^k \\ 0_S & 0_S \end{bmatrix}.$$

In particular, it is easy to see that $\mathcal{M}$ is the set of all (pairwise distinct) elements J^k and IJ^k for $k \in \mathbb{N}_0$, and is presented by $\langle I, J \mid JI = I = I^2 \rangle$. From this one can easily observe that $\mathcal{M}$ is isomorphic to a quotient of the sylvester monoid of rank 2 (since imposing the relations $1^2 = 1 = 21$ on $\mathsf{sylv}_2 = \langle 1, 2 \mid 1211 = 2111, 1221 = 2121 \rangle$, yields the monoid $\langle 1, 2 \mid 21 = 1 = 1^2 \rangle$).

Consider the monoid homomorphism $s : [2]^* \to \mathrm{UT}_2(S)$ defined by $1 \mapsto I$ and $2 \mapsto J$. Note that the image of s is $\mathcal{M}$, specifically:

$$s(w) = \begin{cases} J^k & \text{if } w = 2^k; \\ IJ^k & \text{if } w = w'12^k. \end{cases} \tag{6.1}$$

Lemma 6.2. *The homomorphism* $s : [2]^* \to \mathrm{UT}_2(S)$ *factors to give a homomorphism from* sylv_2 *to the monoid* $\mathcal{M}$.

Proof. Let $u, v \in [2]^*$ be such that $u \equiv_{\mathsf{sylv}} v$. Write $u = u'2^k$ and $v = v'2^j$, where k and j are maximal; thus if either u' or v' is non-empty, then it ends with 1. Then $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ has exactly k consecutive nodes labelled 2 descending from the root, and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ has exactly j consecutive nodes labelled 2 descending from the root. Since $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$, it follows that $j = k$ and hence by (6.1) we have that $s(u) = s(v)$. □

Theorem 6.3. *The sylvester monoid* sylv_n *(respectively,* sylv*) embeds into a direct product of* n *copies (respectively, countably infinite copies) of* $(\mathbb{N}_0, +)$ *and* $\binom{n}{2}$ *copies of the (infinite) monoid* $\mathcal{M}$.

Proof. The direct product of morphisms c_i for $i \in [n]$ (respectively, $i \in \mathbb{N}$) and $s\phi_{i,j}$ for $i, j \in [n]$ (respectively, $i, j \in \mathbb{N}$) with $i < j$ gives a morphism to the appropriate monoid, and it is clear that for each fixed word w, $c_i(w) = 1_S$ for all but finitely many i, and $s\phi_{i,j}(w)$ is the identity matrix for all but finitely many pairs i, j. We show that this is an embedding. Let $u, v \in [n]^*$ and suppose that $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u) \neq \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. By Proposition 6.1 there exists $i < j$ such that $\phi_{i,j}(u) \neq \phi_{i,j}(v)$ in sylv_2. Since each element of sylv_2 can be expressed uniquely in the form $2^a1^b2^c$ where $a, b, c \in \mathbb{N}_0$, we may assume that $\phi_{i,j}(u) \equiv_{\mathsf{sylv}} 2^a1^b2^c \neq 2^p1^q2^r \equiv_{\mathsf{sylv}} \phi_{i,j}(v)$. If u and v do not have the same content, then the fact that $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u) \neq \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ will be detected in the image of c_n. Otherwise, since $\phi_{i,j}(w)$ is the element of sylv_2 obtained from w by replacing each occurrence of i by 1; each occurrence of j by 2; each occurrence of a symbol x, with $i < x < j$, by 21; and erasing each occurrence of any other element, we see that we must have $a + c = p + r$ and $b = q$ with $c \neq r$. In this case $s(\phi_{i,j}(u)) = \alpha^c \neq \alpha^r = s(\phi_{i,j}(u))$. □

Theorem 6.4. *Let* S *be a commutative unital semiring with zero containing an element of infinite multiplicative order. The sylvester monoid of rank* n *admits a faithful representation by upper triangular matrices of size* n^2 *over* S *having block-diagonal structure with largest block of size* 2 *(or size* 1 *if* $n = 1$*).*

Proof. Since $(\mathbb{N}_0, +)$ embeds in $\mathrm{UT}_1(S)$ and $\mathcal{M}$ embeds in $\mathrm{UT}_2(S)$, the result follows by Theorem 6.3. □

Remark 6.5. It is straightforward to prove analogues of Theorem 6.3 and Theorem 6.4 for the #-sylvester monoid, using a strategy similar to the above. Specifically, let

$$I_\# = \begin{bmatrix} \alpha & 0_S \\ 0_S & 1_S \end{bmatrix}, \quad J_\# = \begin{bmatrix} 0_S & 1_S \\ 0_S & 1_S \end{bmatrix}$$

where α is an element of infinite multiplicative order in the commutative semiring S. Let $\mathcal{M}_\#$ be the submonoid of $\mathrm{UT}_2(S)$ generated by $\{I_\#, J_\#\}$. Then one finds that

$$\mathcal{M}_\# = \left\langle I_\#, J_\# \,\middle|\, J_\# I_\# = J_\# = (J_\#)^2 \right\rangle = \{\, (I_\#)^k, (I_\#)^k J : k \in \mathbb{N}_0 \,\}$$

is isomorphic to the quotient of $\mathsf{sylv}^{\#}{}_2 = \langle 1, 2 \mid 2112 = 2121, 2212 = 2221 \rangle$ obtained by imposing the relations $2^2 = 2 = 21$. It is clear that $\mathcal{M}_\#$ is anti-isomorphic to $\mathcal{M}$. The monoid homomorphism $s_\# : [2]^* \to \mathrm{UT}_2(S)$ defined by $1 \mapsto I_\#$ and $2 \mapsto J_\#$ has image $\mathcal{M}_\#$ with

$$s_\#(w) = \begin{cases} I_\#^k & \text{if } w = 1^k, \\ I_\#^k J_\# & \text{if } w = 1^k 2 w'. \end{cases}$$

This homomorphism factors to give a homomorphism from $\mathsf{sylv}^{\#}{}_2$ to the monoid $\mathcal{M}_\#$, and then one can construct (i) an embedding of $\mathsf{sylv}^{\#}{}_n$ into a direct product of n copies of $(\mathbb{N}_0, +)$ with $\binom{n}{2}$ copies of $\mathcal{M}_\#$, and (ii) a faithful upper triangular representation of size n^2 for $\mathsf{sylv}^{\#}{}_n$, in much the same way as in the proofs of Theorem 6.3 and Theorem 6.4 (this time making use of the fact that $\mathsf{sylv}^{\#}{}_n$ embeds into a direct product of copies of $\mathsf{sylv}^{\#}{}_2$; see [9] for details).

Alternatively, one can see that an analogue of Theorem 6.4 holds by first observing that the permutation $\delta_n : i \mapsto n + 1 - i$ extends to give an anti-isomorphism $\Delta_n : \mathsf{sylv}^{\#}{}_n \to \mathsf{sylv}_n$, and so composing Δ_n first with the faithful representation given by Theorem 6.4 and then with the anti-isomorphism given by the transpose map will give a faithful representation of $\mathsf{sylv}^{\#}{}_n$ by *lower* triangular matrices. (From this one can easily obtain an upper triangular representation by conjugating the result by the permutation matrix in $\mathrm{M}_n(S)$ corresponding to the permutation δ_n.)

6.2. Variety generated by the sylvester monoid

The finite basis problem for the sylvester monoid has been studied by Han and Zhang [19, Theorem 4.6]; in this section we deduce here some more information about the corresponding variety.

Proposition 6.6. *Let $n \geq 2$. Then the variety of monoids generated by the sylvester monoid (respectively, #-sylvester monoid) of rank n is:*

(1) *a proper subvariety of* **B***;*
(2) *the variety generated by the (infinite) monoid $\mathcal{M}$ (respectively, $\mathcal{M}_\#$);*
(3) *not contained in the join of* **Comm** *and any variety generated by a finite monoid;*
(4) *equal to the variety generated by the (infinite-rank) sylvester monoid (respectively, infinite rank #-sylvester monoid).*

Proof. We prove the result for the case of the sylvester monoids. The arguments follow in a dual manner for the case of the #-sylvester monoids, using Remark 6.5 and the fact that $\mathsf{sylv}^{\#}{}_n$ and sylv_n are anti-isomorphic.

(1) By Theorem 6.4, sylv_n has a faithful upper-triangular tropical representation with maximum block size 2. Since $UT_2(\mathbb{T})$ satisfies the same identities as the bicyclic monoid [11], the variety generated by sylv_n is a subvariety of $\mathbf{B}$. The containment is proper since sylv_n also satisfies $xyxy = yxxy$ (see [7, Proposition 20]), while the shortest identity satisfied by the bicyclic monoid has length 10 (see [1]).
(2) Since $\mathcal{M}$ is a homomorphic image of sylv_2, which in turn is a homomorphic image of sylv_n, it is clear that the identities satisfied by sylv_n must form a subset of the identities satisfied by $\mathcal{M}$. By Theorem 6.3 the sylvester monoid of rank n embeds in the direct product of copies of $(\mathbb{N}_0, +)$ and copies of $\mathcal{M}$, and so satisfies every balanced identity satisfied by $\mathcal{M}$. Since all identities for sylv_n and $\mathcal{M}$ are balanced, the result follows.
(3) Suppose, with the aim of obtaining a contradiction, that the variety generated by sylv_n is contained in the join of **Comm** and a variety generated by a finite monoid. Then sylv_n is an image of a submonoid L of a direct product of a commutative monoid and copies of the finite monoid under some surjective homomorphism $\phi : L \to \mathsf{sylv}_n$. Let $a, b \in L$ be such that $\phi(a) = [1]_{\mathsf{sylv}}$ and $\phi(b) = [2]_{\mathsf{sylv}}$. Then there exist $p, q \in \mathbb{N}$ with $p \neq q$ where b^p and b^q differ only in the commutative monoid component. Thus ab^p and ab^q also differ only in the commutative monoid component and so commute with each other. Hence so do their images $[12^p]_{\mathsf{sylv}}$ and $[12^q]_{\mathsf{sylv}}$. This is a contradiction, since in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(12^p12^q)$ there are q nodes 2 above the topmost node 1, while in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(12^q12^p)$ there are p nodes 2 above the topmost node 1.
(4) This follows immediately from the facts that sylv embeds into a direct product of copies of sylv_2 and that the sylvester monoid of any finite rank is a submonoid of the sylvester monoid of any higher (or infinite) rank, as observed in [9, Theorem 3.9]. □

While the varieties generated by the hypoplactic, stalactic, and taiga monoids can be expressed as the join of **Comm** and a variety generated by a finite monoid, Proposition 6.6(3) shows that this is not true for a sylvester monoid. Heuristically, this is because there is a bound, dependent only on rank and not on content, on the number of quasi-ribbon tableaux, stalactic tableaux, and binary search trees with multiplicities that have a given content. On the other hand, there is no such bound, independent of content, for binary search trees.

Like the variety generated by the stalactic and taiga monoids, the variety generated by the sylvester monoids turns out to be defined by a single identity, although unlike in those cases the identity in question is not the shortest satisfied by the generating monoid, as the next result (which has been proved by different means in [9, Theorem 4.14] and [19, Theorem 4.6]) demonstrates.

Theorem 6.7. *The variety of the sylvester monoids is defined by the identity $xyzxty = yxzxty$.*

Proof. We shall refer to the identity $xyzxty = yxzxty$ as the *defining identity* in this proof. By Proposition 6.6 it suffices to argue using the monoid $\mathcal{M}$.

First, notice that the defining identity holds in $\mathcal{M}$. Indeed, given any consistent substitution of elements of $\mathcal{M}$ for x, y, z and t, it is clear that if $zxty$ maps to an element in the right-zero subsemigroup $\{ IJ^k : k \geq 0 \}$ then we have $xyzxty = yxzxty$, otherwise each letter must map to a power of J, in which case x and y commute.

Conversely, suppose for a contradiction that some identity holds in $\mathcal{M}$ which is not a consequence of our defining identity. Any non-trivial identity can clearly be written uniquely in the form $uaw = vbw$ where u, v, w are words and $a \neq b$ are distinct letters. Choose an identity (over any alphabet), from among those which hold in $\mathcal{M}$ but are not consequences of our defining identity, so that when it is written as $uaw = vbw$ as above the words u and v are as short as possible. We shall refer to this as the *counterexample identity*.

First, notice that the letters a and b must both appear in the word w, since if w does not contain a (respectively b) the substitution sending a to I, b to J and all other letters to the identity element (respectively, the substitution sending a to J, b to I and all other letters to the identity element) will clearly falsify our counterexample identity in $\mathcal{M}$.

Since the identity $uaw = vbw$ holds in $\mathcal{M}$, the prefixes ua and vb must have the same content, so in particular u contains at least one occurrence of the letter b. Write $u = pbq$ with q as short as possible (so in particular q does not contain any occurrence of the letter b). Notice that every letter appearing in q also appears in w, since if not, the substitution sending a single letter c of q to I, b to J and all other letters to the identity element clearly maps uaw to $IJ^{|w|_b}$, whilst vbw is mapped to IJ^k for some $k > |w|_b$, hence falsifying our counterexample identity in $\mathcal{M}$.

Now since a, b and every letter appearing in q appear in w, we may use the defining identity $|qa|$ times (with the substitutions mapping x to b, y to each letter of qa in turn, z to a suffix of qa followed by a prefix of w, and t to a subword of w) to commute b through qa and deduce the identity $pbqaw = pqabw$. So this latter identity is a consequence of the defining identity, and hence also holds in $\mathcal{M}$. Combining with the counterexample identity, we see that the identity $pqabw = vbw$ holds in $\mathcal{M}$. The words on either side of this identity are of the same length as in the counterexample identity and with a longer common suffix, so by the minimality assumption on the counterexample identity, the new identity $pqabw = vbw$ must be a consequence of the defining identity. But we know that $uaw = pbqaw$ as words, and we have shown that $pbqaw = pqabw$ is a consequence of the defining identity, so we deduce that the counterexample identity $uaw = vbw$ is a consequence of the defining identity, which is the required contradiction. $\square$

Remark 6.8. By Proposition 6.6, the variety of the sylvester (respectively, #-sylvester) monoids is generated by sylv_2 (respectively, $\mathsf{sylv}^{\#}{}_2$). Since $\mathsf{sylv}^{\#}{}_2$ is anti-isomorphic to sylv_2, it is clear that reversal of words gives a bijection between the identities satisfied by the sylvester monoid and the identities satisfied by the #-sylvester monoids.

In particular, the variety of the #-sylvester monoids is defined by the single identity $ytxzyx = ytxzxy$, as observed in [9, Theorem 4.15]; see also [19, Theorem 4.7].

Theorem 6.9. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The Baxter monoid of rank n admits a faithful representation by upper triangular matrices of size $2n^2 - n$ over S having block-diagonal structure with largest block of size 2.*

Proof. Both sylv_n and $\mathsf{sylv}^{\#}{}_n$ have faithful representations by upper triangular matrices of size n^2, each of which is made up of the image of c_n (an $n \times n$ diagonal matrix) and $\binom{n}{2}$ blocks of size 2.

Construct a representation of baxt_n by taking one copy of the image of c_n and the $2\binom{n}{2}$ blocks of size 2 from the representations of sylv_n and $\mathsf{sylv}^{\#}{}_n$. Since elements of baxt_n are equal if and only if their projections into sylv_n and $\mathsf{sylv}^{\#}{}_n$ are equal, this representation is faithful. $\square$

Proposition 6.10. *Let $n \geq 2$. Then the variety of monoids generated by the Baxter monoid* baxt_n *is:*

(1) *the join of the varieties generated by* sylv_n *and* $\mathsf{sylv}^{\#}{}_n$*;*
(2) *the join of the varieties generated by* $\mathcal{M}$ *and* $\mathcal{M}_{\#}$*;*
(3) *a proper subvariety of* $\mathbf{B}$*;*
(4) *not contained in the join of* **Comm** *and any variety generated by a finite monoid; and*
(5) *equal to the variety generated by the (infinite-rank) Baxter monoid.*

Proof. (1) It is easy to see from the definition that an identity is satisfied in baxt_n if and only if it is satisfied in both sylv_n and $\mathsf{sylv}^{\#}{}_n$.
(2) This follows immediately from part (2) of Proposition 6.6.
(3) That the variety generated by baxt_n is a subvariety of $\mathbf{B}$ follows from part (1) and Proposition 6.6(1). To see that it is a proper subvariety, observe that baxt satisfies the identity $xyxyxy = xyyxxy$[7, Proposition 26], while the shortest identity satisfied by semigroups in $\mathbf{B}$ has length 10 on each side (see [1]).
(4) This follows from part (1) and Proposition 6.6(3).
(5) This follows from the facts that baxt embeds into a direct product of copies of baxt_2 and that the Baxter monoid of any finite rank is a submonoid of the Baxter monoid of infinite rank, as shown in [9, Theorem 3.12]. $\square$

Next we provide a finite basis of identities for the variety of Baxter monoids; see also [9, Theorem 4.16] or [19, Theorem 4.10] for an alternative approach.

Theorem 6.11. *The variety generated by the Baxter monoids is defined by the two identities:*

$$xayb\ xy\ cxdy = xayb\ yx\ cxdy \quad \textit{and} \quad xayb\ xy\ cydx = xayb\ yx\ cydx.$$

Proof. We shall refer to the two identities in the statement as the *defining identities*.

The defining identities can clearly be deduced from the identity $xyzxty = yxzxty$ (which by Theorem 6.7 is satisfied in sylv) and separately also from the identity $ytxzyx = ytzxzy$ (which is the reverse of the previous identity, and hence is satisfied in $\mathsf{sylv}^{\#}$ by Remark 6.8). It follows immediately that they are satisfied in baxt.

Conversely, suppose for a contradiction that some identity is satisfied in baxt (and hence also by Proposition 6.10 in $\mathcal{M}$ and $\mathcal{M}_{\#}$), and is not a consequence of our defining identities. As in the proof of Theorem 6.7, choose such an identity (which we shall call the *counterexample identity*) so that when it is written as $uaw = vbw$ with $a \neq b$ single letters the words u and v are as short as possible, and write $u = pbq$ with q as short as possible. Note that ua and vb must have the same evaluation. Since this identity is satisfied in $\mathcal{M}$, we deduce by exactly the same argument as in the proof of Theorem 6.7 that a, b and every letter of q all occur in w.

We claim that b occurs at least once in p. Indeed, if not then since $a \neq b$ and b does not appear in q, we have that b appears only once in the word ua. Since ua has the same evaluation as vb, this means that v does not contain b and so the morphism mapping b to $J_{\#}$, a to $I_{\#}$ and all other letters to the identity element distinguishes vbw and uaw, hence contradicting the assumption that the counterexample identity holds in $\mathcal{M}_{\#}$.

Next we claim that a and every letter of q also occur in p. Indeed, suppose false, say some letter c appears in qa but not in p. Then the leftmost appearance of c in the word $uaw = pbqaw$ lies within the suffix qaw. Since the counterexample identity holds in $\mathcal{M}_{\#}$, the leftmost appearance of c in vbw must be in the same position (as can be seen by considering the morphism mapping c to $J_{\#}$ and all other letters to $I_{\#}$). Since $ua = pbqa$ and vb have the same content and q does not contain a b, this means there are strictly more bs to the left of the leftmost c in uaw than in vbw. But then the morphism mapping c to $J_{\#}$, b to $I_{\#}$ and all other elements to the identity distinguishes the words vbw and uaw, hence contradicting the assumption that the counterexample identity holds in $\mathcal{M}$.

To recap, we have a factorisation (as words) $uaw = pbqaw$ where a, b and every letter of q appears in both p and w. Hence we may use the defining identities $|qa|$ times to commute b through qa and deduce the identity $pbqaw = pqabw$. So this latter identity is a consequence of the defining identities, and hence also holds in baxt. Combining with the counterexample identity, we see that the identity $pqabw = vbw$ holds in baxt. The words on either side of this identity are of the same length as in the counterexample identity and with a longer common suffix, so by the minimality assumption on the counterexample identity, the new identity $pqabw = vbw$ must be a consequence of the defining identities. But we know that $uaw = pbqaw$ as words, and we have shown that $pbqaw = pqabw$ is a consequence of the defining identities, so we deduce that the counterexample identity $uaw = vbw$ is a consequence of the defining identities, which is the required contradiction. $\square$

7. The right patience-sorting monoid

Note that, unlike the sylvester monoid, it is impossible for the right patience sorting monoid of arbitrary rank to be embedded into a direct product of copies of right patience sorting monoid of rank 2. This is because rPS_n does not satisfy any identity of length less than n [10, Proposition 4.8], while it does satisfy the identity $(xy)^{n+1} = (xy)^n yx$ [10, Proposition 4.7]. Since there is no such embedding, and since rPS-tableaux are not characterised by content and a bounded amount of extra information, a very different approach is needed to construct a faithful finite dimensional representation.

The possible bottom rows of rPS-tableaux are clearly in bijective correspondence with subsets of $[n]$ (the correspondence taking each bottom row to the set of elements appearing in it). Let B be the power set of $[n]$, the elements of which we think of as possible bottom rows of rPS-tableaux, and which we also identify with single-row rPS-tableaux in the obvious way. (In particular this means we identify the empty set $\emptyset$ with the identity element in the right patience-sorting monoid.)

If P is an rPS-tableau and z is a generator then, as a direct consequence of the nature of the insertion algorithm, the bottom row of the rPS-tableau $P \leftarrow z$ is uniquely determined by the combination of the bottom row of P and the generator z. Therefore we may define an action of the free monoid $[n]^*$ on the set B, such that for any rPS-tableau P with bottom row $b \in B$ and any $w \in [n]^*$, the bottom row $b \cdot w$ is the bottom row of the rPS-tableau $P \leftarrow w$. Since two words which represent the same element of rPS_n act the same on every tableau, the action of the free monoid induces an action of the right patience-sorting monoid rPS_n, where $b \cdot s$ is defined to be $b \cdot w$ for some $w \in [n]^*$ with $\mathrm{P}_{\mathsf{rPS}}^{\rightarrow}(w) = s$.

Lemma 7.1. *Let $x, y \in [n]$ and $s \in \mathsf{rPS}_n$ and let P be an rPS-tableau. The number of times generator x occurs in column y of $P \leftarrow s$ is uniquely determined by the combination of the element s, the bottom row of P and the number of times x occurs in column y of P.*

Proof. Let $z \in [n]$ and consider the insertion of z into the rPS-tableau P. The column into which z gets inserted is uniquely determined by z and the bottom row of P. Thus, the number of times generator x occurs in column y of $P \leftarrow z$ is uniquely determined by the combination of the generator z (in particular whether it equals x), the bottom row of P and the number of times x occurs in column y of P. The bottom row of $P \leftarrow z$ is clearly also uniquely determined by the generator z and the bottom row of P. The result now follows by induction on the length of a word representing the element s. $\square$

We are now ready to define a faithful finite dimensional representation of the right patience-sorting monoid over any commutative unital semiring S containing a zero element and an element α of infinite multiplicative order. For $x, y \in [n]$, define a function $f_{x,y} : \mathsf{rPS}_n \to \mathrm{M}_B(S)$ as follows. For each $s \in \mathsf{rPS}_n$ and each $p, q \in B$,

- if $p \cdot s = q$ then $[f_{x,y}(s)]_{p,q}$ is equal to α^i where i denotes the number of extra xs added to column y of any rPS-tableau with bottom row p (for example, p itself viewed as a single-row rPS-tableau) and right-multiplying by s. This is well-defined by Lemma 7.1.
- if $p \cdot s \neq q$ then $[f_{x,y}(w)]_{p,q} = 0_S$.

Notice that, by definition, each row of $f_{x,y}(s)$ contains exactly one non-zero entry; namely the entry in column $p \cdot s$. In particular, the identity element of rPS_n is mapped to the usual identity matrix under this map.

Lemma 7.2. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The map $f_{x,y}$ defined above is a morphism from rPS_n to $\mathrm{M}_B(S)$.*

Proof. If $u, v \in \mathsf{rPS}_n$ then

$$[f_{x,y}(u)f_{x,y}(v)]_{p,q} = \sum_{r \in B} \Big(f_{x,y}(u)_{p,r} \cdot f_{x,y}(v)_{r,q} \Big) = f_{x,y}(u)_{p,p\cdot u} f_{x,y}(v)_{p\cdot u,q},$$

where the left-hand equality is the definition of matrix multiplication, and the right-hand equality is because row p of $f_{x,y}(u)$ contains a non-zero entry only in column $p \cdot u$. Now if $f_{x,y}(v)_{p\cdot u,q} = 0_S$ then this means $(p \cdot u) \cdot v = p \cdot (uv) \neq q$, so

$$[f_{x,y}(uv)]_{p,q} = 0_S = [f_{x,y}(u)f_{x,y}(v)]_{p,q}.$$

Otherwise $(p \cdot u) \cdot v = p \cdot (uv) = q$. In this case $[f_{x,y}(u)]_{p,p\cdot u}$ is equal to α^i where i denotes the number of xs added to column y when taking any rPS-tableau with bottom row p and right-multiplying by u, and $[f_{x,y}(v)]_{p\cdot u,q} = f_{x,y}(v)_{p\cdot u,p\cdot(uv)}$ is equal to α^j where j denotes the number of xs added to column y when taking any rPS-tableau with bottom row $p \cdot u$ and right-multiplying by v. Thus

$$[f_{x,y}(u)f_{x,y}(v)]_{p,q} = \alpha^i \alpha^j = \alpha^{i+j} = f_{x,y}(uv)_{p,q},$$

since $i + j$ is clearly the number of xs added to column y when taking any rPS-tableau with bottom row p and right-multiplying by uv, and so α^{i+j} is, by definition, equal to $f_{x,y}(uv)_{p,p\cdot(uv)} = f_{x,y}(uv)_{p,q}$. □

Theorem 7.3. *Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The right patience-sorting monoid of rank n admits a faithful representation by upper-triangular matrices of size $2^{n-1}(n^2 + n)$ over S.*

Proof. Construct a block-diagonal representation by taking the blocks to be the images of the homomorphisms $f_{x,y}$ for $x, y \in [n]$ with $x \geq y$.

To show that the representation is faithful, suppose $u, v \in \mathsf{rPS}_n$ are distinct elements of the right patience-sorting monoid. Then we may choose $x, y \in [n]$ such that the rPS-tableaux corresponding to these two elements differ in the number of times that generator x occurs in column y. Clearly we must have $x \geq y$, since otherwise neither tableau can contain an x in column y. Let $b = \emptyset \cdot u \in B$ be the bottom row of the rPS-tableau corresponding to u. Then by the definition of $f_{x,y}$ the entry in row $\emptyset$ and column b of $f_{x,y}(u)$ contains α^i where i denotes the number of times symbol x occurs in column y of the tableau corresponding to u. In contrast, the corresponding entry in $f_{x,y}(v)$ is different: either it is 0_S (if the tableau of v has a different bottom row) or else it is α^j where j denotes the number of times symbol x occurs in column y of the tableau corresponding to v, where $i \neq j$ by assumption. Hence, $f_{x,y}(u) \neq f_{x,y}(v)$ so $f(u) \neq f(v)$.

Notice that the action of rPS_n on the set B has no cycles except for fixed points: right-multiplying an rPS-tableau by a generator increases the length of the bottom row, or keeps the length the same and decreases the sum along the row, or leaves the row unchanged. It follows that the relation on B defined by $p \leq q$ if and only if $p \cdot s = q$ for some $s \in rPS_n$ is a partial order. For each x and y it is immediate from the definition of $f_{x,y}$ that for any $w \in [n]^*$ we will have $f_{x,y}(w)_{p,q} = 0_S$ unless $p \leq q$. Thus, completing this partial order to a linear order on B yields an order with respect to which $f_{x,y}(w)$ is upper triangular for all $w \in [n]^*$. Using such a linear order within each block of the representation thus ensures that the representation is upper triangular.

For each x and y, the dimension of the representation $f_{x,y}$ is $|B| = 2^n$. The block diagonal representation has $\binom{n+1}{2}$ blocks of this size and so has total dimension $2^n\binom{n+1}{2} = 2^{n-1}(n^2 + n)$. $\square$

7.1. Identities satisfied by the right patience-sorting monoid

An immediate consequence of Theorem 7.3 is that the right patience-sorting monoid rPS_n satisfies every semigroup identity satisfied by $\mathrm{UT}_N(S)$, where $N = 2^{n-1}(n^2 + n)$. Taking $S = \mathbb{T}$ one can therefore determine families of non-trivial identities satisfied by rPS_n by appealing to the known results about $\mathrm{UT}_N(\mathbb{T})$ (see for example [20], [32]). In fact, since the representation f constructed in the proof is block-diagonal where each block has size 2^n, one could instead take $N = 2^n$ in the above statement. Since the variety generated by $\mathrm{UT}_N(\mathbb{T})$ properly contains the variety generated by $\mathrm{UT}_d(\mathbb{T})$ whenever $d < N$ (see [2, Theorem 2.4]), one may be tempted to seek ways to reduce the dimension of the representation, for the purpose of studying identities.

The dimension of the representation in Theorem 7.3 could be refined down slightly by *ad hoc* arguments showing that certain rows and columns are not needed, but any significant reduction in the asymptotics is likely to require a completely different representation, if it is possible at all. Therefore we instead turn our attention to the so-called *chain length* of the representation. Given an N-dimensional upper triangular matrix representation $\phi : M \to \mathrm{UT}_N(S)$ of a semigroup M over a semiring S, let Γ_ϕ denote the transitive closure of the directed graph with node set $\{1, \ldots, N\}$ with an edge from i to

j whenever $i = j$ or there exists $A \in \phi(M)$ with $A_{i,j} \neq 0_S$. Writing $i \preceq j$ if there is an edge from i to j in Γ_ϕ, it is clear that $\preceq$ is a partial order. We define the chain length of the representation ϕ to be the maximal length of a chain in this partial order. In the case where $S = \mathbb{T}$ one can use the results of [11] to show that if $\phi : M \to \mathrm{UT}_N(\mathbb{T})$ is a faithful tropical representation of chain length d, then M is contained in the variety generated by $\mathrm{UT}_d(\mathbb{T})$. To see this, let $\Gamma_\phi(\mathbb{T})$ be the set of all matrices $A \in \mathrm{UT}_N(\mathbb{T})$ such that $A_{i,j} = -\infty$ whenever $i \not\preceq j$ in Γ_ϕ. By construction, the image of the faithful representation ϕ is contained in $\Gamma_\phi(\mathbb{T})$, and so the statement follows from [11, Theorem 5.3] which states that the variety generated by $\Gamma_\phi(\mathbb{T})$ is equal to the variety generated by $\mathrm{UT}_d(\mathbb{T})$.

We now demonstrate that the chain length of our faithful representation of the right patience-sorting monoid turns out to be far smaller than the dimensions of the blocks.

Proposition 7.4. *The maximum chain length of the representation f is $\binom{n+1}{2} + 1$.*

Proof. If P is a tableau and x a generator then the bottom row of $P \leftarrow x$ differs from the bottom row of P if and only if insertion of x into P places the new x in a column not already containing an x. It follows that for $w \in [n]^*$, the number of times the bottom row changes during the iterative construction of the tableau $\emptyset \leftarrow w$ is equal to the sum over columns in $\emptyset \leftarrow w$ of the number of distinct generators appearing in each column. In particular, this number is an invariant of the element of rPS_n (independent of the choice of representative word w). Since the yth column from the left cannot contain more than $n-y+1$ distinct generators (those symbols x with $x \geq y$), this number is bounded above by $\sum_{y=1}^{n} n - y + 1 = \binom{n+1}{2}$.

Now, notice that the block structure of the representation f means that any chain in the partial order Γ_f must lie within $\Gamma_{f_{x,y}}$ for some x and y which is clearly the partial order on B given by $p \leq q$ if and only if $p \cdot s = q$ for some $s \in \mathsf{rPS}_n$. Suppose $b_1, \ldots, b_k \in B$ is a chain of distinct sets such that there exist words $w_1, \ldots w_{k-1} \in [n]^*$ with $f_{x,y}(w_i)_{b_i, b_{i+1}} \neq 0_S$ for each $i \in [k-1]$. By the definition of $f_{x,y}$ this means that $b_i \cdot w_i = b_{i+1}$ for each $i \in [k-1]$. Let $w_0 \in [n]^*$ be a word representing the single-row tableau b_1, and let $w = w_0 w_1 \ldots w_{k-1} \in [n]^*$. Consider the corresponding rPS-tableau $\emptyset \leftarrow w$. Clearly each b_i appears as a bottom row during the iterative insertion of the symbols in w. But the number of bottom rows so appearing exceeds by at most 1 the number of times the bottom row changes. Thus, by the previous paragraph, $k \leq \binom{n+1}{2} + 1$.

Conversely, consider the word

$$w = \prod_{i=1}^{n} \left(\prod_{j=1}^{i} (n - j + 1) \right).$$

It is easy to see that the corresponding tableau $\emptyset \leftarrow w$ is of size $\binom{n+1}{2}$ with distinct entries in every column. Consider the iterative construction of this tableau using the word w.

By the first paragraph of the proof, the bottom row changes $\binom{n+1}{2}$ times, yielding a sequence of $\binom{n+1}{2}+1$ distinct bottom rows which form a chain. □

Corollary 7.5. *The right patience-sorting monoid of rank n satisfies all semigroup identities satisfied by* $\mathrm{UT}_d(\mathbb{T})$ *where* $d = \binom{n+1}{2}+1$.

Remark 7.6. Using some straightforward generalisations of proofs from [11, Section 5], one can establish an analogue of Corollary 7.5 in which $\mathbb{T}$ is replaced by any commutative unital semiring S with both a zero element and an element of infinite multiplicative order. However, to our knowledge the only such semirings where much is known about identities in $\mathrm{UT}_d(S)$ either satisfy no identities (for example, $S = \mathbb{N}$ and hence also S any ring of characteristic 0) in which case the result is vacuous, or satisfy the same identities as $\mathbb{T}$ (for example, the tropical natural number semiring $\mathbb{T} \cap (\mathbb{N} \cup \{-\infty\})$) in which case the stronger statement adds nothing. However, we note that the rPS_n certainly satisfies identities which do not come from $\mathrm{UT}_d(\mathbb{T})$ (for example, those given by [10, Proposition 4.7], which are shorter than any which hold in $\mathrm{UT}_d(\mathbb{T})$), so it may be interesting to investigate representations over other semirings.

Remark 7.7. The right patience-sorting monoid of rank n does not satisfy any identity of length less than n [10, Proposition 4.8]. Thus, in contrast to the monoids considered in the previous sections, the identities satisfied by rPS_n, and hence the variety generated by rPS_n, are dependent on n. Thus there is no direct analogue for right patience-sorting monoids of Corollaries 3.5, 4.6, or 5.8 or Proposition 6.6 (which state that the variety generated by one of our previous plactic-like monoids of infinite rank is equal to the variety generated by any one of the finite rank plactic-like monoids of the same kind). However, there is scope to study other aspects of these varieties further (for example, bases of identities).

Remark 7.8. Since our initial motivation was to study identities for plactic-like monoids via tropical representations, we have not attempted to construct representations of the left patience-sorting monoids; as explained in the introduction, the existence of free submonoids in lPS_n implies that faithful representations by *tropical* matrices simply do not exist. However, one could investigate whether faithful finite-dimensional representations of lPS_n monoids exist over other semirings. A key property of right patience-sorting tableaux used in the construction of our representations is that there are finitely many possible 'bottom rows' in an rPS-tableau. This is not the case for left patience-sorting monoids, and so a completely different approach to that for rPS-monoids would be required.

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