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## A COUPLE OF NON REDUCED BIAS GENERALIZED MEANS IN EXTREME VALUE THEORY: AN ASYMPTOTIC COMPARISON

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Abstract:

- *Lehmer's mean-of-order  $p$  ( $L_p$ )* generalizes the arithmetic mean, and  $L_p$  *extreme value index* (EVI)-estimators can be easily built, as a generalization of the classical Hill EVI-estimators. Apart from a reference to the asymptotic behaviour of this class of estimators, an asymptotic comparison, at optimal levels, of the members of such a class reveals that for the optimal  $(p, k)$  in the sense of minimal mean square error, with  $k$  the number of top order statistics involved in the estimation, they are able to overall outperform a recent and promising generalization of the Hill EVI-estimator, related to the power mean, also known as Hölder's mean-of-order- $p$ . A further comparison with other 'classical' non-reduced-bias estimators still reveals the competitiveness of this class of EVI-estimators.

Key-Words:

- *heavy tails; optimal tuning parameters; semi-parametric estimation; statistical extreme value theory.*

AMS Subject Classification:

- 62G32, 62E20.

**1. GENERALIZED MEANS' ESTIMATORS AND SCOPE OF THE ARTICLE**

Let us consider the notation  $(X_{1:n}, \dots, X_{n:n})$  for the ascending order statistics associated with a random sample of size  $n$ ,  $(X_1, \dots, X_n)$ , from a *cumulative distribution function* (CDF)  $F$ . Let us further assume that there exist sequences of real constants  $\{a_n > 0\}$  and  $\{b_n \in \mathbb{R}\}$  such that the maximum, linearly normalized, i.e.  $(X_{n:n} - b_n)/a_n$ , converges in distribution to a non-degenerate *random variable* (RV). Then (Gnedenko, [24]), the limit distribution is necessarily of the type of the general *extreme value* (EV) CDF, given by

$$(1.1) \quad \text{EV}_\xi(x) := \begin{cases} \exp(-(1 + \xi x)^{-1/\xi}), & 1 + \xi x > 0, & \text{if } \xi \neq 0, \\ \exp(-\exp(-x)), & x \in \mathbb{R}, & \text{if } \xi = 0. \end{cases}$$

The CDF  $F$  is then said to belong to the *max-domain of attraction* of  $\text{EV}_\xi$ , defined in (1.1), we use the notation  $F \in \mathcal{D}_{\mathcal{M}}(\text{EV}_\xi)$ , and the parameter  $\xi$  is the *extreme value index* (EVI), the primary parameter of extreme events. It is well-known that the EVI measures the heaviness of the right-tail function  $\bar{F}(x) := 1 - F(x)$ , and the heavier the right-tail function, the larger  $\xi$  is. From a quantile point of view, with  $F^\leftarrow(x) := \inf\{y: F(y) \geq x\}$  denoting the generalized inverse function of  $F$ , further consider  $U(t) := F^\leftarrow(1 - 1/t)$ ,  $t \geq 1$ , the *reciprocal tail quantile function* (RTQF). Then, with  $\mathcal{R}_a$  denoting the class of regularly varying functions at infinity, with an index of regular variation equal to  $a \in \mathbb{R}$ , i.e. positive measurable functions  $g(\cdot)$  such that for all  $x > 0$ ,  $g(tx)/g(t) \rightarrow x^a$ , as  $t \rightarrow \infty$ , (see Bingham *et al.*, [7], among others),

$$(1.2) \quad \begin{aligned} F \in \mathcal{D}_{\mathcal{M}}^+ &:= \mathcal{D}_{\mathcal{M}}(\text{EV}_\xi)_{\xi > 0} \iff \bar{F} \in \mathcal{R}_{-1/\xi} \quad (\text{Gnedenko, [24]}) \\ &\iff U \in \mathcal{R}_\xi \quad (\text{de Haan, [39]}). \end{aligned}$$

In this article we work with a Pareto-type underlying CDF, satisfying (1.2), i.e. with an associated positive EVI for maxima. These heavy-tailed models are quite common in a large variety of fields of application, like bibliometrics, biostatistics, computer science, insurance, finance, social sciences, statistical quality control and telecommunications, among others. For Pareto-type models, the classical EVI-estimators are the *Hill* (H) estimators (Hill, [44]), which are the averages of the log-excesses, i.e.

$$(1.3) \quad \hat{\xi}^H(k) \equiv H(k) := \frac{1}{k} \sum_{i=1}^k V_{ik}, \quad V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \quad 1 \leq i \leq k < n.$$

One of the interesting facts concerning the H EVI-estimators is that various asymptotically equivalent versions of  $H(k)$  can be derived through essentially different methods, such as the maximum likelihood method or the mean excess function approach, showing that the Hill estimator is quite natural. Details can be found in Beirlant *et al.* ([4]), among others. We merely note that from a quantile point of view, and with  $U(\cdot)$  the RTQF, we can write the distributional identity  $X \stackrel{d}{=} U(Y)$ , with  $Y$  a unit Pareto RV, i.e. an RV with a CDF  $F_Y(y) = 1 - 1/y$ ,  $y \geq 1$ . For the order statistics associated with a random unit Pareto sample  $(Y_1, \dots, Y_n)$ , we have the distributional identity  $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$ ,  $1 \leq i \leq k$ . Moreover,  $kY_{n-k:n}/n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 1$ , i.e.  $Y_{n-k:n} \stackrel{\mathbb{P}}{\sim} n/k$ . Consequently, and provided that  $k = k_n$ ,  $1 \leq k < n$ , is an intermediate sequence of integers, i.e. if

$$(1.4) \quad k = k_n \rightarrow \infty \quad \text{and} \quad k_n = o(n), \quad \text{as } n \rightarrow \infty,$$

we get

$$(1.5) \quad V_{ik} \stackrel{d}{=} \xi \ln Y_{k-i+1:k} + o_{\mathbb{P}}(1) \stackrel{d}{=} \xi E_{k-i+1:k} + o_{\mathbb{P}}(1),$$

with  $E$  denoting a standard exponential RV and the  $o_{\mathbb{P}}(1)$ -term uniform in  $i$ ,  $1 \leq i \leq k$  (see Caeiro *et al.*, [14], among others, for further details on this uniform behaviour). The log-excesses,  $V_{ik}$ ,  $1 \leq i \leq k$ , in (1.3), are thus approximately the  $k$  order statistics of a sample of size  $k$  from an exponential parent with mean value  $\xi$ , motivating the H EVI-estimators in (1.3).

Beyond the average, the  $p$ -moments of log-excesses, i.e.

$$(1.6) \quad M_{k,n}^{(p)} := \frac{1}{k} \sum_{i=1}^k \{ \ln X_{n-i+1:n} - \ln X_{n-k:n} \}^p, \quad p \geq 1,$$

introduced in Dekkers *et al.* ([19]) [ $M_{k,n}^{(1)} \equiv H(k)$ ] have also played a relevant role in the EVI-estimation, and can more generally be parameterized in  $p \in \mathbb{R} \setminus \{0\}$ . Note next that a simple generalization of the mean is Lehmer’s mean-of-order- $p$  (see Havil, [43], p. 121). Given a set of positive numbers  $\mathbf{a} = (a_1, \dots, a_k)$ , such a mean generalizes both the arithmetic mean ( $p = 1$ ) and the harmonic mean ( $p = 0$ ), being defined as

$$L_p(\mathbf{a}) := \sum_{i=1}^k a_i^p / \sum_{i=1}^k a_i^{p-1}, \quad p \in \mathbb{R}.$$

Further note that  $\lim_{p \rightarrow -\infty} L_p(\mathbf{a}) = \min_{1 \leq i \leq k} a_i$  and  $\lim_{p \rightarrow +\infty} L_p(\mathbf{a}) = \max_{1 \leq i \leq k} a_i$ .

The H EVI-estimators can thus be considered as the Lehmer mean-of-order-1 of the  $k$  log-excesses  $\mathbf{V} := (V_{ik}, 1 \leq i \leq k)$ , in (1.3),  $k < n$ . We now more generally consider the Lehmer mean-of-order- $p$  of those statistics. From (1.5), since  $\mathbb{E}(E^p) = \Gamma(p + 1)$  for any real  $p > -1$ , with  $\Gamma(\cdot)$  denoting the complete Gamma function, the law of large numbers enables us to say that

$$\frac{1}{k} \sum_{i=1}^k V_{ik}^p \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \Gamma(p + 1) \xi^p.$$

Hence the reason for the class of *Lehmer mean-of-order- $p$*  ( $L_p$ ) EVI-estimators,

$$(1.7) \quad \hat{\xi}^{L_p(k)} \equiv L_p(k) := \frac{L_p(\mathbf{V})}{p} = \frac{1}{p} \frac{\sum_{i=1}^k V_{ik}^p}{\sum_{i=1}^k V_{ik}^{p-1}} = \frac{M_{k,n}^{(p)}}{p M_{k,n}^{(p-1)}} \quad [L_1(k) \equiv H(k)],$$

consistent for all  $\xi > 0$  and real  $p > 0$ , and where  $M_{k,n}^{(p)}$  is given in (1.6).

As a possible competitive class of EVI-estimators, we further refer the one recently studied in Brillhante *et al.* ([8]), Gomes and Caeiro ([25]) and Caeiro *et al.* ([14]), among others, based on the power mean. Given a set of non-negative numbers  $\mathbf{a} = (a_1, \dots, a_k)$ , such a mean generalizes the arithmetic mean ( $p = 1$ ), the geometric mean ( $p = 0$ ) and the harmonic mean ( $p = -1$ ), being defined as

$$M_p(\mathbf{a}) := \left( \frac{1}{k} \sum_{i=1}^k a_i^p \right)^{1/p}, \quad p \in \mathbb{R}.$$

Further note that  $\lim_{p \rightarrow 0} M_p(\mathbf{a}) \equiv M_0(\mathbf{a}) = (\prod_{i=1}^k a_i)^{1/k}$ ,  $\lim_{p \rightarrow -\infty} M_p(\mathbf{a}) = \min_{1 \leq i \leq k} a_i$  and  $\lim_{p \rightarrow +\infty} M_p(\mathbf{a}) = \max_{1 \leq i \leq k} a_i$ . On the basis of the fact that the Hill EVI-estimator in (1.3) is the logarithm of the geometric mean of

$$(1.8) \quad U_{ik} := X_{n-i+1:n} / X_{n-k:n}, \quad 1 \leq i \leq k < n,$$

the consideration of the power mean, also known as Hölder’s mean-of-order- $p$  ( $MO_p$ ), of those same statistics leads to

$$(1.9) \quad \hat{\xi}^{H_p}(k) \equiv H_p(k) := \begin{cases} \left(1 - \left(\frac{1}{k} \sum_{i=1}^k U_{ik}^p\right)^{-1}\right) / p, & \text{if } p < 1/\xi, \ p \neq 0, \\ H(k), & \text{if } p = 0, \end{cases}$$

the so-called  $MO_p$  EVI-estimators, almost simultaneously considered, for  $p \geq 0$ , in Brillhante *et al.* ([8]), Paulauskas and Vaičiulis ([45]) and Beran *et al.* ([6]). As a measure of comparison, and just as in Gomes and Henriques-Rodrigues ([30]) (see also Gomes and Henriques-Rodrigues, [31]), the *Pareto probability weighted moments* (PPWM) EVI-estimators, introduced in Caeiro and Gomes ([11]), and further studied in Caeiro *et al.* ([17], [15]) will also be considered. The PPWM EVI-estimators, quite common in the areas of climatology and hydrology, are consistent only for  $\xi < 1$ , depend on the statistics  $\hat{a}_j(k) := \frac{1}{k} \sum_{i=1}^k ((i-1)/(k-1))^j X_{n-i+1:n}$ ,  $j = 0, 1$ , and are defined by

$$(1.10) \quad \hat{\xi}^{PPWM}(k) \equiv PPWM(k) := 1 - \frac{\hat{a}_1(k)}{\hat{a}_0(k) - \hat{a}_1(k)}, \quad 1 \leq k < n.$$

We also mention the possibly *reduced-bias* (RB) class of EVI-estimators in Caeiro and Gomes ([10]) (see also, Caeiro and Gomes, [9], [12]),

$$(1.11) \quad \hat{\xi}^{CG_{p,\delta}}(k) \equiv CG_{p,\delta}(k) := \frac{\Gamma(p)}{M_{k,n}^{(p-1)}} \left(\frac{M_{k,n}^{(\delta p)}}{\Gamma(\delta p + 1)}\right)^{1/\delta}, \quad \delta > 0, \ p > 0 \quad [CG_{1,1}(k) \equiv H(k)].$$

For  $\delta = 2$  in (1.11), we obtain a class studied in Caeiro and Gomes ([9]), which generalizes the estimator  $CG_{1,2}(k) = \sqrt{M_{k,n}^{(2)}/2}$ , studied in Gomes *et al.* ([34]), where also  $L_2(k) = M_{k,n}^2 / (2M_{k,n}^{(1)})$  was introduced and studied both asymptotically and for finite samples. And we can also consider the class of  $L_p$  EVI-estimators in (1.7), as a non-RB particular case of (1.11). Indeed,  $L_p(k) \equiv CG_{p,1}(k)$ .

**Remark 1.1.** Note that all the aforementioned EVI-estimators are scale invariant, but not location-invariant. They can however become location-invariant if we apply the *peaks over random threshold* (PORT) methodology, basing them not on the original sample, but on the excesses over a central empirical quantile and even over the minimum of the available sample whenever possible, i.e. when the underlying parent  $F$  has a finite left endpoint. For details on the topic, see, among others, Araújo Santos *et al.* ([1]), where the acronym PORT was introduced, Gomes *et al.* ([27]), and more recently, Gomes and Henriques-Rodrigues ([30]) and Gomes *et al.* ([32]).

In Section 2, after the introduction of a few technical details in the field of *extreme value theory* (EVT), we deal with the asymptotic behaviour of the  $L_p$  EVI-estimators, in (1.7).

In Section 3, it is shown that at optimal  $k$ -levels and for the optimal  $p$ , the members of such a class are able to overall outperform the optimal EVI-estimators in (1.9), which on its turn had been shown in Brillhante *et al.* ([8]) to have a similar behaviour comparatively with the optimal Hill EVI-estimators, for an adequate optimal  $p$  ( $\neq 0$ ). We next compare them, asymptotically and at optimal levels, with the optimal PPWM EVI-estimators, in (1.10). Finally, in Section 4, we advance with an overall comparison of a wide number of EVI-estimators, drawing some concluding remarks.

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## 2. ASYMPTOTIC BEHAVIOUR OF THE EVI-ESTIMATORS

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After a reference, in Section 2.1, to the most common second-order framework for heavy-tailed models, we briefly refer, in Section 2.2, the asymptotic behaviour of the EVI-estimators defined in Section 1. A recent review on the topic of statistical univariate EVT can be found in Gomes and Guillou ([28]). See also Beirlant *et al.* ([2]) and Scarrot and MacDonald ([48]).

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### 2.1. A few technical details in the field of EVT

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In the area of statistical EVT and whenever working with large values, a model  $F$  is commonly said to be heavy-tailed whenever (1.2) holds. The second-order parameter  $\rho$  ( $\leq 0$ ) rules the rate of convergence in any of the first-order conditions, in (1.2), and can be defined as the non-positive parameter appearing in the limiting relation

$$(2.1) \quad \lim_{t \rightarrow \infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} (x^\rho - 1)/\rho, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases}$$

which is assumed to hold for every  $x > 0$ , and where  $|A|$  must then be of regular variation with index  $\rho$ . This condition has been widely accepted as an appropriate condition to specify the right-tail of a Pareto-type distribution in a semi-parametric way. For technical simplicity, we often assume that we are working in Hall–Welsh class of models (Hall and Welsh, [42]), with an RTQF,

$$(2.2) \quad U(t) = C t^\xi \left( 1 + \xi \beta t^{\rho/\rho} + o(t^\rho) \right), \quad \text{as } t \rightarrow \infty,$$

$C > 0$ ,  $\beta \neq 0$  and  $\rho < 0$ . Equivalently, we can say that, with  $(\beta, \rho)$  the vector of second-order parameters, the general second-order condition in (2.1) holds with  $A(t) = \xi \beta t^\rho$ ,  $\rho < 0$ . Further details on second-order conditions can be found in Beirlant *et al.* ([4]), de Haan and Ferreira ([40]) and Fraga Alves *et al.* ([22]), among others.

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### 2.2. Asymptotic behaviour of the EVI-estimators under consideration

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Trivial adaptations of the results in de Haan and Peng ([41]), Caeiro and Gomes ([10]), Caeiro and Gomes ([11]) and Brillhante *et al.* ([8]), respectively for the H,  $CG_{p,\delta}$ , PPWM and  $H_p$  classes of EVI-estimators, enable us to state:

**Theorem 2.1.** Under the validity of the first-order condition, in (1.2), and for intermediate sequences  $k = k_n$ , i.e. if (1.4) holds, the classes of  $H_p$ , PPWM and  $CG_{p,\delta}$  EVI-estimators, respectively defined in (1.9), (1.10), and (1.11), generally denoted by  $\hat{\xi}^\bullet(k)$ , are consistent for the estimation of  $\xi > 0$ , provided that we work in  $\mathcal{S}_\bullet$ , where  $\mathcal{S}_{H_p} = \{(\xi, p) : \xi > 0, p < 1/\xi\}$ ,  $\mathcal{S}_{PPWM} = \{\xi : 0 < \xi < 1\}$  and  $\mathcal{S}_{CG_{p,\delta}} = \{(\xi, p, \delta) : \xi > 0, p > 0, \delta > -1/p\}$ .

Assume further that (2.1) holds. Then, for  $\xi > 0$ , adequate regions of the spaces of parameters and with  $\mathcal{N}(\mu, \sigma^2)$  standing for a normal RV with mean value  $\mu$  and variance  $\sigma^2$ ,

$$(2.3) \quad \sqrt{k} (\hat{\xi}^\bullet(k) - \xi) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(\lambda_A b_\bullet, \sigma_\bullet^2) \quad \text{if} \quad \sqrt{k} A(n/k) \xrightarrow[n \rightarrow \infty]{} \lambda_A, \text{ finite.}$$

Moreover

$$(2.4) \quad b_{H_p} = \frac{1 - p\xi}{1 - p\xi - \rho}, \quad \sigma_{H_p}^2 = \frac{\xi^2(1 - p\xi)^2}{1 - 2p\xi}, \quad \text{if } p < 1/(2\xi),$$

$$b_{PPWM} = \frac{(1 - \xi)(2 - \xi)}{(1 - \xi - \rho)(2 - \xi - \rho)}, \quad \sigma_{PPWM}^2 = \frac{\xi^2(1 - \xi)(2 - \xi)^2}{(1 - 2\xi)(3 - 2\xi)}, \quad \text{if } \xi < 1/2,$$

and

$$(2.5) \quad b_{CG_{p,\delta}} = \frac{(1 - \rho)^{-\delta p} - \delta(1 - \rho)^{-p+1} + \delta - 1}{\delta \rho},$$

$$\sigma_{CG_{p,\delta}}^2 = \frac{\xi^2}{\delta^2} \left\{ \frac{2\Gamma(2\delta p)}{\delta p \Gamma^2(\delta p)} + \frac{\delta^2 \Gamma(2p - 1)}{\Gamma^2(p)} - \frac{2\Gamma((\delta + 1)p)}{p \Gamma(p) \Gamma(\delta p)} - (\delta - 1)^2 \right\},$$

if  $p > 1/2, \delta > 0$ .

For the particular case  $\delta = 1$ , in (1.11), i.e. for the  $L_p$  EVI-estimators in (1.7), we can state:

**Corollary 2.1.** Under the validity of the initial first-order conditions in Theorem 2.1, the class of  $L_p$  EVI-estimators, in (1.7), is consistent for the estimation of  $\xi$ , provided that we work in  $\mathcal{S}_{L_p} = \{(\xi, p) : \xi > 0, p > 0\}$ . Under the second-order conditions of Theorem 2.1, (2.3) holds, with

$$(2.6) \quad b_{L_p} = \frac{1}{(1 - \rho)^p} \quad \text{and} \quad \sigma_{L_p}^2 = \frac{\xi^2 \Gamma(2p - 1)}{\Gamma^2(p)} \quad \text{if } p > 1/2.$$

More specifically, and for all  $\rho \leq 0$ , one can write the asymptotic distributional representation

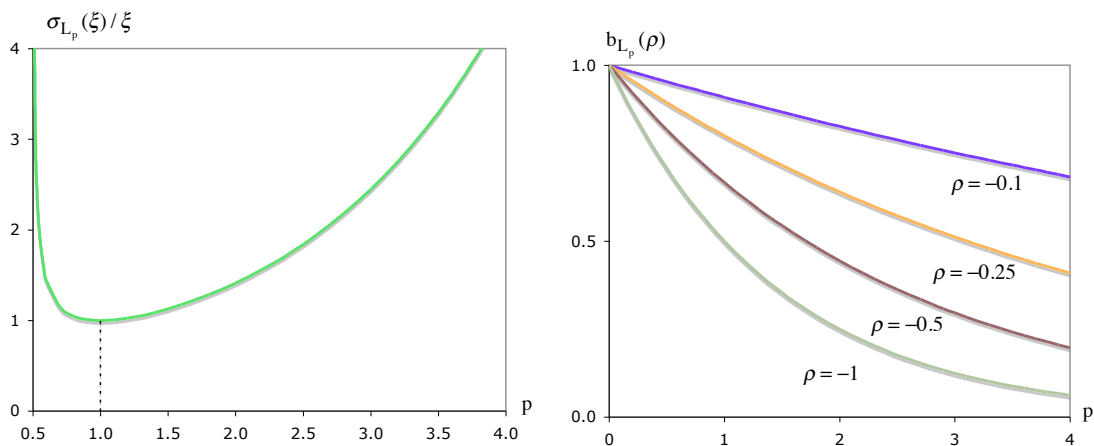
$$(2.7) \quad L_p(k) \stackrel{d}{=} \xi + \frac{\sigma_{L_p} Z_k^{(p)}}{\sqrt{k}} + b_{L_p} A(n/k) + o_{\mathbb{P}}(A(n/k)),$$

with  $(b_{L_p}, \sigma_{L_p}^2)$  given in (2.6), and where  $Z_k^{(p)}$  is an asymptotically standard normal RV.

**Remark 2.1.** Note that regarding the  $L_p$  EVI-estimators, in (1.7), Corollary 2.1 is a particular case of Theorem 1 in Caeiro and Gomes ([10]), but generalizing now consistency for  $p > 0$  and asymptotic normality for  $p > 1/2$  rather than  $p \geq 1$ . Further note that for  $\delta = 1$  there is a full agreement between (2.6) and (2.5), the result provided in Theorem 1 of Caeiro and Gomes ([10]). A detailed proof of Corollary 2.1 can be found in Penalva *et al.* ([47]).

**Remark 2.2.** Further note that for the  $MO_p$  EVI-estimators, denoted by  $H_p$  and defined in (1.9), a distributional representation of the type of the one in (2.7) holds for  $p < 1/(2\xi)$ , with  $(b_{L_p}, \sigma_{L_p}^2)$  replaced by  $(b_{H_p}, \sigma_{H_p}^2)$ , given in (2.4).

For any  $\xi > 0$ , the asymptotic variance  $\sigma_{L_p}^2(\xi)$ , in (2.6), has a minimum at  $p = 1$ . In Figure 1 (left), we present the normalized standard deviation,  $\sigma_{L_p}(\xi)/\xi$ , independent of  $\xi$ , as a function of  $p$ . On another side, the asymptotic bias ruler,  $b_{L_p}(\rho)$ , also in (2.6), is independent of  $\xi$  and always decreasing in  $p$ . Such a performance is shown in Figure 1 (right).



**Figure 1:** Graph of  $\sigma_{L_p}(\xi)/\xi$ , as a function of  $p > 1/2$  (left) and of the asymptotic bias ruler  $b_{L_p}(\rho)$ , for  $\rho = -0.1, -0.25, -0.5$  and  $-1$ , as a function of  $p \geq 0$ .

The aforementioned results claim for an asymptotic study, at optimal  $(k, p)$ , of the class of EVI-estimators in (1.7), a topic to be dealt with in Section 3.

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### 3. ASYMPTOTIC COMPARISON AT OPTIMAL LEVELS

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We next proceed to the comparison of the aforementioned non-RB EVI-estimators, generally denoted by  $\hat{\xi}^\bullet(k)$ , at their optimal levels. This is again done in a way similar to the one used in several articles, among which we refer Dekkers and de Haan ([20]), de Haan and Peng ([41]), Gomes and Martins ([33]), Gomes *et al.* ([36], [37], [35], [26]), Gomes and Neves ([38]), Gomes and Henriques-Rodrigues ([29], [30]), and Brilhante *et al.* ([8]), among others. Let us assume that for any intermediate sequence of integers  $k = k_n$ , (2.3) holds. We write  $\text{Bias}_\infty(\hat{\xi}^\bullet(k)) := b_\bullet A(n/k)$  and  $\text{Var}_\infty(\hat{\xi}^\bullet(k)) := \sigma_\bullet^2/k$ . The so-called *asymptotic mean square error* (AMSE) is then given by  $\text{AMSE}(\hat{\xi}^\bullet(k)) := \sigma_\bullet^2/k + b_\bullet^2 A^2(n/k)$ . Regular variation theory enabled Dekkers and de Haan ([20]) to show that, whenever  $b_\bullet \neq 0$ , there exists a function  $\varphi(n) = \varphi(n, \xi, \rho)$ , such that

$$(3.1) \quad \lim_{n \rightarrow \infty} \varphi(n) \text{AMSE}(\hat{\xi}_0^\bullet) = (\sigma_\bullet^2)^{-\frac{2\rho}{1-2\rho}} (b_\bullet^2)^{\frac{1}{1-2\rho}} =: \text{LMSE}(\hat{\xi}_0^\bullet),$$

where  $\hat{\xi}_0^\bullet := \hat{\xi}^\bullet(k_{0|\bullet}(n))$  and  $k_{0|\bullet}(n) := \arg \min_k \text{MSE}(\hat{\xi}^\bullet(k))$ . Moreover, if we slightly restrict the second-order condition in (2.1), assuming (2.2), we can write

$$k_{0|\bullet}(n) = \arg \min_k \text{MSE}(\hat{\xi}^\bullet(k)) = \left( \frac{\sigma_\bullet^2 n^{-2\rho}}{b_\bullet^2 \xi^2 \beta^2 (-2\rho)} \right)^{1/(1-2\rho)} (1 + o(1)).$$

We consider the following:

**Definition 3.1.** Given two biased estimators  $\hat{\xi}^{(1)}(k)$  and  $\hat{\xi}^{(2)}(k)$ , for which (2.3) holds, with constants  $(\sigma_1, b_1)$  and  $(\sigma_2, b_2)$ ,  $b_1, b_2 \neq 0$ , respectively, both computed at their optimal levels, the asymptotic root efficiency (AREFF) of  $\hat{\xi}_0^{(1)}$  relatively to  $\hat{\xi}_0^{(2)}$  is

$$(3.2) \quad \text{AREFF}_{1|2} \equiv \text{AREFF}_{\hat{\xi}_0^{(1)}|\hat{\xi}_0^{(2)}} := \sqrt{\text{LMSE}(\hat{\xi}_0^{(2)})/\text{LMSE}(\hat{\xi}_0^{(1)})} = \left( \left( \frac{\sigma_2}{\sigma_1} \right)^{-2\rho} \left| \frac{b_2}{b_1} \right| \right)^{\frac{1}{1-2\rho}},$$

with LMSE defined in (3.1).

**Remark 3.1.** Note that the AREFF-indicator, in (3.2), has been conceived so that the highest the AREFF indicator is, the better is the estimator identified with the superscript (1).

The non-RB  $L_p$ ,  $H_p$ , and PPWM EVI-estimators, respectively given in (1.7), (1.9) and (1.10), will be crucially included in the asymptotic comparison in Section 3.1.

### 3.1. Asymptotic comparison of EVI-estimators at optimal levels

Let us now turn back to the  $L_p$  EVI-estimators in (1.7), at optimal  $k$ -levels in the sense of minimum RMSE. We have

$$\text{LMSE}(L_{0|p}) = \left( \xi^2 \Gamma(2p - 1) / \Gamma^2(p) \right)^{-\frac{2\rho}{1-2\rho}} \left( (1 - \rho)^{-2p} \right)^{\frac{1}{1-2\rho}}$$

and

$$(3.3) \quad \text{AREFF}_L(p) \equiv \text{AREFF}_{L_{0|p}|L_{0|1}} = \left( \left( \Gamma(p) / \sqrt{\Gamma(2p - 1)} \right)^{-2\rho} (1 - \rho)^{p-1} \right)^{\frac{1}{1-2\rho}}.$$

**Remark 3.2.** In Gomes *et al.* ([34]) was shown that the AREFF of the optimal  $L_2(k)$  comparatively to the optimal  $L_1(k)$  is given by  $[2^\rho(1 - \rho)]^{1/(1-2\rho)}$ , in agreement with (3.3). As noticed in the aforementioned article,  $\text{AREFF}_L(2) > 1 \iff -1 < \rho < 0$ .

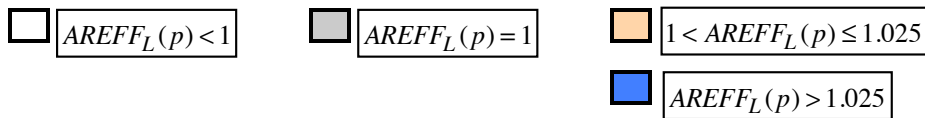
To measure the performance of  $H_{0|p}$ , with  $H_p$  the  $\text{MO}_p$  EVI-estimator in (1.9), Brillhante *et al.* ([8]) computed a similar AREFF-indicator, given by

$$(3.4) \quad \text{AREFF}_H(p) \equiv \text{AREFF}_{H_{0|p}|H_{0|0}} = \left( \left( \frac{\sqrt{1 - 2p\xi}}{1 - p\xi} \right)^{-2\rho} \left| \frac{1 - p\xi - \rho}{(1 - \rho)(1 - p\xi)} \right| \right)^{\frac{1}{1-2\rho}},$$

reparameterized in  $(\rho, a = p\xi < 1/2)$ , and denoted by  $\text{AREFF}_{a|0}^*$ . In Figure 2, we picture  $\text{AREFF}_L(p)$  in (3.3) (*top*) and  $\text{AREFF}_{a|0}^*$  (*bottom*).



$\rho \backslash p$	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20	2.25
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
-0.20	1.00	1.01	1.01	1.02	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
-0.30	1.00	1.01	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.03
-0.40	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02
-0.50	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02
-0.60	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01
-0.70	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00
-0.80	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99
-0.90	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98
-1.00	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.01	1.00	0.99	0.99	0.98	0.98	0.97
-1.10	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.96
-1.20	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96
-1.30	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.95	0.95
-1.40	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.94
-1.50	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.94	0.93
-1.60	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93	0.92
-1.70	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.95	0.94	0.93	0.92
-1.80	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.92	0.91
-1.90	1.00	1.01	1.02	1.02	1.02	1.03	1.03	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93	0.92	0.91	0.91
-2.00	1.00	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.91	0.90	0.90



$\rho \backslash a$	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.42	0.44	0.46	0.48	0.49	
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
-0.10	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.96	0.91	0.87
-0.20	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.95	0.91	0.84	0.77	
-0.30	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	0.99	0.98	0.96	0.93	0.88	0.79	0.70	
-0.40	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.97	0.94	0.90	0.85	0.74	0.65
-0.50	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	0.99	0.98	0.95	0.93	0.88	0.82	0.71	0.61
-0.60	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.98	0.97	0.94	0.91	0.87	0.80	0.68	0.57
-0.70	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.96	0.93	0.90	0.85	0.78	0.65	0.54
-0.80	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	1.00	0.98	0.97	0.95	0.92	0.89	0.83	0.76	0.63	0.52
-0.90	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.96	0.94	0.91	0.87	0.82	0.74	0.62	0.50
-1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.97	0.96	0.93	0.90	0.87	0.81	0.73	0.60	0.48	
-1.10	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.00	0.99	0.98	0.97	0.95	0.93	0.90	0.86	0.80	0.72	0.59	0.47	
-1.20	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.01	1.01	1.01	1.00	0.99	0.98	0.96	0.95	0.92	0.89	0.85	0.79	0.71	0.57	0.46	0.46	
-1.30	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.96	0.94	0.92	0.88	0.84	0.78	0.70	0.56	0.45	0.45	
-1.40	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.97	0.96	0.94	0.91	0.88	0.83	0.78	0.69	0.55	0.44	
-1.50	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.95	0.93	0.91	0.87	0.83	0.77	0.68	0.55	0.43	
-1.60	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.95	0.93	0.90	0.87	0.82	0.76	0.68	0.54	0.42	
-1.70	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	0.99	0.99	0.98	0.96	0.95	0.92	0.90	0.86	0.82	0.76	0.67	0.53	0.41	
-1.80	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.86	0.81	0.75	0.66	0.52	0.41	
-1.90	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.85	0.81	0.75	0.66	0.52	0.40	
-2.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.85	0.80	0.74	0.65	0.51	0.40	

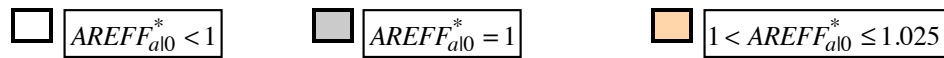


Figure 2:  $AREFF_L(p)$ , in (3.3) (top) and  $AREFF_{a|0}^*$  (bottom).

The gain in efficiency is not terribly high, but, at optimal levels, there is a wide region of the  $(p, \rho)$ -plane where the new class of  $L_p$  EVI-estimators performs better than the Hill EVI-estimators, with efficiencies slightly higher than the ones associated with the comparison of  $H_p$  and the Hill, in the  $(a, \rho)$ -plane. This result together with the fact that as far as we know, the EVI-estimators in (1.9) computed at the optimal  $(k, p)$  in the sense of maximal  $\text{AREFF}_H(p)$ , with  $\text{AREFF}_H(p)$  given in (3.4), i.e. computed at  $p_{M|H} \equiv p_{M|H}(\rho) := \arg \max_p \text{AREFF}_H(p)$ , explicitly given by

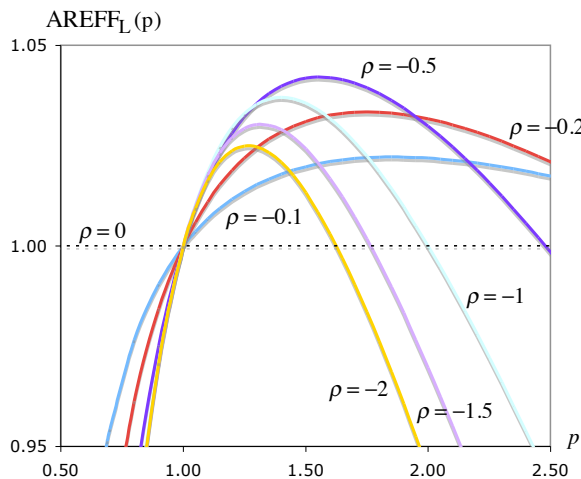
$$(3.5) \quad p_{M|H} = \varphi_\rho / \xi, \quad \text{with} \quad \varphi_\rho := 1 - \rho/2 - \sqrt{\rho^2 - 4\rho + 2} / 2,$$

$b_{p_{M|H}} \neq 0$ , is, as expected, a non-RB EVI-estimator which is able to beat the Hill EVI-estimator in the whole  $(\xi, \rho)$ -plane, immediately leads us to think on what happens for the optimal value of  $p$  associated with the  $L_p$  EVI-estimation. Contrarily to the explicit expression for  $p_{M|H}$ , in (3.5), the value of  $p_{M|L} = p_{M|L}(\rho) := \arg \max_p \text{AREFF}_L(p)$ , with  $\text{AREFF}_L(p)$  given in (3.3), is an implicit function of  $\rho$ , easy to evaluate numerically. Some of those values are presented in Table 1.

**Table 1:** Values of  $p_{M|L} = p_{M|L}(\rho) := \arg \max_p \text{AREFF}_L(p)$  for a few values of  $|\rho|$ .

$ \rho $	$0^+$	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.5	2	$+\infty$
$p_{M L}$	1	1.98	1.86	1.75	1.67	1.61	1.56	1.52	1.45	1.40	1.32	1.27	1

In Figure 3, we picture the indicator  $\text{AREFF}_L(p)$ , as a function of  $p$  for a few values of  $\rho$ .

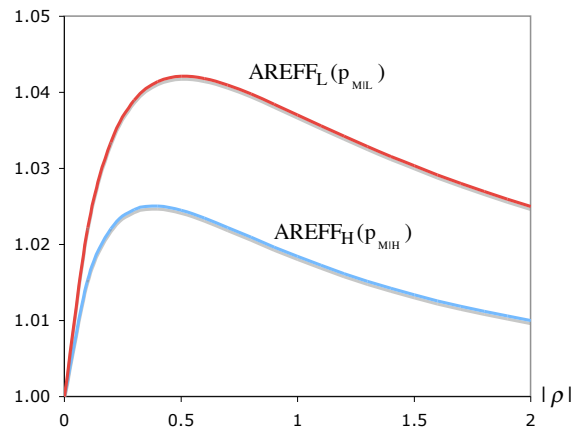


**Figure 3:**  $\text{AREFF}_L(p)$ , as a function of  $p$ , for  $|\rho| = 0, 0.1, 0.2, 0.5(0.5)2$ .

Indeed, just as  $\text{AREFF}_H(p_{M|H}) > 1$ , for any  $\rho < 0$  and  $\xi > 0$ , also  $\text{AREFF}_L(p_{M|L}) > 1$ , for any  $\rho < 0$  and  $\xi > 0$ . Moreover,

$$\text{AREFF}_L(p_{M|L}) > \text{AREFF}_H(p_{M|H}),$$

as illustrated in Figure 4.



**Figure 4:**  $AREFF_L(p_{M|L})$  and  $AREFF_H(p_{M|H})$  as a function of  $|\rho| = 0(0.1)2$ .

Just as done in Gomes and Henriques-Rodrigues ([30]), and due to the competitive behaviour of the PPWM EVI-estimators, we still compare the  $L_p$  with the PPWM EVI-estimators, in (1.10), again at optimal levels. Whereas the gain in efficiency of the PPWM comparatively to the optimal  $H_p$  EVI-estimator happens in a wide region of the  $(\xi, \rho)$ -plane,  $L^* := L_{p_{M|L}}$  beats the optimal PPWM EVI-estimator (now denoted P, for sake of simplicity) in a wider region of the  $(\xi, \rho)$ -plane, as can be seen in Figure 5 (bottom). Indeed, in Figure 5 (top), we reproduce the Figure in Gomes and Henriques-Rodrigues ([31]), related to the comparative behaviour between  $H^* := H_{p_{M|H}}$  and the optimal PPWM EVI-estimator.

So far, asymptotically and for a heavy right-tail, the class of Lehmer’s EVI-estimators, in (1.7), seems indeed to be the most competitive class of non-RB EVI-estimators in the literature. Note however that further classes of generalized means, among which we mention the ones studied in Paulauskas and Vaičiulis ([46]), may possibly provide even more astonishing results.

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#### 4. AN ASYMPTOTIC COMPARISON WITH OTHER EVI-ESTIMATORS AT OPTIMAL LEVELS

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As mentioned above, the optimal  $MO_p$  EVI-estimator ( $H^*$ ), associated with a value  $p_{M|H} \neq 0$ , can beat the optimal Hill EVI-estimator in the whole  $(\xi, \rho)$ -plane. But it is now beaten by the optimal Lehmer EVI-estimator ( $L^*$ ), also in the whole  $(\xi, \rho)$ , an atypical behaviour among other classical EVI-estimators. We thus consider now sensible to compare  $H^*$  and  $L^*$  with the most common EVI-estimators in the literature, non generally RB, but possibly RB in some regions of the  $(\xi, \rho)$ -plane.

We shall take into account the *moment* (M) EVI-estimators, studied in Dekkers *et al.* ([19]), based on  $(M_{k,n}^{(1)}, M_{k,n}^{(2)})$ , with  $M_{k,n}^{(p)}$  defined in (1.6). They are consistent for all  $\xi \in \mathbb{R}$ , being given by

$$(4.1) \quad \hat{\xi}^M(k) \equiv M(k) := M_{k,n}^{(1)} + \frac{1}{2} \left\{ 1 - \left( M_{k,n}^{(2)} / (M_{k,n}^{(1)})^2 - 1 \right)^{-1} \right\}.$$

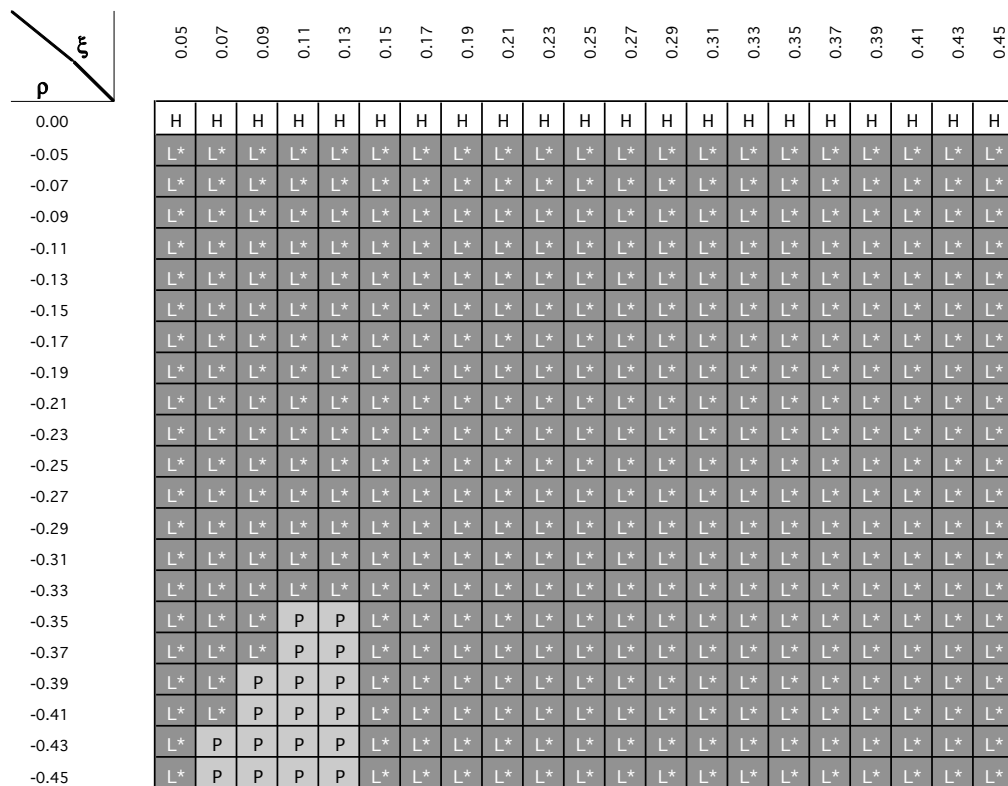
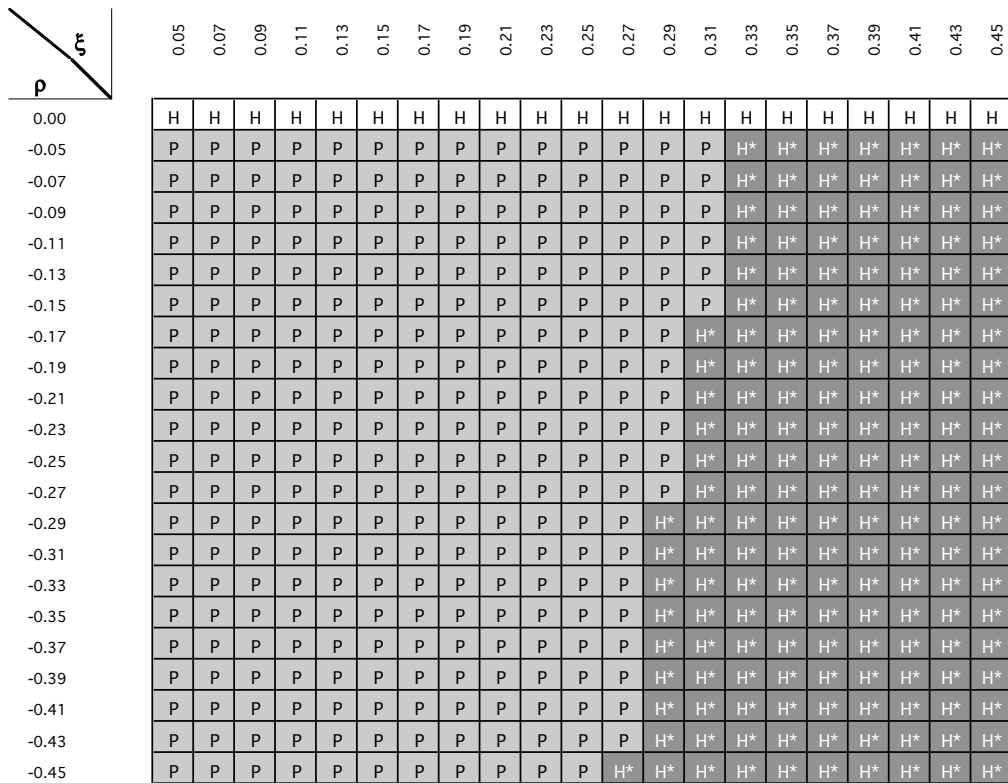


Figure 5: Best EVI-estimator asymptotically and at optimal levels for a choice between  $H^*$  and PPWM (top) and between  $L^*$  and PPWM (bottom).

We additionally consider the *generalized Hill* (GH) EVI-estimators (Beirlant *et al.*, [5]), based on the Hill estimators in (1.3) and with the functional form

$$(4.2) \quad \hat{\xi}^{\text{GH}}(k) \equiv \text{GH}(k) := \hat{\xi}^{\text{H}}(k) + \frac{1}{k} \sum_{i=1}^k \left\{ \ln \hat{\xi}^{\text{H}}(i) - \ln \hat{\xi}^{\text{H}}(k) \right\},$$

further studied in Beirlant *et al.* ([3]). Just as in de Haan and Ferreira ([40]), we also consider, for  $\xi < 1$ , the *generalized Pareto* (GP) PWM (GPPWM) EVI-estimators, based on the sample of exceedances over the high random level  $X_{n-k:n}$  and defined by

$$(4.3) \quad \hat{\xi}^{\text{GPPWM}}(k) \equiv \text{GPPWM}(k) := 1 - \frac{2 \hat{a}_1^*(k)}{\hat{a}_0^*(k) - 2 \hat{a}_1^*(k)},$$

with  $k = 1, \dots, n - 1$ , and

$$\hat{a}_s^*(k) := \frac{1}{k} \sum_{i=1}^k \left( \frac{i-1}{k-1} \right)^s (X_{n-i+1:n} - X_{n-k:n}), \quad s = 0, 1.$$

Finally, with  $U_{ik}$ ,  $1 \leq i \leq k$ , given in (1.8), and the notation

$$L_{k,n}^{(j)} := \frac{1}{k} \sum_{i=1}^k \left( 1 - U_{ik}^{-1} \right)^j, \quad j \geq 1,$$

we further consider the *mixed moment* (MM) EVI-estimators (Fraga Alves *et al.*, [23]), defined by

$$(4.4) \quad \hat{\xi}^{\text{MM}}(k) \equiv \text{MM}(k) := \frac{\hat{\varphi}_{k,n} - 1}{1 + 2 \min(\hat{\varphi}_{k,n} - 1, 0)}, \quad \text{with } \hat{\varphi}_{k,n} := \frac{M_{k,n}^{(1)} - L_{k,n}^{(1)}}{(L_{k,n}^{(1)})^2}.$$

The estimators in (4.3) are consistent only for  $0 < \xi < 1$ . The estimators in (4.1), (4.2) and (4.4) are consistent for any  $\xi \in \mathbb{R}$ , but will be here considered only for  $\xi > 0$ .

**Remark 4.1.** Note that the MM EVI-estimators, in (4.4), are, for a wide class of models with  $\xi > 0$ , very close to the implicit ML EVI-estimators, based on the excesses  $W_{ik} := X_{n-i+1:n} - X_{n-k:n}$ ,  $1 \leq i \leq k < n$  (see Fraga Alves *et al.*, [23], for details on the topic). A comprehensive study of the asymptotic properties of the aforementioned ML EVI-estimators has been undertaken in Drees *et al.* ([21]).

**Remark 4.2.** Further note that all the aforementioned EVI-estimators in this section are scale invariant. The GPPWM and the ML EVI-estimators are also location invariant, and can be regarded as classes of PORT EVI-estimators. We can further consider PORT-M, GH and MM EVI-estimators.

Under the validity of the second-order condition in (2.1), and for intermediate  $k = k_n$ , (2.3) holds, with

$$b_{\text{M}} = b_{\text{GH}} = \frac{\xi - \xi\rho + \rho}{\xi(1 - \rho)^2}, \quad \sigma_{\text{M}}^2 = \sigma_{\text{GH}}^2 = 1 + \xi^2,$$

$$b_{\text{MM}} = b_{\text{ML}} = \frac{(1 + \xi)(\xi + \rho)}{\xi(1 - \rho)(1 + \xi - \rho)}, \quad \sigma_{\text{MM}}^2 = \sigma_{\text{ML}}^2 = (1 + \xi)^2,$$

and for  $\xi < 1/2$ ,

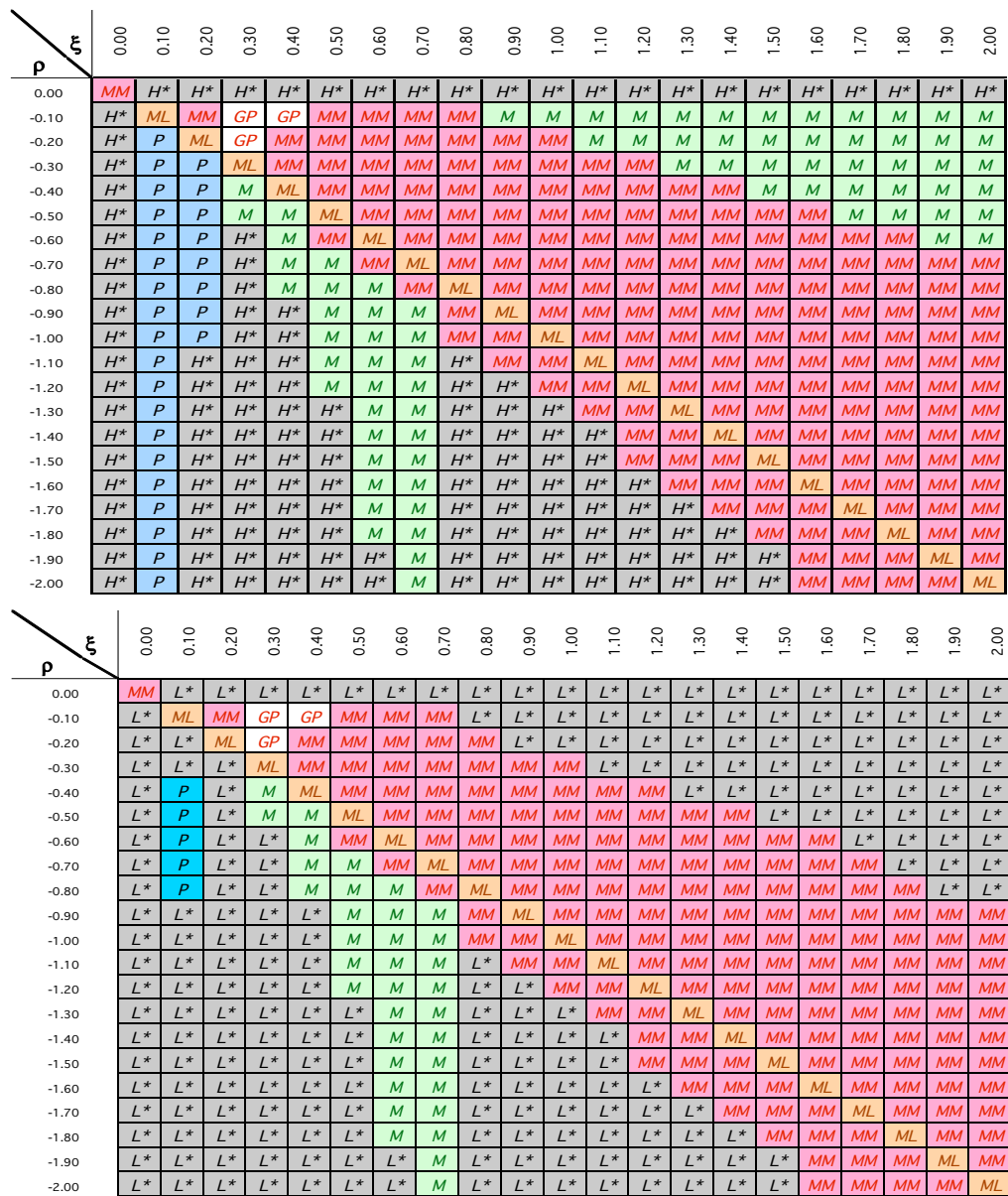
$$b_{\text{GPPWM}} = \frac{(\xi + \rho) b_{\text{PPWM}}}{\xi} \quad \text{and} \quad \sigma_{\text{GPPWM}}^2 = \frac{(1 - \xi + 2\xi^2)(1 - \xi)(2 - \xi)^2}{(1 - 2\xi)(3 - 2\xi)}.$$

As happened before with the optimal  $\text{MO}_p$  EVI-estimator, the optimal Lehmer EVI-estimator can be beaten by the optimal M (and GH) EVI-estimator in a region close to  $\xi = -\rho/(1 - \rho)$ , where  $b_{\text{M}} = b_{\text{GH}} = 0$ . The optimal MM EVI-estimator in (4.4), asymptotically equivalent to the optimal ML-estimator, unless  $\xi + \rho = 0$  and  $(\xi, \rho) \neq (0, 0)$ , outperforms the M EVI-estimator at optimal levels, in a region around  $\xi + \rho = 0$ , and can even outperform the optimal Lehmer EVI-estimator. The GPPWM EVI-estimator, in (4.3), is RB for  $\xi + \rho = 0$ , and can beat the MM EVI-estimator in a short region of the  $(\xi, \rho)$ -plane, as can be seen in Figure 6, where we exhibit the comparative behaviour of all ‘classical’ EVI-estimators under consideration, including both the  $L^*$  and the  $H^*$  classes (Figure 6, *bottom*), after including only the  $H^*$  class (Figure 6, *top*), as done in Brillhante *et al.* ([8]). The GPPWM and PPWM EVI-estimators are respectively denoted by GP and P. The PPWM, despite of non-RB, can beat even the optimal Lehmer for a few values of  $\xi$  around 0.1, as detected before (see also Figure 5, *bottom*).

**Remark 4.3.** As already mentioned in Brillhante *et al.* ([8]), note that in the region  $\xi + \rho \neq 0$  and  $\xi \neq -\rho/(1 - \rho)$ , where a further study under the third-order framework is needed, all RB EVI-estimators, like the corrected-Hill EVI-estimators in Caeiro *et al.* ([16]), overpass at optimal levels all classical and non-RB EVI-estimators available in the literature. They were thus not included in Figure 6, so that we can see the comparative behaviour of the non-RB EVI-estimators. A similar comment applies to the optimal  $\text{CG}_{p,\delta}$  EVI-estimators, in (1.11).

**Remark 4.4.** As expected, none of the estimators can always dominate the alternatives, but the  $L_p$  EVI-estimators have a quite interesting performance, being unexpectedly able to beat the  $\text{MO}_p \equiv H_p$  EVI-estimators at optimal levels in the whole  $(\xi, \rho)$ -plane.

**Remark 4.5.** For a final adaptive EVI-estimation, i.e. for the choice of  $(k, p)$  in (1.7), a double-bootstrap algorithm, of the type of Algorithm 4.1 in Brillhante *et al.* ([8]), now based on the asymptotic behaviour in (2.7), can be used. Such an algorithm relies on the minimization of a bootstrap estimate of the AMSE. Also, the slight modification of the semi-parametric bootstrap method in Caers *et al.* ([18]), provided in the Algorithm 4.3 of Caeiro and Gomes ([13]) is expected to provide an adequate estimation of the bootstrap MSE. Alternatively, one can use any of the available methods based on sample-path stability (see also Caeiro and Gomes, [13], among others).



**Figure 6:** Comparative overall behaviour of the EVI-estimators under study, considering only the optimal  $H_p$ , denoted  $H^*$  (top) and including both  $H^*$  and the optimal  $L_p$ , denoted  $L^*$  (bottom).

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