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MULTIPLICATION IS AN OPEN BILINEAR MAPPING IN THE BANACH ALGEBRA OF FUNCTIONS OF BOUNDED WIENER p-VARIATION

Abstract

Let $BV_p[0, 1]$, $1 \le p < \infty$, be the Banach algebra of functions of bounded p-variation in the sense of Wiener. Recently, Kowalczyk and Turowska [9] proved that the multiplication in $BV_1[0, 1]$ is an open bilinear mapping. We extend this result for all values of $p \in [1, \infty)$.

1 Introduction.

Let $\mathcal{A}$ be a Banach algebra with a Banach algebra norm $\|\cdot\|_{\mathcal{A}}$. We denote by $B_{\mathcal{A}}(a, \varepsilon)$ the open ball in $\mathcal{A}$ centered at a of radius $\varepsilon > 0$, that is,

$$B_{\mathcal{A}}(a, \varepsilon) := \{b \in \mathcal{A} : \|a - b\|_{\mathcal{A}} < \varepsilon\}.$$

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We say that the multiplication in $\mathcal{A}$ is a bilinear mapping locally open at a pair $(a,b) \in \mathcal{A}^2 := \mathcal{A} \times \mathcal{A}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$B_{\mathcal{A}}(a \cdot b, \delta) \subset B_{\mathcal{A}}(a, \varepsilon) \cdot B_{\mathcal{A}}(b, \varepsilon),$$

where

$$B_{\mathcal{A}}(a, \varepsilon) \cdot B_{\mathcal{A}}(b, \varepsilon) := \{c \cdot d \in \mathcal{A} \ : \ c \in B_{\mathcal{A}}(a, \varepsilon), \ d \in B_{\mathcal{A}}(b, \varepsilon)\}.$$

Following [9], the multiplications in $\mathcal{A}$ is called an open bilinear mapping if it is locally open at every pair $(a,b) \in \mathcal{A}^2$.

Note that the multiplication might not be an open bilinear mapping even in very simple situations. For instance, if $\mathcal{A} = C[0,1]$ is the algebra of real continuous functions with the supremum norm

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|, \tag{1.1}$$

then for the function $g = x - 1/2$ one has

$$g^2 \in (B_{\mathcal{A}}(g, 1/2))^2 \setminus \operatorname{int}\big((B_{\mathcal{A}}(g, 1/2))^2\big),$$

where $\operatorname{int}(S)$ denotes the interior of a set S (see [3]). Thus, the multiplication is not an open bilinear mapping in the algebra $C[0,1]$. This result was extended in [11] to the case of the algebra $C^n[0,1]$ of n times continuously differentiable functions.

The aim of this paper is to show that the multiplication is an open biliniear mapping in the Banach algebra $BV_p[0,1]$, $1 \le p < \infty$, of functions of bounded Wiener p-variation, extending the recent result by Kowlaczyk and Turowska [9] for $p = 1$ to all values $p \in [1, \infty)$.

Let us recall the definition of functions of bounded Wiener p-variation. Suppose that $0 \le \alpha \le \beta \le 1$. Let $\mathcal{P}[\alpha, \beta]$ be the set of all partitions $P = \{t_0, \dots, t_m\}$ of the segment $[\alpha, \beta]$ of the form

$$\alpha = t_0 < t_1 < \dots < t_m = \beta.$$

Following [13] and [2, Definition 1.31], for a given a real number $p \in [1, \infty)$, a partition $P = \{t_0, \dots, t_m\} \in \mathcal{P}[\alpha, \beta]$ and a function $f : [\alpha, \beta] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the nonnegative number

$$\operatorname{Var}_p(f, P, [\alpha, \beta]) := \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^p$$

is called the Wiener p-variation of f on $[\alpha, \beta]$ with respect to P, while the (possibly infinite) number

$$\mathrm{Var}_p(f, [\alpha, \beta]) := \sup\{\mathrm{Var}_p(f, P, [\alpha, \beta]) : P \in \mathcal{P}[\alpha, \beta]\},$$

where the supremum is taken over all partitions of $[\alpha, \beta]$, is called the total Wiener p-variation of f on $[\alpha, \beta]$. Let

$$BV_p[0, 1] := \{f : [0, 1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\} : \mathrm{Var}_p(f, [0, 1] < \infty\}$$

be the set of all functions of bounded Wiener p-variation. It is well known that $BV_p[0, 1]$ is a Banach algebra with respect to the pointwise multiplication and the norm

$$\|f\|_{BV_p} := \|f\|_\infty + \big(\mathrm{Var}_p(f, [0, 1])\big)^{1/p}, \tag{1.2}$$

where $\|f\|_\infty$ is given by (1.1) (for instance, this result follows from [6, Theorem 3.7 and Corollary 3.8] with $\Phi(t) = t^p$, $1 \le p < \infty$).

Theorem 1.1 (Main result)**.** *Let $1 \le p < \infty$. Then the multiplication in the Banach algebra $BV_p[0, 1]$ is an open bilinear mapping.*

The paper is organized as follows. In Section 2, following the main lines of the proof of [9, Theorem 2.4], we show that the multiplication in a Banach algebra continuously embedded into the Banach algebra $B[0, 1]$ of bounded functions and satisfying natural assumptions (the so-called symmetry property, the inverse closedness property and the selection principle) is locally open at every pair of functions (F, G) such that $|F| + |G|$ is bounded away from zero. We call such functions F and G jointly nondegenerate. Further, we show that the Banach algebra $BV_p[0, 1]$ of functions of bounded p-variation in the Wiener sense and the Banach algebra $\Lambda_p BV[0, 1]$ of functions of bounded variation in the Shiba-Waterman sense (see [7, 10, 12]) satisfy the hypotheses of the above result. In Section 3, we extend [9, Lemma 2.1] from the setting of $BV_1[0, 1]$ to the setting of $BV_p[0, 1]$ with an arbitrary $p \ge 1$. We should note that the passage from $p = 1$ to an arbitrary $p \ge 1$ is not trivial. In Section 4, with the aid of the main result of Section 3 and following the scheme of the proof of [9, Theorem 2.2], we show that an arbitrary pair of functions $(F, G) \in (BV_p[0, 1])^2$ can be approximated by a pair of jointly nondegenerate functions $(F_1, G_1) \in (BV_p[0, 1])^2$ such that $F \cdot G = F_1 \cdot G_1$. In Section 5, we prove Theorem 1.1 combining the results of Sections 2 and 4. As a corollary of Theorem 1.1 and [5, Propositions 2.4 and 4.1] we get that the set of all invertible elements $\mathcal{G}BV_p[0, 1]$ of the algebra $BV_p[0, 1]$ is dense in $BV_p[0, 1]$ for all $p \in [1, \infty)$. We conclude the paper with the conjecture that multiplication is an open bilinear mapping also in the Banach algebra $\Lambda_p BV[0,1]$ of functions of bounded variation in the sense of Shiba-Waterman.

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2 Local openness of multiplication in algebras of bounded functions

Let $B[0,1]$ denote the Banach algebra of all bounded functions $f:[0,1]\to\mathbb{F}$, where $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, with the norm given by (1.1). We say that functions $f,g\in B[0,1]$ are jointly nondegenerate if

$$\inf_{x\in[0,1]}\big(|f(x)|+|g(x)|\big)>0.$$

Let $\mathcal{F}[0,1]$ be a Banach algebra equipped with a norm $\|\cdot\|_{\mathcal{F}}$ and continuously embedded into the algebra $B[0,1]$. We will say that the algebra $\mathcal{F}[0,1]$ satisfies the symmetry property if for every function $f\in\mathcal{F}[0,1]$, its complex conjugate $\overline{f}$ also belongs to $\mathcal{F}[0,1]$ and $\|\overline{f}\|_{\mathcal{F}}=\|f\|_{\mathcal{F}}$. It is clear that every real algebra $\mathcal{F}[0,1]$ has the symmetry property.

Further, we will say that $\mathcal{F}[0,1]$ satisfies the inverse closedness property if for every $f\in\mathcal{F}[0,1]$, the inequality

$$\inf_{x\in[0,1]}|f(x)|>0$$

implies that $1/f\in\mathcal{F}[0,1]$ and

$$\left\|\frac{1}{f}\right\|_{\mathcal{F}}\le\left(\inf_{x\in[0,1]}|f(x)|\right)^{-2}\|f\|_{\mathcal{F}}.$$

Finally, we will say that $\mathcal{F}[0,1]$ satisfies the selection principle if from every sequence of functions $\{f_n\}$ satisfying

$$\sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{F}}<\infty$$

one can extract a subsequence $\{f_{n_k}\}$ that converges pointwise on $[0,1]$ to a function $f\in\mathcal{F}[0,1]$.

Theorem 2.1. *Let $\mathcal{F}[0,1]$ be a Banach algebra continuously embedded into the Banach algebra $B[0,1]$. Suppose that the algebra $\mathcal{F}[0,1]$ satisfies the symmetry property, the inverse closedness property and the selection principle. Then the multiplication in $\mathcal{F}[0,1]$ is locally open at every pair of jointly nondegenerate functions $(F,G)\in(\mathcal{F}[0,1])^2$.*

PROOF. The proof is analogous to that of [9, Theorem 2.4]. Since $\mathcal{F}[0,1]$ is continuously embedded into $B[0,1]$, there is a constant $C \geq 1$ such that for all $f \in \mathcal{F}[0,1]$,

$$\sup_{x\in[0,1]} |f(x)| \leq C\|f\|_{\mathcal{F}}. \tag{2.1}$$

Without loss of generality, we can suppose that $\varepsilon \in (0,1)$. Take

$$\delta := \min\left\{1, \frac{1}{2}\inf_{x\in[0,1]}\big(|F(x)|+|G(x)|\big)\right\} \tag{2.2}$$

and

$$K := 2\max\big\{\|F\|_{\mathcal{F}}, \|G\|_{\mathcal{F}}, 1\big\}. \tag{2.3}$$

Let $h \in \mathcal{F}[0,1]$ be such that

$$\|h\|_{\mathcal{F}} < \varepsilon \cdot \frac{\delta^8}{128CK^6}. \tag{2.4}$$

Consider

$$F_0 := F, \quad G_0 := G, \quad h_0 := h \tag{2.5}$$

and define sequences $\{F_n\}_{n=0}^{\infty}$, $\{G_n\}_{n=0}^{\infty}$, and $\{h_n\}_{n=0}^{\infty}$ inductively by

$$F_{n+1} := F_n + h_n \cdot \frac{\overline{G_n}}{|F_n|^2+|G_n|^2}, \tag{2.6}$$

$$G_{n+1} := G_n + h_n \cdot \frac{\overline{F_n}}{|F_n|^2+|G_n|^2}, \tag{2.7}$$

$$h_{n+1} := -h_n^2 \cdot \frac{\overline{F_n G_n}}{(|F_n|^2+|G_n|^2)^2}. \tag{2.8}$$

We claim that for $n \in \mathbb{N}\cup\{0\}$,

(i)
$$F_nG_n + h_n = FG + h,$$

(ii)
$$\|F_n\|_{\mathcal{F}} \leq \frac{K}{2} + 1 - 2^{-n}, \quad \|G_n\|_{\mathcal{F}} \leq \frac{K}{2} + 1 - 2^{-n},$$

(iii)
$$\inf_{x\in[0,1]}\big(|F_n(x)|+|G_n(x)|\big) \geq \delta + \delta\cdot 2^{-n},$$

(iv)

$$\|h_n\|_{\mathcal{F}} \le \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6}.$$

We will prove these claims by induction. It follows from (2.5) that

$$F_0G_0 + h_0 = FG + h.$$

We obtain from (2.2)–(2.5) that

$$\|F_0\|_{\mathcal{F}} = \|F\|_{\mathcal{F}} \le \frac{K}{2}, \quad \|G_0\|_{\mathcal{F}} = \|G\|_{\mathcal{F}} \le \frac{K}{2}, \quad \|h_0\|_{\mathcal{F}} = \|h\|_{\mathcal{F}} < \varepsilon \cdot \frac{\delta^8}{128CK^6},$$

$$\inf_{x\in[0,1]} \big(|F_0(x)| + |G_0(x)|\big) = \inf_{x\in[0,1]} \big(|F(x)| + |G(x)|\big) \ge 2\delta.$$

That is, (i)–(iv) are satisfied for $n = 0$.

Now we assume that (i)–(iv) are fulfilled for some $n \in \mathbb{N} \cup \{0\}$. Then, taking into account (2.3), we see that $K/2 \ge 1$ and

$$F_nG_n + h_n = FG + h, \tag{2.9}$$

$$\|F_n\|_{\mathcal{F}} \le \frac{K}{2} + 1 - 2^{-n} < K, \tag{2.10}$$

$$\|G_n\|_{\mathcal{F}} \le \frac{K}{2} + 1 - 2^{-n} < K, \tag{2.11}$$

$$\inf_{x\in[0,1]} \big(|F_n(x)| + |G_n(x)|\big) \ge \delta + \delta \cdot 2^{-n} > \delta, \tag{2.12}$$

$$\|h_n\|_{\mathcal{F}} \le \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6}. \tag{2.13}$$

Let us show that (i)–(iv) are fulfilled for $n + 1$.

(i) It follows from (2.6)–(2.9) that

$$\begin{aligned}
&F_{n+1}G_{n+1} + h_{n+1} \\
&\quad = \left(F_n + \frac{h_n \cdot \overline{G_n}}{|F_n|^2 + |G_n|^2}\right)\left(G_n + \frac{h_n \cdot \overline{F_n}}{|F_n|^2 + |G_n|^2}\right) - \frac{h_n^2 \cdot \overline{F_nG_n}}{(|F_n|^2 + |G_n|^2)^2} \\
&\quad = F_nG_n + h_n\frac{F_n\overline{F_n} + G_n\overline{G_n}}{|F_n|^2 + |G_n|^2} + h_n^2\frac{\overline{F_nG_n}}{(|F_n|^2 + |G_n|^2)^2} - h_n^2\frac{\overline{F_nG_n}}{(|F_n|^2 + |G_n|^2)^2} \\
&\quad = F_nG_n + h_n = FG + h.
\end{aligned}$$

Hence, (i) is satisfied for $n + 1$.

(ii) Since $\mathcal{F}[0,1]$ is a Banach algebra satisfying the symmetry property, we obtain from (2.10) and (2.11) that

$$\begin{aligned}\||F_n|^2+|G_n|^2\|_{\mathcal{F}} &\le \|F_n\cdot\overline{F_n}\|_{\mathcal{F}}+\|G_n\cdot\overline{G_n}\|_{\mathcal{F}}\\ &\le \|F_n\|_{\mathcal{F}}\|\overline{F_n}\|_{\mathcal{F}}+\|G_n\|_{\mathcal{F}}\|\overline{G_n}\|_{\mathcal{F}}\\ &= \|F_n\|_{\mathcal{F}}^2+\|G_n\|_{\mathcal{F}}^2\le K^2+K^2=2K^2. \qquad (2.14)\end{aligned}$$

It follows from (2.12) that for every $x\in[0,1]$,

$$\begin{aligned}\delta^2 &\le \big(|F_n(x)|+|G_n(x)|\big)^2=|F_n(x)|^2+2|F_n(x)|\cdot|G_n(x)|+|G_n(x)|^2\\ &\le 2\big(|F_n(x)|^2+|G_n(x)|^2\big).\end{aligned}$$

Hence

$$\inf_{x\in[0,1]}\big(|F_n(x)|^2+|G_n(x)|^2\big)\ge\frac{\delta^2}{2}. \qquad (2.15)$$

Taking into account that $\mathcal{F}[0,1]$ is a Banach algebra with the symmetry property, it follows from (2.6) and (2.10)–(2.11) that

$$\begin{aligned}\|F_{n+1}\|_{\mathcal{F}} &\le \|F_n\|_{\mathcal{F}}+\|h_n\|_{\mathcal{F}}\|G_n\|_{\mathcal{F}}\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_{\mathcal{F}}\\ &\le \left(\frac{K}{2}+1-2^{-n}\right)+\|h_n\|_{\mathcal{F}}K\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_{\mathcal{F}}. \qquad (2.16)\end{aligned}$$

Since $\mathcal{F}[0,1]$ has the inverse closedness property, we deduce from (2.14)–(2.15) that

$$\begin{aligned}\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_{\mathcal{F}} &\le \left(\inf_{x\in[0,1]}\big(|F_n(x)|^2+|G_n(x)|^2\big)\right)^{-2}\||F_n|^2+|G_n|^2\|_{\mathcal{F}}\\ &\le \left(\frac{2}{\delta^2}\right)^2 2K^2=\frac{8K^2}{\delta^4}. \qquad (2.17)\end{aligned}$$

Combining (2.16)–(2.17) with (2.13) and taking into account that $\varepsilon\in(0,1)$ and $C\ge 1$, we obtain

$$\begin{aligned}\|F_{n+1}\|_{\mathcal{F}} &\le \frac{K}{2}+1-2^{-n}+\frac{8K^3}{\delta^4}\cdot\varepsilon\cdot 2^{-n}\cdot\frac{\delta^8}{128CK^6}\\ &< \frac{K}{2}+1-2^{-n}+2^{-n}\frac{\delta^4}{16K^3}. \qquad (2.18)\end{aligned}$$

It follow from (2.2)–(2.3) that $\delta\le 1\le K/2$. Therefore

$$\frac{\delta^4}{16K^3}=\frac{\delta}{16}\left(\frac{\delta}{K}\right)^3\le\frac{\delta}{16}\cdot\frac{1}{8}=\frac{\delta}{128}<\frac{1}{2}. \qquad (2.19)$$

In view of (2.18)–(2.19) we obtain

$$\|F_{n+1}\|_{\mathcal{F}} < \frac{K}{2} + 1 - 2^{-n} + 2^{-n-1} = \frac{K}{2} + 1 - 2^{-n-1}.$$

Analogously it can be shown that

$$\|G_{n+1}\|_{\mathcal{F}} < \frac{K}{2} + 1 - 2^{-n-1}.$$

Thus, (ii) is fulfilled for $n+1$.

(iii) Since $\mathcal{F}[0,1]$ is a Banach algebra and $\varepsilon \in (0,1)$, it follows from (2.6), (2.1), (2.11), (2.13), (2.17), and (2.19) that for $x \in [0,1]$,

$$\begin{aligned}
|F_n(x)| &\le |F_{n+1}(x)| + |h_n(x)| \frac{|G_n(x)|}{|F_n(x)|^2 + |G_n(x)|^2} \\
&\le |F_{n+1}(x)| + C\|h_n\|_{\mathcal{F}} \|G_n\|_{\mathcal{F}} \left\| \frac{1}{|F_n|^2 + |G_n|^2} \right\|_{\mathcal{F}} \\
&\le |F_{n+1}(x)| + C\varepsilon \cdot 2^{-n} \frac{\delta^8}{128CK^6} \cdot K \cdot \frac{8K^2}{\delta^4} \\
&< |F_{n+1}(x)| + 2^{-n} \cdot \frac{\delta^4}{16K^3} \\
&< |F_{n+1}(x)| + 2^{-n} \cdot \frac{\delta}{4}.
\end{aligned}$$

Hence

$$|F_{n+1}(x)| > |F_n(x)| - 2^{-n-2}\delta, \quad x \in [0,1]. \tag{2.20}$$

Analogously,

$$|G_{n+1}(x)| > |F_n(x)| - 2^{-n-2}\delta, \quad x \in [0,1]. \tag{2.21}$$

We conclude from (2.12) and (2.20)–(2.21) that

$$\begin{aligned}
\inf_{x\in[0,1]} \big(|F_{n+1}(x)| + |G_{n+1}(x)|\big) &\ge \inf_{x\in[0,1]} \big(|F_n(x)| + |G_n(x)|\big) - 2 \cdot 2^{-n-2}\delta \\
&\ge \delta + \delta \cdot 2^{-n} - \delta \cdot 2^{-n-1} = \delta + \delta \cdot 2^{-n-1}.
\end{aligned}$$

Hence (iii) is fulfilled for $n+1$.

(iv) Since $\mathcal{F}[0,1]$ is a Banach algebra with the symmetry property, $\varepsilon \in (0,1)$

and $C \geq 1$, it follows from (2.8), (2.10)–(2.11), (2.13) and (2.17) that

$$\begin{aligned}
\|h_{n+1}\|_{\mathcal{F}} &\leq \|h_n\|_{\mathcal{F}}^2 \|\overline{F_n}\|_{\mathcal{F}} \|\overline{G_n}\|_{\mathcal{F}} \left\| \frac{1}{|F_n|^2 + |G_n|^2} \right\|_{\mathcal{F}}^2 \\
&= \|h_n\|_{\mathcal{F}}^2 \|F_n\|_{\mathcal{F}} \|G_n\|_{\mathcal{F}} \left\| \frac{1}{|F_n|^2 + |G_n|^2} \right\|_{\mathcal{F}}^2 \\
&\leq \left(\varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6} \right)^2 K^2 \left(\frac{8K^2}{\delta^4} \right)^2 \\
&= \varepsilon^2 \cdot 2^{-2n-1} \cdot \frac{\delta^8}{128C^2K^6} \\
&< \varepsilon \cdot 2^{-n-1} \cdot \frac{\delta^8}{128CK^6}.
\end{aligned}$$

Hence (iv) is fulfilled for $n+1$.

Thus, we have verified properties (i)–(iv) by induction for all $n \in \mathbb{N} \cup \{0\}$.

In view of (ii), the terms of the sequences $\{F_n\}_{n=0}^{\infty}$ and $\{G_n\}_{n=0}^{\infty}$ have uniformly bounded norms. By the selection principle, there exist a subsequence $\{F_{n_k}\}_{k=0}^{\infty}$ of $\{F_n\}_{n=0}^{\infty}$ and a subsequence $\{G_{n_k}\}_{k=0}^{\infty}$ of $\{G_n\}_{n=0}^{\infty}$ such that for every $x \in [0,1]$,

$$\lim_{k\to\infty} F_{n_k}(x) = f(x), \quad \lim_{k\to\infty} G_{n_k}(x) = g(x), \tag{2.22}$$

where $f, g \in \mathcal{F}[0,1]$. It follows from (2.1) and (iv) that for all $x \in [0,1]$,

$$\lim_{n\to\infty} |h_n(x)| \leq C \lim_{n\to\infty} \|h_n\|_{\mathcal{F}} \leq \frac{\varepsilon\delta^8}{128CK^6} \lim_{n\to\infty} 2^{-n} = 0. \tag{2.23}$$

In view of (i) and (2.22)–(2.23), we obtain for $x \in [0,1]$,

$$\begin{aligned}
f(x)g(x) &= \lim_{k\to\infty} F_{n_k}(x)G_{n_k}(x) = \lim_{k\to\infty} \big(F_{n_k}(x)G_{n_k}(x) + h_{n_k}(x) \big) \\
&= F(x)G(x) + h(x). \qquad (2.24)
\end{aligned}$$

Since

$$\begin{aligned}
f(x) - F(x) &= \lim_{k\to\infty} (F_{n_k}(x) - F(x)) = \lim_{k\to\infty} \sum_{j=0}^{n_k} (F_{j+1}(x) - F_j(x)) \\
&= \sum_{n=0}^{\infty} (F_{n+1}(x) - F_n(x)),
\end{aligned}$$

$\mathcal{F}[0,1]$ is a Banach algebra with the symmetry property, $\varepsilon \in (0,1)$ and $C \geq 1$, we obtain from (2.6), (2.11), (2.13), (2.17), and (2.19) that

$$\begin{aligned} \|f-F\|_{\mathcal{F}} &\leq \sum_{n=0}^{\infty} \|F_{n+1}-F_n\|_{\mathcal{F}} \\ &\leq \sum_{n=0}^{\infty} \|h_n\|_{\mathcal{F}} \|G_n\|_{\mathcal{F}} \left\| \frac{1}{|F_n|^2+|G_n|^2} \right\|_{\mathcal{F}} \\ &\leq \sum_{n=0}^{\infty} \varepsilon \cdot 2^{-n} \cdot \frac{\delta^8}{128CK^6} \cdot K \cdot \frac{8K^2}{\delta^4} \\ &= \frac{\varepsilon}{C} \cdot \frac{\delta^4}{16K^3} \sum_{n=0}^{\infty} 2^{-n} < \varepsilon. \end{aligned} \tag{2.25}$$

Analogously we can show that

$$\|g-G\|_{\mathcal{F}} < \varepsilon. \tag{2.26}$$

So, for every $h \in \mathcal{F}[0,1]$ satisfying (2.4), there exist f and g in $\mathcal{F}[0,1]$ such that (2.25) and (2.26) hold, and $FG+h=fg$ (see (2.24)). This means that

$$B_{\mathcal{F}[0,1]}(F\cdot G,\eta) \subset B_{\mathcal{F}[0,1]}(F,\varepsilon)\cdot B_{\mathcal{F}[0,1]}(G,\varepsilon)$$

with $\eta := \varepsilon \cdot \frac{\delta^8}{128CK^6}$. Hence, the multiplication in the Banach algebra $\mathcal{F}[0,1]$ is locally open at the pair $(F,G) \in (\mathcal{F}[0,1])^2$. □ □

Corollary 2.2. *Let $1 \leq p < \infty$. Then the multiplication in $BV_p[0,1]$ is locally open at every pair of jointly nondegenerate functions $(F,G) \in (BV_p[0,1])^2$.*

Proof. We have to verify the hypotheses of Theorem 2.1. The definitions of the norms (1.2) and (1.1) immediately imply that the Banach algebra $BV_p[0,1]$ is continuously embedded into the Banach algebra $B[0,1]$ (with the embedding constant 1) and that the algebra $BV_p[0,1]$ satisfies the symmetry property. It follows from the Helly-type selection theorem [2, Theorem 2.49] with $\Phi(t)=t^p$, $1 \leq p < \infty$, that $BV_p[0,1]$ satisfies the selection principle.

Let us show that $BV_p[0,1]$ has the inverse closedness property. Take a function $f \in BV_p[0,1]$ such that

$$\inf_{x\in[0,1]} |f(x)| > 0 \tag{2.27}$$

and a partition $P = \{t_0, \ldots, t_m\} \in \mathcal{P}[0,1]$. Then $f(t_j) \neq 0$ for $j \in \{0, \ldots, m\}$ in view of (2.27) and

$$\begin{aligned}\mathrm{Var}_p(1/f, P, [0,1]) &= \sum_{j=1}^{m} \left| \frac{1}{f(t_j)} - \frac{1}{f(t_{j-1})} \right|^p = \sum_{j=1}^{m} \left| \frac{f(t_j) - f(t_{j-1})}{f(t_j) f(t_j)} \right|^p \\ &\leq \left(\inf_{x \in [0,1]} |f(x)| \right)^{-2p} \mathrm{Var}_p(f, P, [0,1]).\end{aligned}$$

Therefore

$$\mathrm{Var}_p(1/f, [0,1]) \leq \left(\inf_{x \in [0,1]} |f(x)| \right)^{-2p} \mathrm{Var}_p(f, [0,1]). \tag{2.28}$$

On the other hand,

$$\|1/f\|_\infty = \sup_{x \in [0,1]} |1/f(x)| = \left(\inf_{x \in [0,1]} |f(x)| \right)^{-1}. \tag{2.29}$$

Combining (2.28) and (2.29), we arrive at the following:

$$\begin{aligned}\|1/f\|_{BV_p} &= \|1/f\|_\infty + \big(\mathrm{Var}_p(1/f, [0,1]) \big)^{1/p} \\ &\leq \left(\inf_{x \in [0,1]} |f(x)| \right)^{-1} + \left(\inf_{x \in [0,1]} |f(x)| \right)^{-2} \big(\mathrm{Var}_p(f, [0,1] \big)^{1/p} \\ &\leq \left(\inf_{x \in [0,1]} |f(x)| \right)^{-2} \Big(\|f\|_\infty + \big(\mathrm{Var}_p(f, [0,1] \big)^{1/p} \Big) \\ &= \left(\inf_{x \in [0,1]} |f(x)| \right)^{-2} \|f\|_{BV_p}.\end{aligned} \tag{2.30}$$

Thus $BV_p[0,1]$ satisfies the inverse closedness property. It remains to apply Theorem 2.1. □ □

Let us show that the hypotheses of Theorem 2.1 are also satisfied in the case of Banach algebras of functions of generalized variation in the Shiba-Waterman sense. Shiba [10] introduced the class $\Lambda_p BV[0,1]$ with $1 \leq p < \infty$, extending the concept of the bounded Λ-variation in the sense of Waterman [12]. Let $\Lambda = \{\lambda_i\}_{i=1}^\infty$ be a nondecreasing sequence of positive numbers such that $\sum_{i=1}^\infty \frac{1}{\lambda_i} = +\infty$ and let $1 \leq p < \infty$. A function $f : [0,1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is said to be of bounded Λ_p-variation in the Shiba-Waterman sense if

$$\mathrm{Vap}_{\Lambda_p}(f, [0,1]) := \sup \sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping intervals on $[0,1]$ and $f(I_i) := f(\sup I_i) - f(\inf I_i)$. Let $\Lambda_p BV[0,1]$ be the set of all functions $f : [0,1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ of bounded Λ_p-variation. Kantorowitz [8, Theorem 1] proved that $\Lambda_p BV[0,1]$ is a Banach algebra with respect to the pontwise multiplication and the norm

$$\|f\|_{\Lambda_p BV} := \|f\|_\infty + \big(\operatorname{Vap}_{\Lambda_p}(f,[0,1])\big)^{1/p}. \tag{2.31}$$

Corollary 2.3. *Let $1 \le p < \infty$. Then the multiplication in the Banach algebra $\Lambda_p BV[0,1]$ is locally open at every pair of jointly nondegenerate functions $(F,G) \in (\Lambda_p BV[0,1])^2$.*

Proof. As in the proof of the previous corollary, we have to verify the hypotheses of Theorem 2.1. The definitions of the norms (2.31) and (1.1) immediately imply that the Banach algebra $\Lambda_p BV[0,1]$ is continuously embedded into the Banach algebra $B[0,1]$ (with the embedding constant 1) and that the algebra $\Lambda_p BV[0,1]$ satisfies the symmetry property. The selection principle for the algebra $\Lambda_p BV[0,1]$ is proved in [7, Theorem 3.2].

If $f \in \Lambda_p BV[0,1]$ satisifies (2.27), then for every interval $I \subset [0,1]$,

$$|(1/f)(I)| \le \left(\inf_{x\in[0,1]} |f(x)|\right)^{-2} |f(I)|.$$

Therefore

$$\operatorname{Var}_{\Lambda,p}(1/f,[0,1]) \le \left(\inf_{x\in[0,1]} |f(x)|\right)^{-2p} \operatorname{Vap}_{\Lambda_p}(f,[0,1]). \tag{2.32}$$

Combining (2.32) and (2.29), similarly to (2.30), we obtain

$$\|1/f\|_{\Lambda_p BV} \le \left(\inf_{x\in[0,1]} |f(x)|\right)^{-2} \|f\|_{\Lambda_p BV}.$$

Thus $\Lambda_p BV[0,1]$ satisfies the inverse closedness property. It remains to apply Theorem 2.1. □

3 Key lemma

The aim of this section is to prove an extension of [9, Lemma 2.1] for the Banach algebras $BV_p[0,1]$ with arbitrary $p \in [1,\infty)$.

Let us start with several elementary inequalities.

Lemma 3.1. *Let* $1 \le p < \infty$*. Then*

$$(1+x)^p \le 1 + p2^{p-1}x \quad \textit{for all} \quad x \in [0,1]. \tag{3.1}$$

PROOF. Integrating both sides of the inequality

$$(1+t)^{p-1} \le 2^{p-1}, \quad t \in [0,1]$$

from 0 to x, one gets

$$\frac{1}{p}\left((1+x)^p - 1\right) \le 2^{p-1}x,$$

which is equivalent to (3.1). □

Lemma 3.2. *Let* $1 \le p < \infty$*. Then*

$$(a+b)^p \le a^p + \max\{p,2\}\, 2^{p-1}b \quad \textit{for all} \quad a,b \in [0,1]. \tag{3.2}$$

PROOF. If $a = 0$ then (3.2) holds because $b^p \le b$. Suppose $a > 0$. If $b \le a$, then it follows from Lemma 3.1 that

$$\begin{aligned}(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p &\le a^p\left(1+p2^{p-1}\,\frac{b}{a}\right)\\ &= a^p + p2^{p-1}a^{p-1}b \le a^p + p2^{p-1}b.\end{aligned} \tag{3.3}$$

If $b > a$, then

$$(a+b)^p < (2b)^p = 2^p b^p < a^p + 2^p b^p \le a^p + 2^p b. \tag{3.4}$$

Estimate (3.2) follows from (3.3) and (3.4). □

Corollary 3.3. *Let* $1 \le p < \infty$ *and* $u, v \in \mathbb{C}$ *be such that* $|u-v|, |v| \le 1$*. Then*

$$|u-v|^p \ge |u|^p - \max\{p,2\}\, 2^{p-1}|v|. \tag{3.5}$$

PROOF. Using (3.2) with $a = |u-v|$ and $b = |v|$, one gets

$$|u|^p \le (|u-v| + |v|)^p \le |u-v|^p + \max\{p,2\}\, 2^{p-1}|v|,$$

which immediately implies (3.5). □

The following lemma is a special case of the desired result for functions with values in the segment $[0,1]$.

Lemma 3.4. *Let* $1 \le p < \infty$ *and let* $f \in BV_p[0,1]$ *be such that* $f : [0,1] \to [0,1]$. *For any* $\varepsilon > 0$ *there exist* $\eta > 0$ *such that if*

$$0 \le x_1 < x_2 < \cdots < x_m \le 1 \quad \textit{and} \quad f(x_j) < \eta,\ j = 1, \dots, m, \tag{3.6}$$

then

$$\left(\sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} < \varepsilon. \tag{3.7}$$

PROOF. Choose a partition $0 = y_1 < y_2 < \cdots < y_n = 1$ such that

$$\sum_{k=1}^{n-1} |f(y_{k+1}) - f(y_k)|^p > \mathrm{Var}_p(f, [0,1]) - \frac{\varepsilon^p}{2}.$$

Set

$$\eta = \min \left\{ 1, \frac{\varepsilon^p}{n(p+2)2^{p+1}} \right\}.$$

Suppose (3.6) holds. If $[y_k, y_{k+1}]$ contains some of the points $x_1, \dots, x_m$, let

$$j_k := \min\{j : \ x_j \in [y_k, y_{k+1}]\}, \quad J_k := \max\{j : \ x_j \in [y_k, y_{k+1}]\}.$$

Note that since $f \ge 0$, one has

$$(f(y_k))^p + (f(y_{k+1}))^p \ge (\max\{f(y_k), f(y_{k+1})\})^p \ge |f(y_{k+1}) - f(y_k)|^p.$$

Then using Corollary 3.3, one gets

$$\begin{aligned}
&|f(x_{j_k}) - f(y_k)|^p + |f(x_{j_k+1}) - f(x_{j_k})|^p + \cdots + |f(x_{J_k}) - f(x_{J_k-1})|^p \\
&\qquad + |f(y_{k+1}) - f(x_{J_k})|^p \\
&\quad \ge (f(y_k))^p - \max\{p,2\}\, 2^{p-1} f(x_{j_k}) + \sum_{j=j_k}^{J_k-1} |f(x_{j+1}) - f(x_j)|^p + (f(y_{k+1}))^p \\
&\quad\quad - \max\{p,2\}\, 2^{p-1} f(x_{J_k}) \\
&\quad \ge |f(y_{k+1}) - f(y_k)|^p - \max\{p,2\}\, 2^p \eta + \sum_{j=j_k}^{J_k} |f(x_{j+1}) - f(x_j)|^p - \eta^p \\
&\quad \ge |f(y_{k+1}) - f(y_k)|^p - (p+2)2^p \eta + \sum_{j=j_k}^{J_k} |f(x_{j+1}) - f(x_j)|^p,
\end{aligned}$$

where we take $f(x_{m+1}) = 0$ if $J_k = m$. In the last inequality above, we have used the following inequality

$$\max\{p, 2\} + 1 \leq p + 2.$$

Summing over k from 1 to $n-1$, one obtains

$$\begin{aligned}
&\mathrm{Var}_p(f, [0,1]) \\
&\geq \sum_{k=1}^{n-1} |f(y_{k+1}) - f(y_k)|^p - (n-1)(p+2)2^p\eta + \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \\
&> \mathrm{Var}_p(f, [0,1]) - \frac{\varepsilon^p}{2} - \frac{\varepsilon^p}{2} + \sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p,
\end{aligned}$$

which proves (3.7). □

We are now in a position to prove the main result of this section. For $p = 1$ the following lemma was proved in [9, Lemma 2.1].

Lemma 3.5 (Key lemma)**.** *Let* $1 \leq p < \infty$ *and* $f \in BV_p[0,1]$. *For any* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that if*

$$0 \leq x_1 < x_2 < \cdots < x_m \leq 1 \quad \textit{and} \quad |f(x_j)| < \delta \quad \textit{for} \quad j \in \{1, \ldots, m\},$$

then

$$\left(\sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} < \varepsilon.$$

Proof. There is nothing to prove if $f = 0$. So, we assume that $f \neq 0$. Let $M := \|f\|_\infty$, $f_0 := \frac{1}{M} f$. Let u and v be the real and the imaginary parts of f_0. Hence $f_0 = u + iv$. Consider the functions

$$w_1 = u_+ := \max\{u, 0\} = \frac{|u| + u}{2}, \quad w_2 = u_- := (-u)_+ = \frac{|u| - u}{2} = u_+ - u$$

and $w_3 = v_+$, $w_4 = v_-$. Then $f_0 = w_1 - w_2 + i(w_3 - w_4)$ and

$$0 \leq w_l \leq \|f_0\|_\infty = 1 \quad \text{for} \quad l \in \{1, 2, 3, 4\}.$$

Since $|a_+ - b_+| \leq |a - b|$ for all $a, b \in \mathbb{R}$, one also has

$$\mathrm{Var}_p(w_l, [0,1]) \leq \mathrm{Var}_p(f_0, [0,1]) = \frac{1}{M^p} \mathrm{Var}_p(f, [0,1]) \quad \text{for} \quad l \in \{1, 2, 3, 4\}.$$

Take an arbitrary $\varepsilon > 0$. It follows from Lemma 3.4 that for every $l \in \{1, 2, 3, 4\}$, there exists $\eta_l > 0$ such that

$$0 \le x_1 < x_2 < \cdots < x_m \le 1 \quad \text{and} \quad w_l(x_j) < \eta_l, \quad j = 1, \ldots, m$$

imply

$$\left(\sum_{j=1}^{m-1} |w_l(x_{j+1}) - w_l(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{4M}.$$

Let $\eta := M \min\{\eta_l : \; l = 1, 2, 3, 4\}$. If

$$0 \le x_1 < x_2 < \cdots < x_m \le 1 \quad \text{and} \quad |f(x_j)| < \eta, \; j = 1, \ldots, m,$$

then

$$w_l(x_j) < \frac{1}{M}\, \eta \le \eta_l, \; j = 1, \ldots, m,$$

and it follows from the above that

$$\begin{aligned} \left(\sum_{j=1}^{m-1} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} &= M \left(\sum_{j=1}^{m-1} |f_0(x_{j+1}) - f_0(x_j)|^p \right)^{1/p} \\ &\le M \sum_{l=1}^{4} \left(\sum_{j=1}^{m-1} |w_l(x_{j+1}) - w_l(x_j)|^p \right)^{1/p} \\ &< M \sum_{l=1}^{4} \frac{\varepsilon}{4M} = \varepsilon, \end{aligned}$$

which completes the proof. □

4 Approximating in $BV_p[0, 1]$ an arbitrary pair of functions by a pair of jointly nondegenerate functions

Let us start this section with two simple lemmas.

Lemma 4.1. *Let $1 \le p < \infty$ and $f \in BV_p[0, 1]$. Then f possesses a limit from the left and from the right at each point. Moreover f has a most countably many discontinuities.*

This statement can be proved as in the case $p = 1$ (see, e.g., [4, Proposition 1.32 and Corollary 1.33]).

Lemma 4.2. *Let* $1 \le p < \infty$, $\rho > 0$, *and* $f : (a,b) \to \mathbb{C}$ *be such that*

$$\inf_{x\in(a,b)} |f(x)| < \rho.$$

Then

$$\sup_{x\in(a,b)} |f(x)| \le \rho + \sup_{[\alpha,\beta]\subset(a,b)} \big(\operatorname{Var}_p(f,[\alpha,\beta])\big)^{1/p}. \tag{4.1}$$

Proof. There exists $x_0 \in (a,b)$ such that $|f(x_0)| < \rho$. Consider an arbitrary $x \in (a,b)$. Let $I_x \subset (a,b)$ be the segment with the endpoints x and x_0. By [2, Proposition 1.32(c)],

$$|f(x) - f(x_0)| \le \big(\operatorname{Var}_p(f,I_x)\big)^{1/p} \le \sup_{[\alpha,\beta]\subset(a,b)} \big(\operatorname{Var}_p(f,[\alpha,\beta])\big)^{1/p}.$$

Hence

$$\begin{aligned}|f(x)| &\le |f(x_0)| + \sup_{[\alpha,\beta]\subset(a,b)} \big(\operatorname{Var}_p(f,[\alpha,\beta])\big)^{1/p}\\ &< \rho + \sup_{[\alpha,\beta]\subset(a,b)} \big(\operatorname{Var}_p(f,[\alpha,\beta])\big)^{1/p}.\end{aligned}$$

Since $x \in (a,b)$ is arbitrary,

$$\sup_{x\in(a,b)} |f(x)| \le \rho + \sup_{[\alpha,\beta]\subset(a,b)} \big(\operatorname{Var}_p(f,[\alpha,\beta])\big)^{1/p},$$

which completes the proof. □

The next theorem says that an arbitrary pair of functions in $(BV_p[0,1])^2$ can be approximated by a pair of jointly nondegenerate functions with the same product.

Theorem 4.3. *Suppose that* $1 \le p < \infty$. *For every* $\varepsilon > 0$ *and every pair of functions* $(F,G) \in (BV_p[0,1])^2$ *there is a pair of jointly nondegenerate functions* $(F_1,G_1) \in (BV_p[0,1])^2$ *such that* $F\cdot G = F_1 \cdot G_1$ *and*

$$\|F - F_1\|_{BV_p} < \varepsilon, \quad \|G - G_1\|_{BV_p} < \varepsilon.$$

Proof. The idea of the proof is borrowed from the proof of [9, Theorem 2.2]. Fix $\varepsilon > 0$. By Lemma 3.5, we can find some $\delta > 0$ such that for every partition

$$0 \le x_1 < x_2 < \cdots < x_m \le 1,$$

we have

$$|F(x_j)| < \delta \text{ for } j \in \{1, \dots, m\} \;\Rightarrow\; \left(\sum_{j=1}^{m-1} |F(x_{j+1}) - F(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{48} \quad (4.2)$$

and

$$|G(x_j)| < \delta \text{ for } j \in \{1, \dots, m\} \Rightarrow \left(\sum_{j=1}^{m-1} |G(x_{j+1}) - G(x_j)|^p \right)^{1/p} < \frac{\varepsilon}{48}. \quad (4.3)$$

Take

$$\eta := \min \left\{ \delta, \frac{\varepsilon}{24}, \frac{1}{2} \sup_{x \in [0,1]} \big(|F(x)| + |G(x)| \big) \right\}. \quad (4.4)$$

By the representation theorem for open sets on the real line (see, e.g., [1, Theorem 3.11]), the interior of the set $\{x \in [0,1] : |F(x)| + |G(x)| < \eta\}$ is the union of at most countable collection of disjoint open intervals. Let A_0 be the collection of those open intervals $U = (a, b)$, $a < b$, in this union such that

$$\inf_{x \in U} \big(|F(x)| + |G(x)| \big) < \frac{\eta}{2}.$$

We claim that there are only finitely many intervals in A_0. Indeed, assume the contrary:

$$A_0 = \big\{ U_i = (a_i, b_i) \ : \ i \in \mathbb{N}, \ a_i < b_i \big\}.$$

Without loss of generality, we can assume that $b_i \le a_{i+1}$ for every $i \in \mathbb{N}$. Let $H := |F| + |G|$. By the definition of the infimum, for every $i \in \mathbb{N}$, there exists $x_i \in (a_i, b_i)$ such that $H(x_i) < \eta/2$. On the other hand, there is at least one point y_i such that $b_i \le y_i \le a_{i+1}$ and $H(y_i) \ge \eta$. Hence

$$\mathrm{Var}_p(H, [0,1]) \ge \sum_{i=1}^{\infty} |H(y_i) - H(x_i)|^p \ge \sum_{i=1}^{\infty} \left(\eta - \frac{\eta}{2} \right)^p = +\infty,$$

which is impossible since $H = |F| + |G| \in BV_p[0,1]$. Thus, for some $N \in \mathbb{N}$, we have

$$A_0 = \big\{ (a_1, b_1), \dots, (a_N, b_N) \big\}.$$

Let

$$\rho := \min \left\{ \frac{\eta}{2}, \frac{\varepsilon}{48N} \right\} \quad (4.5)$$

and let A be the part of A_0 consisting of the intervals (a_i, b_i) such that

$$\inf_{x\in(a_i,b_i)} \big(|F(x)| + |G(x)|\big) < \rho. \tag{4.6}$$

Relabelling $(a_i, b_i) \in A$ if necessary, we can assume

$$A = \big\{(a_1, b_1), \dots, (a_n, b_n)\big\},$$

where $n \le N$.

For $i \in \{1, \dots, n\}$, put

$$c_i := \max\left\{\sup_{x\in(a_i,b_i)} |F(x)|, \frac{\varepsilon}{24n}\right\},\ d_i := \max\left\{\sup_{x\in(a_i,b_i)} |G(x)|, \frac{\varepsilon}{24n}\right\}. \tag{4.7}$$

It follows from definitions (4.7), (4.4) and the definition of the collection A that

$$\max_{1\le i\le n} \max\{c_i, d_i\} \le \frac{\varepsilon}{24}. \tag{4.8}$$

Taking into account the definition of the collection A and (4.4), we see that for every $i \in \{1, \dots, n\}$, every interval $[\alpha, \beta] \subset (a_i, b_i)$ and every its partition $\alpha = x_1 < \cdots < x_m = \beta$, one has $|F(x_j)| < \delta$ and $|G(x_j)| < \delta$ for $j \in \{1, \dots, m\}$. Then (4.2)–(4.3) imply that

$$\sum_{i=1}^{n} \sup_{[\alpha,\beta]\subset(a_i,b_i)} \mathrm{Var}_p(F, [\alpha, \beta]) \le \left(\frac{\varepsilon}{48}\right)^p, \tag{4.9}$$

$$\sum_{i=1}^{n} \sup_{[\alpha,\beta]\subset(a_i,b_i)} \mathrm{Var}_p(G, [\alpha, \beta]) \le \left(\frac{\varepsilon}{48}\right)^p. \tag{4.10}$$

It follows from Lemma 4.2, definition (4.5), estimates (4.9)–(4.10), and the inequality

$$(t + \tau)^p \le 2^{p-1} \left(t^p + \tau^p\right), \quad t, \tau \ge 0 \tag{4.11}$$

that

$$\begin{aligned}
\sum_{i=1}^{n} \left(\sup_{x\in(a_i,b_i)} |F(x)|\right)^p &\le \sum_{i=1}^{n} \left(\rho + \sup_{[\alpha,\beta]\subset(a_i,b_i)} (\mathrm{Var}_p(F, [\alpha, \beta]))^{1/p}\right)^p \\
&\le \sum_{i=1}^{n} 2^{p-1} \left(\left(\frac{\varepsilon}{48N}\right)^p + \sup_{[\alpha,\beta]\subset(a_i,b_i)} \mathrm{Var}_p(F, [\alpha, \beta])\right) \\
&\le \left(\frac{\varepsilon}{24}\right)^p,
\end{aligned} \tag{4.12}$$

and

$$\sum_{i=1}^{n}\left(\sup_{x\in(a_i,b_i)}|G(x)|\right)^p \le \left(\frac{\varepsilon}{24}\right)^p.$$

Combining (4.7) and (4.12), we see that

$$\begin{aligned}\left(\sum_{i=1}^{n} c_i^p\right)^{1/p} &\le \left(\sum_{i=1}^{n}\left(\sup_{x\in(a_i,b_i)}|F(x)|\right)^p + \sum_{i=1}^{n}\left(\frac{\varepsilon}{24n}\right)^p\right)^{1/p}\\ &\le \left(\left(\frac{\varepsilon}{24}\right)^p + n\left(\frac{\varepsilon}{24n}\right)^p\right)^{1/p}\\ &\le \frac{\varepsilon}{24}+\frac{\varepsilon}{24}=\frac{\varepsilon}{12} \qquad (4.13)\end{aligned}$$

and, similarly,

$$\left(\sum_{i=1}^{n} d_i^p\right)^{1/p} \le \frac{\varepsilon}{12}. \qquad (4.14)$$

Define $f,g:[0,1]\to\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ by

$$f(x):=\begin{cases} F(x), & x\notin\bigcup_{i=1}^{n}(a_i,b_i),\\ c_i+d_i, & x\in(a_i,b_i), \qquad i\in\{1,\dots,n\},\end{cases} \qquad (4.15)$$

$$g(x):=\begin{cases} G(x), & x\notin\bigcup_{i=1}^{n}(a_i,b_i),\\ \dfrac{F(x)G(x)}{c_i+d_i}, & x\in(a_i,b_i), \qquad i\in\{1\ \dots,n\}.\end{cases} \qquad (4.16)$$

It follows from (4.7)–(4.8) and (4.15) that

$$\begin{aligned}\|F-f\|_\infty &= \max_{1\le i\le n}\sup_{x\in(a_i,b_i)}|F(x)-(c_i+d_i)|\\ &< \max_{1\le i\le n} 2(c_i+d_i) \le 2\left(\frac{\varepsilon}{24}+\frac{\varepsilon}{24}\right)=\frac{\varepsilon}{6} \qquad (4.17)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}_p(F-f,[0,1]) \le &\sum_{i=1}^{n}\sup_{[\alpha,\beta]\subset(a_i,b_i)}\mathrm{Var}_p(F-(c_i+d_i),[\alpha,\beta])\\ &+\sum_{i=1}^{n}\lim_{x\to a_i^+}|F(x)-(c_i+d_i)|^p\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n} \lim_{x \to b_i^-} |F(x) - (c_i + d_i)|^p \\
&\le \sum_{i=1}^{n} \sup_{[\alpha,\beta] \subset (a_i,b_i)} \operatorname{Var}_p(F, [\alpha,\beta]) \\
&\quad + 2 \sum_{i=1}^{n} \sup_{x \in (a_i,b_i)} \big(|F(x)| + c_i + d_i\big)^p \\
&< \sum_{i=1}^{n} \sup_{[\alpha,\beta] \subset (a_i,b_i)} \operatorname{Var}_p(F, [\alpha,\beta]) + 4^p \sum_{i=1}^{n} (c_i + d_i)^p. \qquad (4.18)
\end{aligned}
$$

Combining (4.17)–(4.18) with (4.9) and (4.13)–(4.14), we see that

$$
\begin{aligned}
\|F - f\|_{BV_p} &= \|F - f\|_\infty + \big(\operatorname{Var}_p(F - f, [0,1])\big)^{1/p} \\
&< \frac{\varepsilon}{6} + \left(\left(\frac{\varepsilon}{48}\right)^p + 4^p \sum_{i=1}^{n} (c_i + d_i)^p \right)^{1/p} \\
&\le \frac{\varepsilon}{6} + \frac{\varepsilon}{48} + 4 \left(\sum_{i=1}^{n} c_i^p \right)^{1/p} + 4 \left(\sum_{i=1}^{n} d_i^p \right)^{1/p} \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{24} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{7\varepsilon}{8}. \qquad (4.19)
\end{aligned}
$$

Analogously, it follows from (4.7)–(4.8) and (4.16) that

$$
\begin{aligned}
\|G - g\|_\infty &= \max_{1 \le i \le n} \sup_{x \in (a_i,b_i)} \left| G(x) - \frac{F(x)G(x)}{c_i + d_i} \right| \\
&\le \max_{1 \le i \le n} \left(\sup_{x \in (a_i,b_i)} |G(x)| + \sup_{x \in (a_i,b_i)} |G(x)| \sup_{x \in (a_i,b_i)} \frac{|F(x)|}{c_i + d_i} \right) \\
&\le \max_{1 \le i \le n} \left(d_i + \frac{d_i \cdot c_i}{c_i + d_i} \right) \\
&< 2 \max_{1 \le i \le n} d_i \\
&\le \frac{2\varepsilon}{24} = \frac{\varepsilon}{12}. \qquad (4.20)
\end{aligned}
$$

If $i \in \{1, \ldots, n\}$ and $[\alpha,\beta] \subset (a_i,b_i)$, then taking into account inequality (4.11) and definitions (4.7), we get

$$
\begin{aligned}
&\mathrm{Var}_p\left(G\left(1-\frac{F}{c_i+d_i}\right),[\alpha,\beta]\right)\\
&\le 2^{p-1}\left\{\sup_{x\in[\alpha,\beta]}|G(x)|^p\cdot\mathrm{Var}_p\left(1-\frac{F}{c_i+d_i},[\alpha,\beta]\right)\right.\\
&\qquad\left.+\mathrm{Var}_p(G,[\alpha,\beta])\cdot\sup_{x\in[\alpha,\beta]}\left|1-\frac{F(x)}{c_i+d_i}\right|^p\right\}\\
&\le 2^p\left\{\left(\frac{\sup_{x\in(a_i,b_i)}|G(x)|}{c_i+d_i}\right)^p\cdot\mathrm{Var}_p(F,[\alpha,\beta])\right.\\
&\qquad\left.+\mathrm{Var}_p(G,[\alpha,\beta])\cdot\left(1+\frac{\sup_{x\in(a_i,b_i)}|F(x)|}{c_i+d_i}\right)^p\right\}\\
&\le 2^p\left\{\left(\frac{d_i}{c_i+d_i}\right)^p\mathrm{Var}_p(F,[\alpha,\beta])+\mathrm{Var}_p(G,[\alpha,\beta])\left(1+\frac{c_i}{c_i+d_i}\right)^p\right\}\\
&\le 2^p\,\mathrm{Var}_p(F,[\alpha,\beta])+4^p\,\mathrm{Var}_p(G,[\alpha,\beta]).
\end{aligned}
\tag{4.21}
$$

Further, definitions (4.7) imply that for $i\in\{1,\dots,n\}$,

$$
\begin{aligned}
&\lim_{x\to a_i^+}\left|G(x)\left(1-\frac{F(x)}{c_i+d_i}\right)\right|^p+\lim_{x\to b_i^-}\left|G(x)\left(1-\frac{F(x)}{c_i+d_i}\right)\right|^p\\
&\le 2\sup_{x\in(a_i,b_i)}|G(x)|^p\cdot\sup_{x\in(a_i,b_i)}\left|1-\frac{F(x)}{c_i+d_i}\right|^p\\
&\le 2d_i^p\left(1+\frac{c_i}{c_i+d_i}\right)^p\le 2^{p+1}d_i^p\le 4^pd_i^p.
\end{aligned}
\tag{4.22}
$$

It follows from (4.21)–(4.22) that

$$
\begin{aligned}
\mathrm{Var}_p(G-g,[0,1])\le&\sum_{i=1}^n\sup_{[\alpha,\beta]\subset(a_i,b_i)}\mathrm{Var}_p\left(G\left(1-\frac{F}{c_i+d_i}\right),[\alpha,\beta]\right)\\
&+\sum_{i=1}^n\lim_{x\to a_i^+}\left|G(x)\left(1-\frac{F(x)}{c_i+d_i}\right)\right|^p\\
&+\sum_{i=1}^n\lim_{x\to b_i^-}\left|G(x)\left(1-\frac{F(x)}{c_i+d_i}\right)\right|^p
\end{aligned}
$$

$$\begin{aligned}&\leq 2^p \sum_{i=1}^{n} \sup_{[\alpha,\beta]\subset(a_i,b_i)} \mathrm{Var}_p(F,[\alpha,\beta])\\&\quad + 4^p \sum_{i=1}^{n} \sup_{[\alpha,\beta]\subset(a_i,b_i)} \mathrm{Var}_p(G,[\alpha,\beta]) + 4^p \sum_{i=1}^{n} d_i^p. \end{aligned} \tag{4.23}$$

Combining (4.20) and (4.23) with (4.9)–(4.10) and (4.14), we see that

$$\begin{aligned}\|G-g\|_{BV_p} &= \|G-g\|_\infty + \big(\mathrm{Var}_p(G-g,[0,1])\big)^{1/p}\\&< \frac{\varepsilon}{12} + \left(2^p\left(\frac{\varepsilon}{48}\right)^p + 4^p\left(\frac{\varepsilon}{48}\right)^p + 4^p\sum_{i=1}^{n} d_i^p\right)^{1/p}\\&\leq \frac{\varepsilon}{12} + \frac{\varepsilon}{24} + \frac{\varepsilon}{12} + 4\left(\sum_{i=1}^{n} d_i^p\right)^{1/p}\\&< \frac{\varepsilon}{4} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned} \tag{4.24}$$

It follows from (4.19) and (4.24) that $f, g \in BV_p[0,1]$, whence

$$h := |f| + |g| \in BV_p[0,1].$$

In view of Lemma 4.1, the set J of jumps of h is at most countable. Let ∂S and $\mathrm{int}(S)$ denote the boundary and the interior of a set $S \subset [0,1]$, respectively. Consider the sets

$$S_\eta := \{x \in [0,1] : h(x) < \eta\}, \quad B_\eta := \{x \in [0,1] : h(x) \geq \eta\}.$$

Note that in view of the choice of η in (4.4), the set B_η is nonempty. Then we have $\partial(S_\eta) \setminus J \subset B_\eta$. Consider the set

$$J_\eta := \partial(S_\eta) \setminus B_\eta \subset J.$$

We have

$$[0,1] = B_\eta \cup S_\eta = B_\eta \cup \mathrm{int}(S_\eta) \cup J_\eta, \tag{4.25}$$

where the sets B_η, $\mathrm{int}(S_\eta)$ and J_η are pairwise disjoint.

We claim that the set

$$J_\eta^s := \{y \in J_\eta : h(y) < \eta/2\}$$

is finite. Indeed, since $J_\eta^s \subset J_\eta \subset J$, the set J_η^s is at most countable. Assume the contrary, that is, that the set J_η^s is infinite. Let $J_\eta^s = \{y_j\}_{j=1}^\infty$ and $y_j <$

y_{j+1} for all $j \in \mathbb{N}$. Then for every $j \in \mathbb{N}$, there exists $x_j \in B_\eta$ such that $y_{2j-1} < x_j < y_{2j+1}$. Therefore

$$\mathrm{Var}_p(h,[0,1]) \geq \sum_{j=1}^{\infty} |h(x_j) - h(y_{2j-1})|^p \geq \sum_{j=1}^{\infty} \Big(\eta - \frac{\eta}{2}\Big)^p = +\infty,$$

which is impossible since $h \in BV_p[0,1]$. Thus, the set J_η^s is finite.

Consider the (obviously, finite) set

$$J_\eta^0 := \{y \in J_\eta^s : h(y) = 0\}.$$

Let k be the cardinality of J_η^0. Define the functions $F_1, G_1 : [0,1] \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ by

$$F_1(x) := \begin{cases} f(x), & x \in [0,1] \setminus J_\eta^0, \\ \dfrac{\varepsilon}{24k}, & x \in J_\eta^0, \end{cases} \tag{4.26}$$

and

$$G_1(x) := g(x), \quad x \in [0,1]. \tag{4.27}$$

It is clear that

$$f(x) = g(x) = 0, \quad x \in J_\eta^0. \tag{4.28}$$

It follows from (4.15)–(4.16) and (4.26)–(4.28) that

$$F(x)G(x) = f(x)g(x) = F_1(x)G_1(x), \quad x \in [0,1]. \tag{4.29}$$

Moreover,

$$\begin{aligned} \|F_1 - f\|_{BV_p} &= \|F_1 - f\|_\infty + \big(\mathrm{Var}_p(F_1 - f, [0,1])\big)^{1/p} \\ &= \frac{\varepsilon}{24k} + \Big(2k\Big(\frac{\varepsilon}{24k}\Big)^p\Big)^{1/p} \leq \frac{2k+1}{24k}\varepsilon \leq \frac{\varepsilon}{8}. \end{aligned} \tag{4.30}$$

Combining (4.19) and (4.30), we arrive at the following:

$$\|F - F_1\|_{BV_p} \leq \|F - f\|_{BV_p} + \|f - F_1\|_{BV_p} < \frac{7\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon. \tag{4.31}$$

In view of (4.24) and (4.27), we have

$$\|G - G_1\|_{BV_p} = \|G - g\|_{BV_p} < \varepsilon. \tag{4.32}$$

For a set $S \subset [0,1]$, let

$$I(S) := \inf_{x \in S} \big(|F_1(x)| + |G_1(x)|\big).$$

Then it follows from (4.26)–(4.28) that

$$I_1 := I(B_\eta) = \inf_{x \in B_\eta} \big(|f(x)| + |g(x)|\big) \geq \eta > 0, \tag{4.33}$$

$$I_2 := I(J_\eta \setminus J_\eta^s) = \inf_{y \in J_\eta \setminus J_\eta^s} \big(|f(y)| + |g(y)|\big) \geq \frac{\eta}{2} > 0, \tag{4.34}$$

$$I_3 := I(J_\eta^s \setminus J_\eta^0) = \min_{y \in J_\eta^s \setminus J_\eta^0} \big(|f(y)| + |g(y)|\big) > 0 \tag{4.35}$$

(recall that the set $J_\eta^s \setminus J_\eta^0$ is finite), and

$$I_4 := I(J_\eta^0) \geq \frac{\varepsilon}{24k} > 0. \tag{4.36}$$

By the definition of the collection A and definitions (4.15)–(4.16) and (4.26)–(4.27), we have

$$\begin{aligned} I_5 &:= I\left(\operatorname{int}(S_\eta) \setminus \left(\bigcup_{i=1}^n (a_i, b_i)\right)\right) \\ &= \inf_{x \in \operatorname{int}(S_\eta) \setminus \left(\bigcup_{i=1}^n (a_i,b_i)\right)} \big(|F(x)| + |G(x)|\big) \geq \rho > 0 \end{aligned} \tag{4.37}$$

(see (4.5) and (4.6)) and, in view of (4.7), we see that

$$\begin{aligned} I_6 &:= I\left(\bigcup_{i=1}^n (a_i, b_i)\right) \geq \min_{1 \leq i \leq n} \inf_{x \in (a_i, b_i)} \big(|f(x)| + |g(x)|\big) \\ &\geq \min_{1 \leq i \leq n} (c_i + d_i) \geq \frac{\varepsilon}{12n} > 0. \end{aligned} \tag{4.38}$$

It follows from (4.25) and (4.33)–(4.38) that

$$I([0,1]) \geq \min_{1 \leq j \leq 6} I_j > 0.$$

Thus, functions $F_1, G_1 \in BV_p[0,1]$ are jointly nondegenerate. Combining this observation with (4.29) and (4.31)–(4.32), we arrive at the conclusion of the theorem. □

5 Proof of the main result and final remarks

Proof of Theorem 1.1

Take an arbitrary pair $(F, G) \in (BV_p[0,1])^2$. Fix $\varepsilon > 0$. It follows from Theorem 4.3 that there exists a pair of jointly nondegenerate functions $(F_1, G_1) \in (BV_p[0,1])^2$ such that

$$F \cdot G = F_1 \cdot G_1 \tag{5.1}$$

and

$$\|F - F_1\|_{BV_p} < \varepsilon/2, \quad \|G - G_1\|_{BV_p} < \varepsilon/2. \tag{5.2}$$

By Corollary 2.2, there exists a $\delta > 0$ such that

$$B_{BV_p[0,1]}(F_1 \cdot G_1, \delta) \subset B_{BV_p[0,1]}(F_1, \varepsilon/2) \cdot B_{BV_p[0,1]}(G_1, \varepsilon/2). \tag{5.3}$$

Combining (5.1)–(5.3), we arrive at the following:

$$\begin{aligned} B_{BV_p[0,1]}(F \cdot G, \delta) &\subset B_{BV_p[0,1]}(F_1, \varepsilon/2) \cdot B_{BV_p[0,1]}(G_1, \varepsilon/2) \\ &\subset B_{BV_p[0,1]}(F, \varepsilon) \cdot B_{BV_p[0,1]}(G, \varepsilon). \end{aligned}$$

Thus, the multiplication in the Banach algebra $BV_p[0,1]$ is locally open at the pair (F, G). Since $(F, G) \in (BV_p[0,1])^2$ is an arbitrary pair, we conclude that the multiplication in $BV_p[0,1]$ is an open bilinear mapping. □

Combining Theorem 1.1 with [5, Propositions 2.4 and 4.1], we arrive at the following.

Corollary 5.1. *Let $1 \le p < \infty$. Then the set of all invertible elements $\mathcal{G}BV_p[0,1]$ of the Banach algebra $BV_p[0,1]$ is dense in $BV_p[0,1]$.*

Let $1 \le p < \infty$ and $\Lambda_p BV[0,1]$ be the Banach algebra of all functions of bounded variation in the Shiba-Waterman sense. We conclude the paper with the following.

Conjecture 5.2. *The multiplication in the Banach algebra $\Lambda_p BV[0,1]$ is an open bilinear mapping.*

In view of Corollary 2.3, to confirm this conjecture, one has to prove that every pair of functions $(f, g) \in (\Lambda_p BV[0,1])^2$ can be approximated by a pair of jointly nondegenerate functions $(f_1, g_1) \in (\Lambda_p BV[0,1])^2$ such that $f \cdot g = f_1 \cdot g_1$.

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