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A lower estimate for weak-type Fourier multipliers

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ABSTRACT

Asmar et al. [Note on norm convergence in the space of weak type multipliers. *J Operator Theory*. 1998;39(1):139–149] proved that the space of weak-type Fourier multipliers acting from L^p into $L^{p,\infty}$ is continuously embedded into L^∞ . We obtain a sharper result in the setting of abstract Lorentz spaces $\Lambda_q(X)$ with $0 < q \leq \infty$ built upon a Banach function space X on \mathbb{R}^n . We consider a source space \mathfrak{S} and a target space \mathfrak{T} in the class of admissible spaces $\mathcal{A} := \{X, \Lambda_q(X) : 0 < q \leq \infty\}$. Let $\mathcal{M}_{\mathfrak{S},\mathfrak{T}}^0$ denote the space of Fourier multipliers acting from \mathfrak{S} to \mathfrak{T} . We show that if the space X satisfies the weak doubling property, then the space $\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}^0$ is continuously embedded into L^∞ for every $0 < q \leq \infty$. This implies that $\mathcal{M}_{\mathfrak{S},\mathfrak{T}}^0$ is a quasi-Banach space for all choices of source and target spaces $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$.

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1. Introduction

Let S and S' denote the Schwartz spaces of rapidly decaying functions and of tempered distributions on \mathbb{R}^n , respectively. The action of a distribution $a \in S'$ on a function $u \in S$ is denoted by $\langle a, u \rangle := a(u)$. A Fourier multiplier on \mathbb{R}^n with symbol $a \in S'$ is defined as the operator

$$u \mapsto F^{-1}aFu, \quad (1)$$

where

$$(Fu)(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\xi} dx$$

is the Fourier transform of $u \in S$, F^{-1} denotes the inverse Fourier transform, and $x\xi$ denotes the scalar product of $x, \xi \in \mathbb{R}^n$. We observe that since $u \in S$ and $a \in S'$, the function Fu belongs to the space S and aFu is a tempered distribution. Thus $F^{-1}aFu$ is well defined and it belongs to S' . In fact, we have $F^{-1}aFu = (F^{-1}a) * u$, and therefore, $F^{-1}aFu \in C_{\text{poly}}^\infty$ (see, e.g. [1, Theorem 2.3.20]), where C_{poly}^∞ denotes the set of all infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that for every $\alpha \in \mathbb{Z}_+^n$ there exist $m_\alpha \in \mathbb{Z}_+ :=$

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$\{0, 1, 2, \dots\}$ and $C_\alpha > 0$ satisfying $|\partial_x^\alpha f(x)| \leq C_\alpha(1 + |x|)^{m_\alpha}$ for all $x \in \mathbb{R}^n$. Thus, if $u \in S$ and $a \in S'$, then $F^{-1}aFu$ is a regular tempered distribution, whose action on $v \in S$ is evaluated as follows:

$$\langle F^{-1}aFu, v \rangle = \int_{\mathbb{R}^n} (F^{-1}aFu)(x)v(x) \, dx \quad \text{for all } v \in S.$$

The aim of this paper is to study the Fourier multiplier operator (1) as an operator acting from a source space \mathfrak{S} to a target space \mathfrak{T} , where $\mathfrak{S}, \mathfrak{T}$ belong to the class of admissible spaces \mathcal{A} consisting of a given Banach function space X on \mathbb{R}^n and all abstract Lorentz spaces $\Lambda_q(X), 0 < q \leq \infty$, built upon X . Our paper is inspired by the work by Asmar et al. [2], where the operator (1) was considered as acting from the Lebesgue space $X = L^p(G), 1 \leq p < \infty$, to the Marcinkiewicz space $\Lambda_\infty(X) = L^{p,\infty}(G)$ over a locally compact abelian group G . It is closely related to our recent work [3], where we treated the operator (1) as acting from a Banach function space X on \mathbb{R}^n into itself. To formulate our results, we need several definitions.

Let \mathcal{M} denote the set of all Lebesgue measurable extended complex-valued functions on \mathbb{R}^n , that is, the functions of the form $f = f_1 + if_2$, where f_1 and f_2 are Lebesgue measurable extended real-valued functions (see, e.g. [4, Definition 16.1]). Let \mathcal{M}^+ be the subset of functions in \mathcal{M} whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by χ_E and the Lebesgue measure of E is denoted by $|E|$. Following [5, p. 3] (see also [6, Chap. 1, Definition 1.1]), a mapping $\rho : \mathcal{M}^+ \rightarrow [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_j (j \in \mathbb{N})$ in \mathcal{M}^+ , for all constants $a \geq 0$, and for all measurable subsets E of \mathbb{R}^n , the following properties hold:

- (A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f), \quad \rho(f + g) \leq \rho(f) + \rho(g),$
- (A2) $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),
- (A3) $0 \leq f_j \uparrow f$ a.e. $\Rightarrow \rho(f_j) \uparrow \rho(f)$ (the Fatou property),
- (A4) E is bounded $\Rightarrow \rho(\chi_E) < \infty,$
- (A5) E is bounded $\Rightarrow \int_E f(x) \, dx \leq C_E \rho(f)$

with $C_E \in (0, \infty)$ that may depend on E and ρ but is independent of f . When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ becomes a Banach space under the norm

$$\|f\|_X := \rho(|f|)$$

and under the natural linear space operations (see [5, Chap. 1, § 1, Theorem 1] or [6, Chap. 1, Theorems 1.4 and 1.6]). It is called a Banach function space. The class of Banach function spaces is very large. It includes, for example, classical Lebesgue spaces $L^p, 1 \leq p \leq \infty$ [6, p. 3], Orlicz spaces L^Φ [6, Chap. 4, Theorem 8.9], variable Lebesgue spaces $L^{p(\cdot)}$ [7, Section 2.10.3].

We note that our definition of a Banach function space is slightly different from that found in [6, Chap. 1, Definition 1.1]. In particular, in Axioms (A4) and (A5), we assume that E is a bounded set, whereas it is sometimes assumed that E merely satisfies $|E| < \infty$. We do this so that the weighted Lebesgue spaces with Muckenhoupt weights satisfy Axioms

(A4) and (A5). Moreover, it is well known that all main elements of the general theory of Banach function spaces work with (A4) and (A5) stated for bounded sets [5] (see also the discussion at the beginning of Chapter 1 on page 2 of [6]). Unfortunately, we overlooked that the definition of a Banach function space in our previous work [3] had to be changed by replacing in Axioms (A4) and (A5) the requirement of $|E| < \infty$ by the requirement that E is a bounded set to include weighted Lebesgue spaces in our considerations. However, the results proved in the above paper remain true.

Let \mathcal{M}_0 (resp., \mathcal{M}_0^+) denote the set of all a.e. finite functions in \mathcal{M} (resp., in \mathcal{M}^+). For $0 < p, q \leq \infty$, the classical Lorentz space $L^{p,q}$ consists of all functions $f \in \mathcal{M}_0$ such that the quantity

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty, \end{cases} \tag{2}$$

is finite, where f^* denotes the non-increasing rearrangement of f (see, e.g. [6, Chap. 2, Section 1] or [1, Section 1.4.1]). Note that if $1 \leq q \leq p < \infty$, then $L^{p,q}$ is a Banach function space with respect to the Banach function norm $\|\cdot\|_{L^{p,q}}$. On the other hand, $L^{\infty,q} = \{0\}$ for $0 < q < \infty$.

For $f \in \mathcal{M}$ and $\lambda > 0$, let

$$\chi_\lambda^f := \chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}}.$$

The quantity $\|f\|_{L^{p,q}}$ for $0 < p < \infty$ can also be written as

$$\|f\|_{L^{p,q}} = \begin{cases} p^{1/q} \left(\int_0^\infty (\lambda \|\chi_\lambda^f\|_{L^p})^q \frac{d\lambda}{\lambda} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{\lambda>0} \lambda \|\chi_\lambda^f\|_{L^p}, & q = \infty, \end{cases} \tag{3}$$

(see, e.g. [1, Proposition 1.4.9] or [8, Theorem 6.6]).

Bearing in mind formula (3), for a given Banach function space X on \mathbb{R}^n and $0 < q \leq \infty$, we define the abstract Lorentz space $\Lambda_q(X)$ built upon X as the set of all functions $f \in \mathcal{M}$ such that

$$\|f\|_{\Lambda_q(X)} := \begin{cases} q^{1/q} \left(\int_0^\infty (\lambda \|\chi_\lambda^f\|_X)^q \frac{d\lambda}{\lambda} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{\lambda>0} \lambda \|\chi_\lambda^f\|_X, & q = \infty \end{cases} \tag{4}$$

is finite. The normalising factor $q^{1/q}$ in the above definition is taken different from $p^{1/q}$ in (3) to guarantee that for every bounded measurable set $E \subset \mathbb{R}^n$, one has $\|\chi_E\|_{\Lambda_q(X)} = \|\chi_E\|_X$ for $0 < q \leq \infty$ (see Lemma 2.4 below).

It follows from (3)–(4) that for $0 < q \leq \infty$ and $1 \leq p < \infty$, one has $\Lambda_q(L^p) = L^{p,q}$ and

$$\|f\|_{\Lambda_q(L^p)} = (q/p)^{1/q} \|f\|_{L^{p,q}}.$$

Further, $\Lambda_\infty(L^\infty) = L^\infty$ and $\|f\|_{\Lambda_\infty(L^\infty)} = \|f\|_{L^\infty}$. Finally, $\Lambda_q(L^\infty) = \{0\}$ whenever $0 < q < \infty$.

The abstract Lorentz space $\Lambda_1(X)$ built upon a rearrangement-invariant Banach function space X is considered, e.g. in [6, Chap. 2, Section 5] (see Definition 5.12 there and Appendix below). For variable Lebesgue spaces $L^{p(\cdot)}$, the spaces $\Lambda_q(L^{p(\cdot)})$ were introduced by Kempka and Vybíral in [9, Definition 2.2]. For an arbitrary Banach function space X , the space $\Lambda_\infty(X)$ was considered by Ho in [10, Section 2].

We collect basic properties of abstract Lorentz spaces $\Lambda_q(X)$, $0 < q \leq \infty$, built upon a Banach function space X in the following statement.

Theorem 1.1. *Let X be a Banach function space on \mathbb{R}^n .*

- (a) *If $f \in X$, then $f \in \Lambda_\infty(X)$ and $\|f\|_{\Lambda_\infty(X)} \leq \|f\|_X$.*
- (b) *If $f \in \Lambda_1(X)$, then $f \in X$ and $\|f\|_X \leq \|f\|_{\Lambda_1(X)}$.*
- (c) *If $0 < q \leq r < \infty$, then for all $f \in \mathcal{M}$,*

$$\|f\|_{\Lambda_\infty(X)} \leq \|f\|_{\Lambda_r(X)} \leq (r/q)^{1/r} \|f\|_{\Lambda_q(X)}.$$

- (d) *If $0 < q \leq \infty$, then $\Lambda_q(X)$ is a quasi-Banach space such that for all $f, g \in \mathcal{M}$,*

$$|f| \leq |g| \text{ a.e.} \Rightarrow \|f\|_{\Lambda_q(X)} \leq \|g\|_{\Lambda_q(X)}.$$

Part (a) and part (d) for $q = \infty$ were proved in [10, Theorem 2.7]. Other statements will be proved in Section 2.

For a given Banach function space X on \mathbb{R}^n , denote by

$$\mathcal{A} := \{X, \Lambda_q(X) : 0 < q \leq \infty\}$$

the collection of admissible spaces. Suppose that a source space \mathfrak{S} and a target space \mathfrak{T} belong to the collection \mathcal{A} . We say that a distribution $a \in \mathcal{S}'$ belongs to the set $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}$ of Fourier multipliers from \mathfrak{S} to \mathfrak{T} if

$$\|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}} := \sup \left\{ \frac{\|F^{-1}aFu\|_{\mathfrak{T}}}{\|u\|_{\mathfrak{S}}} : u \in (S \cap \mathfrak{S}) \setminus \{0\} \right\} < \infty.$$

A function $a \in L^\infty$ is said to belong to the set $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$ of Fourier multipliers from \mathfrak{S} to \mathfrak{T} if

$$\|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0} := \sup \left\{ \frac{\|F^{-1}aFu\|_{\mathfrak{T}}}{\|u\|_{\mathfrak{S}}} : u \in (L^2 \cap \mathfrak{S}) \setminus \{0\} \right\} < \infty.$$

Here, $F^{\pm 1}$ are understood as mappings on L^2 . Since $S \subset L^2$, it is clear that

$$\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0 \subseteq \mathcal{M}_{\mathfrak{S}, \mathfrak{T}} \cap L^\infty \subseteq \mathcal{M}_{\mathfrak{S}, \mathfrak{T}} \tag{5}$$

and

$$\|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}} \leq \|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0} \quad \text{for all } a \in \mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0. \tag{6}$$

Since $\|\cdot\|_{\mathfrak{T}}$ is a quasi-norm, it is not difficult to see that the sets $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$ are quasi-normed linear spaces with respect to the quasi-norms $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0}$, respectively.

The quasi-normed space $\mathcal{M}_{L^p(G), L^{p,\infty}(G)}^0$ for $1 \leq p < \infty$ and a locally compact abelian group G was studied in [2], where it was shown that it is continuously embedded into $L^\infty(\Gamma)$, where Γ is the dual group of G . As a consequence of this continuous embedding, it was shown there that $\mathcal{M}_{L^p(G), L^{p,\infty}(G)}^0$ is a quasi-Banach space.

For $y \in \mathbb{R}^n$ and $R > 0$, let $B(y, R) := \{x \in \mathbb{R}^n : |x - y| < R\}$ be the open ball of radius R centred at y . Following [3, Definition 1.2], we say that a Banach function space X on \mathbb{R}^n satisfies the weak doubling property if there exists a number $\tau > 1$ such that

$$\liminf_{R \rightarrow \infty} \left(\inf_{y \in \mathbb{R}^n} \frac{\|\chi_{B(y, \tau R)}\|_X}{\|\chi_{B(y, R)}\|_X} \right) < \infty.$$

This condition is satisfied, for instance, if X is translation-invariant [3, Corollary 3.5]. Another sufficient condition for the weak doubling property can be stated in terms of the Hardy-Littlewood maximal operator given for $f \in L^1_{\text{loc}}$ by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes that contain x . If M is bounded from X to $\Lambda_\infty(X)$, then X satisfies the weak doubling property. This fact follows from the combination of [3, Lemma 3.3] and [10, Lemma 2.9]. For further discussion of the weak doubling property, see [3, Sections 3.2 and 3.5].

For $\sigma \in \mathbb{R}$, we will say that a function $f \in \mathcal{M}$ belongs to the space $L_{1,\sigma}$ if

$$\int_{\mathbb{R}^n} (1 + |\xi|)^{-\sigma} |f(\xi)| \, d\xi < \infty.$$

Let

$$\mathcal{L} := \bigcup_{\sigma \in \mathbb{R}} L_{1,\sigma}.$$

Theorem 1.2 (Main result). *Let X be a Banach function space satisfying the weak doubling property and $0 < q \leq \infty$. If $a \in \mathcal{L} \cap \mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}$, then*

$$\|a\|_{L^\infty} \leq \|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}}. \tag{7}$$

The constant 1 on the right-hand side of inequality (7) is best possible.

Since $L^\infty \subset L_{1,\sigma}$ for $\sigma > n$, Theorem 1.2 and inequality (6) imply the following generalisation and refinement of [2, Theorem 1.1] for $G = \Gamma = \mathbb{R}^n$.

Corollary 1.3. *Let X be a Banach function space satisfying the weak doubling property and $0 < q \leq \infty$. If $a \in \mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}^0$, then*

$$\|a\|_{L^\infty} \leq \|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}^0}. \tag{8}$$

The constant 1 on the right-hand side of inequality (8) is best possible.

Inequality (8) and Theorem 1.1(c) imply the following.

Corollary 1.4. *Let X be a Banach function space satisfying the weak doubling property. Suppose that $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$. Then $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$ and $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}} \cap L^\infty$ are quasi-Banach spaces with respect to the quasi-norms $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}}$, respectively. Moreover, $\mathcal{M}_{\mathfrak{S}, X}^0$ and $\mathcal{M}_{\mathfrak{S}, X} \cap L^\infty$ are Banach spaces with respect to the norms $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, X}^0}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, X}}$, respectively.*

The paper is organized as follows. In Section 2, we collect basic properties of abstract Lorentz spaces built upon Banach function spaces and prove Theorem 1.1. In Section 3, we recall some auxiliary results proved in our recent paper [3] and then prove Theorem 1.2 and Corollary 1.4. We conclude the paper stating two open problems in Section 4. In Appendix, we provide a formula for the quasi-norm $\|f\|_{\Lambda_q(X)}$ in the case of a rearrangement-invariant Banach function space X in terms of the non-increasing rearrangement f^* , which generalises (2).

It is our pleasure to dedicate this paper to Professor Vladimir Rabinovich on his eightieth birthday.

2. Abstract Lorentz spaces built upon Banach function spaces

2.1. Inclusion $\Lambda_q(X) \subset \mathcal{M}_0$

Lemma 2.1. *Let X be a Banach function space and $0 < q \leq \infty$. If $f \in \Lambda_q(X)$, then $f \in \mathcal{M}_0$.*

Proof: If $f \in \mathcal{M} \setminus \mathcal{M}_0$, then there exists a measurable set $E \subset \mathbb{R}^n$ of positive measure such that $|f(x)| = +\infty$ for a.e. $x \in E$. For every $\lambda > 0$, one has $\chi_\lambda^f \geq \chi_E$. Hence

$$\|f\|_{\Lambda_q(X)} \geq \begin{cases} q^{1/q} \left(\int_0^\infty (\lambda \| \chi_E \|_X)^q \frac{d\lambda}{\lambda} \right)^{1/q}, & q < \infty, \\ \sup_{\lambda > 0} \lambda \| \chi_E \|_X, & q = \infty. \end{cases}$$

Since $\| \chi_E \|_X > 0$, the right-hand side is infinite, which completes the proof. ■

2.2. Proof of Theorem 1.1(a),(b)

Since $\lambda \chi_\lambda^f \leq |f|$ for all $\lambda > 0$, one has $\|f\|_{\Lambda_\infty(X)} \leq \|f\|_X$, which completes the proof of part (a).

Suppose now that $f \in \Lambda_1(X)$. For $\lambda > 0$, one has

$$\chi_\lambda^f(y) = \begin{cases} 1, & \text{if } |f(y)| > \lambda, \\ 0, & \text{if } |f(y)| \leq \lambda, \end{cases}$$

and hence

$$\int_0^\infty \chi_\lambda^f(y) \, d\lambda = \int_{0 \leq \lambda < |f(y)|} 1 \, d\lambda = |f(y)|, \quad y \in \mathbb{R}^n.$$

To proceed further, we need the notion of the associate space X' of the Banach function space X defined by a Banach function norm ρ . Its associate norm ρ' is defined on \mathcal{M}^+ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x) dx : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}.$$

It is a Banach function norm itself (see [5, Chap. 1, § 1] or [6, Chap. 1, Theorem 2.2]). The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X . The space X'' is defined similarly.

It follows from the Lorentz-Luxemburg theorem (see [5, Chap. 1, Theorem 4] or [6, Chap. 1, Theorem 2.9]) and Tonelli's theorem (see, e.g. [11, Theorem 5.28]) that

$$\begin{aligned} \|f\|_X = \|f\|_{X''} &= \sup_{h \in X', \|h\|_{X'} \leq 1} \int_{\mathbb{R}^n} |f(y)| |h(y)| dy \\ &= \sup_{h \in X', \|h\|_{X'} \leq 1} \int_{\mathbb{R}^n} \left(\int_0^\infty \chi_\lambda^f(y) d\lambda \right) |h(y)| dy \\ &= \sup_{h \in X', \|h\|_{X'} \leq 1} \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_\lambda^f(y) |h(y)| dy \right) d\lambda \\ &\leq \int_0^\infty \left(\sup_{h \in X', \|h\|_{X'} \leq 1} \int_{\mathbb{R}^n} \chi_\lambda^f(y) |h(y)| dy \right) d\lambda \\ &= \int_0^\infty \|\chi_\lambda^f\|_{X''} d\lambda = \int_0^\infty \|\chi_\lambda^f\|_X d\lambda = \|f\|_{\Lambda_1(X)}, \end{aligned}$$

which completes the proof of part (b).

2.3. Proof of Theorem 1.1(c)

The proof is almost identical to that of [6, Chap. 4, Proposition 4.2]. Since $\|\chi_\lambda^f\|_X$ is a non-increasing function of λ , one has

$$\begin{aligned} \lambda \|\chi_\lambda^f\|_X &= r^{1/r} \left(\int_0^\lambda \left(\tau \|\chi_\tau^f\|_X \right)^r \frac{d\tau}{\tau} \right)^{1/r} \\ &\leq r^{1/r} \left(\int_0^\lambda \left(\tau \|\chi_\tau^f\|_X \right)^r \frac{d\tau}{\tau} \right)^{1/r} \leq \|f\|_{\Lambda_r(X)}. \end{aligned}$$

Taking the supremum over all $\lambda > 0$, one obtains

$$\|f\|_{\Lambda_\infty(X)} \leq \|f\|_{\Lambda_r(X)}. \quad (9)$$

If $r < \infty$, then using (9) with q in place of r , one gets

$$\begin{aligned} \|f\|_{\Lambda_r(X)} &= r^{1/r} \left(\int_0^\infty \left(\lambda \|\chi_\lambda^f\|_X \right)^{r-q+q} \frac{d\lambda}{\lambda} \right)^{1/r} \\ &\leq \|f\|_{\Lambda_\infty(X)}^{1-q/r} r^{1/r} \left(\int_0^\infty \left(\lambda \|\chi_\lambda^f\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{1/r} \end{aligned}$$

$$\begin{aligned} &\leq \|f\|_{\Lambda_q(X)}^{1-q/r} \left(\frac{r}{q}\right)^{1/r} \|f\|_{\Lambda_q(X)}^{q/r} \\ &= \left(\frac{r}{q}\right)^{1/r} \|f\|_{\Lambda_q(X)}, \end{aligned}$$

which completes the proof.

2.4. Quasi-triangle inequality

The quasi-triangle inequality for $\Lambda_\infty(X)$ with the constant 2 can be found in the proof of [10, Theorem 2.7]. We will need a slightly stronger version of this inequality for $\Lambda_\infty(X)$, which is also true for $\Lambda_q(X)$ with $0 < q < \infty$.

Lemma 2.2. *Let X be a Banach function space, $0 < q \leq \infty$, and let $q_* := \min\{1, q\}$. Then for every $f, g \in \Lambda_q(X)$ and every $\kappa \in (0, 1)$, one has*

$$\|f + g\|_{\Lambda_q(X)}^{q_*} \leq \frac{1}{\kappa^{q_*}} \|f\|_{\Lambda_q(X)}^{q_*} + \frac{1}{(1 - \kappa)^{q_*}} \|f\|_{\Lambda_q(X)}^{q_*}.$$

Proof: For $\lambda > 0$ and $f, g \in \mathcal{M}_0$, we have

$$\{x \in \mathbb{R}^n : |f(x) + g(x)| > \lambda\} \subseteq \{x \in \mathbb{R}^n : |f(x)| > \kappa\lambda\} \cup \{x \in \mathbb{R}^n : |g(x)| > (1 - \kappa)\lambda\}.$$

Therefore,

$$\chi_\lambda^{f+g} \leq \chi_{\kappa\lambda}^f + \chi_{(1-\kappa)\lambda}^g.$$

This inequality and Lemma 2.1 imply that

$$\begin{aligned} \|f + g\|_{\Lambda_\infty(X)} &\leq \sup_{\lambda > 0} \lambda \left\| \chi_{\kappa\lambda}^f + \chi_{(1-\kappa)\lambda}^g \right\|_X \\ &\leq \sup_{\lambda > 0} \lambda \left\| \chi_{\kappa\lambda}^f \right\|_X + \sup_{\lambda > 0} \lambda \left\| \chi_{(1-\kappa)\lambda}^g \right\|_X \\ &= \frac{1}{\kappa} \sup_{\tau > 0} \tau \left\| \chi_\tau^f \right\|_X + \frac{1}{1 - \kappa} \sup_{\tau > 0} \tau \left\| \chi_\tau^g \right\|_X \\ &= \frac{1}{\kappa} \|f\|_{\Lambda_\infty(X)} + \frac{1}{1 - \kappa} \|g\|_{\Lambda_\infty(X)}, \end{aligned}$$

as well as

$$\begin{aligned} \|f + g\|_{\Lambda_q(X)}^{q_*} &= q^{q_*/q} \left(\int_0^\infty \left(\lambda \left\| \chi_\lambda^{f+g} \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{q_*/q} \\ &\leq q^{q_*/q} \left(\int_0^\infty \left(\lambda \left\| \chi_{\kappa\lambda}^f + \chi_{(1-\kappa)\lambda}^g \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{q_*/q} \\ &\leq q^{q_*/q} \left(\int_0^\infty \left(\lambda \left\| \chi_{\kappa\lambda}^f \right\|_X + \lambda \left\| \chi_{(1-\kappa)\lambda}^g \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{q_*/q}. \end{aligned}$$

Using the Minkowski inequality for $1 < q < \infty$ and the subadditivity of the concave function $\varphi(t) = t^q$ for $0 < q \leq 1$ and $t \geq 0$, we further get

$$\begin{aligned} \|f + g\|_{\Lambda_q(X)}^{q^*} &\leq q^{q^*/q} \left(\int_0^\infty \left(\lambda \| \chi_{\kappa\lambda}^f \|_X \right)^q \frac{d\lambda}{\lambda} \right)^{q^*/q} \\ &\quad + q^{q^*/q} \left(\int_0^\infty \left(\lambda \| \chi_{(1-\kappa)\lambda}^g \|_X \right)^q \frac{d\lambda}{\lambda} \right)^{q^*/q} \\ &= \frac{q^{q^*/q}}{\kappa^{q^*}} \left(\int_0^\infty \left(\tau \| \chi_\tau^f \|_X \right)^q \frac{d\tau}{\tau} \right)^{q^*/q} \\ &\quad + \frac{q^{q^*/q}}{(1-\kappa)^{q^*}} \left(\int_0^\infty \left(\tau \| \chi_\tau^g \|_X \right)^q \frac{d\tau}{\tau} \right)^{q^*/q} \\ &= \frac{1}{\kappa^{q^*}} \|f\|_{\Lambda_q(X)}^{q^*} + \frac{1}{(1-\kappa)^{q^*}} \|g\|_{\Lambda_q(X)}^{q^*}, \end{aligned}$$

which completes the proof. ■

2.5. Fatou’s lemma

We will need Fatou’s lemma for abstract Lorentz spaces $\Lambda_q(X)$, $0 < q \leq \infty$, built upon a Banach function space X .

Lemma 2.3. *Let X be a Banach function space and $0 < q \leq \infty$. If $f_k \rightarrow f$ a.e. as $k \rightarrow \infty$ and $\liminf_{k \rightarrow \infty} \|f_k\|_{\Lambda_q(X)} < \infty$, then $f \in \Lambda_q(X)$ and*

$$\|f\|_{\Lambda_q(X)} \leq \liminf_{k \rightarrow \infty} \|f_k\|_{\Lambda_q(X)}.$$

Proof: For $q = \infty$, this lemma is proved in [10, Lemma 2.5]. Although for $0 < q < \infty$ the proof is analogous, we provide details here for the reader’s convenience. Write $h_j := \inf_{k \geq j} |f_k|$. Then $h_j \geq 0$ for all $j \in \mathbb{N}$ and $h_j \uparrow |f|$ a.e. as $j \rightarrow \infty$. Therefore, $\chi_\lambda^{h_j} \uparrow \chi_\lambda^f$ a.e. as $j \rightarrow \infty$ for every $\lambda > 0$. By the Fatou property of X (Axiom (A3)),

$$\| \chi_\lambda^f \|_X = \lim_{j \rightarrow \infty} \| \chi_\lambda^{h_j} \|_X \leq \liminf_{j \rightarrow \infty} \liminf_{k \geq j} \| \chi_\lambda^{f_k} \|_X = \liminf_{j \rightarrow \infty} \| \chi_\lambda^{f_j} \|_X.$$

Then

$$\| \chi_\lambda^f \|_X^q \leq \liminf_{j \rightarrow \infty} \| \chi_\lambda^{f_j} \|_X^q.$$

Integrating this inequality over $(0, \infty)$ and applying the classical Fatou lemma (see, e.g. [12, Lemma 1.7]), we get

$$\begin{aligned} \|f\|_{\Lambda_q(X)} &= q^{1/q} \left(\int_0^\infty \left(\lambda \| \chi_\lambda^f \|_X \right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &\leq q^{1/q} \left(\int_0^\infty \left(\lambda^q \liminf_{j \rightarrow \infty} \| \chi_\lambda^{f_j} \|_X^q \right) \frac{d\lambda}{\lambda} \right)^{1/q} \end{aligned}$$

$$\begin{aligned} &\leq q^{1/q} \left(\liminf_{j \rightarrow \infty} \int_0^\infty \left(\lambda \left\| \chi_\lambda^{f_j} \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= \liminf_{j \rightarrow \infty} \|f_j\|_{\Lambda_q(X)}. \end{aligned} \tag{10}$$

But f is certainly measurable (being the pointwise limit of a sequence of measurable functions), so (10) shows that f belongs to $\Lambda_q(X)$, which completes the proof. ■

2.6. Proof of Theorem 1.1(d)

For $q = \infty$, the proof is given in [10, Theorem 2.7]. For $0 < q < \infty$ the proof is similar. Since the proof of completeness of $\Lambda_\infty(X)$ in [10] contains a minor inaccuracy, we provide details for the reader’s convenience. It follows immediately from the definition of the quantity $\| \cdot \|_{\Lambda_q(X)}$ that (i) $\|f\|_{\Lambda_q(X)} = 0$ if and only if $f = 0$ a.e.; (ii) for every $\mu \in \mathbb{C}, \lambda > 0$ and $f \in \Lambda_q(X)$, one has $\chi_\lambda^{\mu f} = \chi_{\lambda/|\mu|}^f$. Therefore, making the change of variables $\tau = \lambda/|\mu|$, we get

$$\begin{aligned} \|\mu f\|_{\Lambda_q(X)} &= q^{1/q} \left(\int_0^\infty \left(\lambda \left\| \chi_\lambda^{\mu f} \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= q^{1/q} \left(\int_0^\infty \left(\lambda \left\| \chi_{\lambda/|\mu|}^f \right\|_X \right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= q^{1/q} \left(\int_0^\infty \left(|\mu| \tau \left\| \chi_\tau^f \right\|_X \right)^q \frac{d\tau}{\tau} \right)^{1/q} \\ &= |\mu| \|f\|_{\Lambda_q(X)}. \end{aligned}$$

(iii) The quasi-triangle inequality for $\Lambda_q(X)$ follows from Lemma 2.2 with $\kappa = 1/2$. Thus $\Lambda_q(X)$ is a quasi-normed space.

If $|f| \leq |g|$ a.e., then for all $\lambda > 0$ one has $\chi_\lambda^f \leq \chi_\lambda^g$. By Axiom (A2) for the space X , one gets $\|\chi_\lambda^f\|_X \leq \|\chi_\lambda^g\|_X$, whence

$$\left(\lambda \left\| \chi_\lambda^f \right\|_X \right)^q \leq \left(\lambda \left\| \chi_\lambda^g \right\|_X \right)^q.$$

Integrating this inequality over $(0, \infty)$, we easily get $\|f\|_{\Lambda_q(X)} \leq \|g\|_{\Lambda_q(X)}$.

It remains to show that the quasi-normed space $\Lambda_q(X)$ is complete with respect to the quasi-norm $\| \cdot \|_{\Lambda_q(X)}$. Let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $\Lambda_q(X)$. It follows from Theorem 1.1(c) that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\Lambda_\infty(X)$ as well.

For every $k \in \mathbb{N}$, let $B_k = \{x \in \mathbb{R}^n : |x| \leq k\}$. Axiom (A5) implies that for every $k \in \mathbb{N}$, there exists $C_k \in (0, \infty)$ such that for all $j, m \in \mathbb{N}$ and all $\lambda > 0$,

$$C_k \left\| \chi_\lambda^{f_j - f_m} \right\|_X \geq \int_{B_k} \chi_\lambda^{f_j - f_m}(x) \, dx = |\{x \in \mathbb{R}^n : |f_j(x) - f_m(x)| > \lambda\} \cap B_k|.$$

Since $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\Lambda_\infty(X)$, it follows from the above inequality and the definition of quasi-norm in $\Lambda_\infty(X)$ that

$$|\{x \in \mathbb{R}^n : |f_j(x) - f_m(x)| > \lambda\} \cap B_k| \rightarrow 0$$

as $j, m \rightarrow \infty$ for each $k \in \mathbb{N}$ and $\lambda > 0$. Thus the sequence $\{f_j\}_{j=1}^\infty$ is locally Cauchy in measure. It follows from [13, Proposition 1.2.2(ii)] that there exists a function $f \in \mathcal{M}$ and a subsequence $\{f_{j_k}\}_{k=1}^\infty$ such that $f_{j_k} \rightarrow f$ a.e. as $k \rightarrow \infty$.

Since $\|\cdot\|_{\Lambda_q(X)}$ is a quasi-norm, by the Aoki-Rolewicz theorem (see, e.g. [14, Theorem 1.3]), there exists $p > 0$ such that for any $g, h \in \Lambda_q(X)$, one has

$$\|g + h\|_{\Lambda_q(X)}^p \leq \|g\|_{\Lambda_q(X)}^p + \|h\|_{\Lambda_q(X)}^p.$$

This inequality implies that for all $j, m \in \mathbb{N}$,

$$\left| \|f_j\|_{\Lambda_q(X)}^p - \|f_m\|_{\Lambda_q(X)}^p \right| \leq \|f_j - f_m\|_{\Lambda_q(X)}^p.$$

Therefore, $\{\|f_j\|_{\Lambda_q(X)}^p\}_{j=1}^\infty$ is a Cauchy sequence in \mathbb{R} . Hence, the limit $\lim_{j \rightarrow \infty} \|f_j\|_{\Lambda_q(X)}$ exists and it is finite. Since $f_{j_k} \rightarrow f$ a.e. as $k \rightarrow \infty$, in view of Lemma 2.3, we conclude that $f \in \Lambda_q(X)$.

Fix $\varepsilon > 0$. Since $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\Lambda_q(X)$, there exists $N \in \mathbb{N}$ such that for all $j, m > N$, one has $\|f_j - f_m\|_{\Lambda_q(X)} < \varepsilon$. We have $f_{j_k} - f_m \rightarrow f - f_m$ a.e. as $k \rightarrow \infty$ for all $m > N$. By Lemma 2.3, for $m > N$,

$$\|f - f_m\|_{\Lambda_q(X)} \leq \liminf_{k \rightarrow \infty} \|f_{j_k} - f_m\|_{\Lambda_q(X)} \leq \varepsilon,$$

that is, $f_m \rightarrow f$ in $\Lambda_q(X)$ as $m \rightarrow \infty$. Thus, the quasi-normed space $\Lambda_q(X)$ is complete with respect to the quasi-norm $\|\cdot\|_{\Lambda_q(X)}$.

2.7. Quasi-norms of characteristic functions of bounded measurable sets

We observe that the quasi-norms of the characteristic function of a bounded measurable set in $\Lambda_q(X)$ for all $0 < q \leq \infty$ coincide with its norm in X .

Lemma 2.4. *Let X be a Banach function space and $0 < q \leq \infty$. For a bounded measurable set $E \subset \mathbb{R}^n$, one has $\chi_E \in \Lambda_q(X)$ and*

$$\|\chi_E\|_{\Lambda_q(X)} = \|\chi_E\|_X.$$

Proof: The proof for $q = \infty$ can be found in [10, Lemma 2.6]. The proof for $q < \infty$ is quite similar. We include the details here for the reader’s convenience. Since

$$\{x \in \mathbb{R}^n : |\chi_E(x)| > \lambda\} = \begin{cases} E, & 0 < \lambda < 1, \\ \emptyset, & \lambda \geq 1, \end{cases}$$

one has

$$\chi_\lambda^{\chi_E} = \begin{cases} \chi_E, & 0 < \lambda < 1, \\ 0, & \lambda \geq 1. \end{cases}$$

Hence

$$\begin{aligned} \|\chi_E\|_{\Lambda_q(X)} &= q^{1/q} \left(\int_0^\infty (\lambda \|\chi_{\lambda^E}\|_X)^q \frac{d\lambda}{\lambda} \right)^{1/q} = q^{1/q} \left(\int_0^1 (\lambda \|\chi_E\|_X)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= \|\chi_E\|_X q^{1/q} \left(\int_0^1 \lambda^{q-1} d\lambda \right)^{1/q} = \|\chi_E\|_X, \end{aligned}$$

which completes the proof. ■

3. Proofs of the main results

3.1. Lemma on approximation at Lebesgue points

Given $\delta > 0$ and a function ψ on \mathbb{R}^n , we define the function ψ_δ by

$$\psi_\delta(\xi) := \delta^{-n} \psi(\xi/\delta), \quad \xi \in \mathbb{R}^n.$$

Recall that a point $x \in \mathbb{R}^n$ is said to be a Lebesgue point of a function $f \in L^1_{\text{loc}}$ if

$$\lim_{R \rightarrow 0^+} \frac{1}{|B(x, R)|} \int_{B(x, R)} |f(y) - f(x)| dy = 0.$$

Lemma 3.1 ([3, Lemma 2.16].). *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ be such that $\sigma_2 \geq \sigma_1$ and $\sigma_2 > n$. Suppose ψ is a measurable function on \mathbb{R}^n satisfying*

$$|\psi(\xi)| \leq C(1 + |\xi|)^{-\sigma_2} \quad \text{for almost all } \xi \in \mathbb{R}^n \tag{11}$$

with some constant $C \in (0, \infty)$. Then for every Lebesgue point $\eta \in \mathbb{R}^n$ of a function a belonging to the space L_{1, σ_1} , one has

$$\int_{\mathbb{R}^n} |a(\xi) - a(\eta)| |\psi_\delta(\eta - \xi)| d\xi \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

3.2. The infimum of the doubling constants

For a Banach function space X and $\tau > 1$, consider the doubling constant

$$D_{X, \tau} := \liminf_{R \rightarrow \infty} \left(\inf_{y \in \mathbb{R}^n} \frac{\|\chi_{B(y, \tau R)}\|_X}{\|\chi_{B(y, R)}\|_X} \right). \tag{12}$$

Lemma 3.2 ([3, Lemma 3.1].). *If a Banach function space X satisfies the weak doubling property, then*

$$\inf_{\tau > 1} D_{X, \tau} = 1.$$

3.3. Proof of Theorem 1.2

We follow the scheme of the proof of [3, Theorem 4.1]. Let $D_{X,\varrho}$ be defined for all $\varrho > 1$ by (12). If, for some $\varrho > 1$, the quantity $D_{X,\varrho}$ is infinite, then it is obvious that

$$\|a\|_{L^\infty} \leq D_{X,\varrho} \|a\|_{\mathcal{M}_{\Lambda_\varrho(X),\Lambda_\infty(X)}}. \tag{13}$$

Since X satisfies the weak doubling property, there exists $\varrho > 1$ such that $D_{X,\varrho}$ is finite.

Take an arbitrary Lebesgue point $\eta \in \mathbb{R}^n$ of the function a . Let an even function $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy the following conditions:

$$0 \leq \varphi \leq 1, \quad \varphi(x) = 1 \text{ for } |x| \leq 1, \quad \varphi(x) = 0 \text{ for } |x| \geq \varrho.$$

Let

$$f_{\delta,\eta}(x) := e^{i\eta x} \varphi(\delta x), \quad x \in \mathbb{R}^n, \delta > 0,$$

and

$$f_{\delta,\eta,y}(x) := f_{\delta,\eta}(x - y), \quad y \in \mathbb{R}^n.$$

Then

$$\begin{aligned} (Ff_{\delta,\eta,y})(\xi) &= e^{-i\xi y} (Ff_{\delta,\eta})(\xi) = e^{-i\xi y} \delta^{-n} (F\varphi) \left(\frac{\xi - \eta}{\delta} \right) \\ &= e^{-i\xi y} (F\varphi)_\delta(\xi - \eta) = e^{-i\xi y} (F\varphi)_\delta(\eta - \xi) \end{aligned}$$

and

$$\begin{aligned} (F^{-1}aFf_{\delta,\eta,y})(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(\xi) (F\varphi)_\delta(\eta - \xi) \, d\xi, \\ a(\eta)f_{\delta,\eta,y}(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(\eta) (F\varphi)_\delta(\eta - \xi) \, d\xi. \end{aligned}$$

Hence, for all $x, y \in \mathbb{R}^n$ and $\delta > 0$,

$$\begin{aligned} |(F^{-1}aFf_{\delta,\eta,y})(x) - a(\eta)f_{\delta,\eta,y}(x)| &= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} e^{i(x-y)\xi} (a(\xi) - a(\eta)) (F\varphi)_\delta(\eta - \xi) \, d\xi \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |a(\xi) - a(\eta)| |(F\varphi)_\delta(\eta - \xi)| \, d\xi. \end{aligned}$$

Since $F\varphi \in S$ and η is a Lebesgue point of a , it follows from Lemma 3.1 that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $x, y \in \mathbb{R}^n$ and all $\delta \in (0, \delta_\varepsilon)$,

$$|(F^{-1}aFf_{\delta,\eta,y})(x) - a(\eta)f_{\delta,\eta,y}(x)| < \varepsilon.$$

It is clear that $[f_{\delta,\eta,y}]_{\chi_{B(y,1/\delta)}} = \chi_{B(y,1/\delta)}$. Then the above inequality implies that for all $y \in \mathbb{R}^n$ and $\delta \in (0, \delta_\varepsilon)$,

$$|a(\eta)|_{\chi_{B(y,1/\delta)}} \leq |F^{-1}aFf_{\delta,\eta,y}| + \varepsilon \chi_{B(y,1/\delta)}.$$

Hence, it follows from Lemma 2.4, Theorem 1.1(d) for $\Lambda_\infty(X)$ and Lemma 2.2 that for all $\kappa \in (0, 1)$,

$$\begin{aligned} |a(\eta)| \|\chi_{B(y,1/\delta)}\|_X &= |a(\eta)| \|\chi_{B(y,1/\delta)}\|_{\Lambda_\infty(X)} \\ &\leq \| |F^{-1}aFf_{\delta,\eta,y}| + \varepsilon \chi_{B(y,1/\delta)} \|_{\Lambda_\infty(X)} \\ &\leq \frac{1}{\kappa} \|F^{-1}aFf_{\delta,\eta,y}\|_{\Lambda_\infty(X)} + \frac{\varepsilon}{1-\kappa} \|\chi_{B(y,1/\delta)}\|_{\Lambda_\infty(X)} \\ &\leq \frac{1}{\kappa} \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \|f_{\delta,\eta,y}\|_{\Lambda_q(X)} + \frac{\varepsilon}{1-\kappa} \|\chi_{B(y,1/\delta)}\|_X. \end{aligned}$$

Taking into account that $|f_{\delta,y,\eta}| \leq \chi_{B(y,\varrho/\delta)}$, it follows from the above inequality, Theorem 1.1(d) for $\Lambda_q(X)$ and Lemma 2.4 that for all $\kappa \in (0, 1)$,

$$\begin{aligned} |a(\eta)| \|\chi_{B(y,1/\delta)}\|_X &\leq \frac{1}{\kappa} \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \|\chi_{B(y,\varrho/\delta)}\|_{\Lambda_q(X)} + \frac{\varepsilon}{1-\kappa} \|\chi_{B(y,1/\delta)}\|_X \\ &= \frac{1}{\kappa} \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \|\chi_{B(y,\varrho/\delta)}\|_X + \frac{\varepsilon}{1-\kappa} \|\chi_{B(y,1/\delta)}\|_X. \end{aligned} \tag{14}$$

Since $D_{X,\varrho} < \infty$, the definition of $D_{X,\varrho}$ given in (12) implies that there exist $\delta \in (0, \delta_\varepsilon)$ and $y \in \mathbb{R}^n$ such that

$$\frac{\|\chi_{B(y,\varrho/\delta)}\|_X}{\|\chi_{B(y,1/\delta)}\|_X} \leq D_{X,\varrho} + \varepsilon.$$

Choosing these δ and y , and dividing both sides of inequality (14) by $\|\chi_{B(y,1/\delta)}\|_X$, we get

$$|a(\eta)| \leq \frac{1}{\kappa} (D_{X,\varrho} + \varepsilon) \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} + \frac{\varepsilon}{1-\kappa} \quad \text{for all } \varepsilon > 0, \kappa \in (0, 1).$$

Passing to the limit as $\varepsilon \rightarrow 0$, we obtain for all Lebesgue points $\eta \in \mathbb{R}^n$ of the function a ,

$$|a(\eta)| \leq \frac{1}{\kappa} D_{X,\varrho} \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \quad \text{for all } \kappa \in (0, 1).$$

Passing to the limit as $\kappa \rightarrow 1$, we get

$$|a(\eta)| \leq D_{X,\varrho} \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}}.$$

Since $a \in L_{1,\sigma} \subset L^1_{loc}$, almost all points $\eta \in \mathbb{R}^n$ are Lebesgue points of the function a in view of the Lebesgue differentiation theorem (see, e.g. [1, Corollary 2.1.16 and Exercise 2.1.10]). Therefore $a \in L^\infty$ and inequality (13) holds for all $\varrho > 1$. It is now left to apply Lemma 3.2.

Suppose that there is a constant $D_X > 0$ such that

$$\|a\|_{L^\infty} \leq D_X \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}}$$

for all $a \in \mathcal{L} \cap \mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}$. It follows from Theorem 1.1(c) that $a_0 \equiv 1$ belongs to $\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}$ and $\|a_0\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \leq 1$. Since $L^\infty \subset \mathcal{L}$, we conclude that

$$1 = \|a_0\|_{L^\infty} \leq D_X \|a_0\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}} \leq D_X.$$

So, the constant $D_X = 1$ in the estimate $\|a\|_{L^\infty} \leq \|a\|_{\mathcal{M}_{\Lambda_q(X),\Lambda_\infty(X)}}$ is best possible.

3.4. Proof of Corollary 1.4

Let $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$. We already know that $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$ and $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}$ are quasi-normed spaces with respect to the quasi-norms $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}}$, respectively. Moreover, $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, X}^0}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, X}}$ are norms. So, it remains only to show that the spaces $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$ and $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}} \cap L^\infty$ are complete with respect to $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0}$ and $\|\cdot\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}}$, respectively.

It follows from Theorem 1.1(a),(c) that

$$\|a\|_{\mathcal{M}_{\mathfrak{S}, \Lambda_\infty(X)}^0} \leq \|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0} \quad \text{for all } \mathfrak{S}, \mathfrak{T} \in \mathcal{A}. \tag{15}$$

By definition, every $\mathfrak{S} \in \mathcal{A}$ is equal to either X or $\Lambda_q(X)$ for some $q \in (0, \infty]$. Taking $q = 1$ in the former case and using Theorem 1.1(b), we conclude that

$$\|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}^0} \leq \|a\|_{\mathcal{M}_{\mathfrak{S}, \Lambda_\infty(X)}^0} \quad \text{for some } q \in (0, \infty]. \tag{16}$$

It follows from (15), (16) that for every $\mathfrak{S} \in \mathcal{A}$ there exists $q \in (0, \infty]$ such that

$$\|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}^0} \leq \|a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0} \tag{17}$$

for all $\mathfrak{T} \in \mathcal{A}$.

Suppose that $\{a_m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$. It follows from Corollary 1.3 and inequality (17) that $\{a_m\}_{m=1}^\infty$ is a Cauchy sequence in L^∞ . Since L^∞ is complete, there exists $a \in L^\infty$ such that $\|a_k - a\|_{L^\infty} \rightarrow 0$ as $k \rightarrow \infty$. Hence, in view of [1, Theorem 2.5.10], for each $u \in L^2 \cap X$,

$$\|F^{-1}a_kFu - F^{-1}aFu\|_{L^2} \leq \|a_k - a\|_{L^\infty} \|u\|_{L^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

In view of the standard fact on L^p spaces (see, e.g. [11, Section 7.23]), there is a subsequence $\{a_{k_s}\}_{s=1}^\infty$ such that $F^{-1}a_{k_s}Fu \rightarrow F^{-1}aFu$ a.e. as $s \rightarrow \infty$.

Fix $\varepsilon > 0$. Since $\{a_m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$, there exists $N \in \mathbb{N}$ such that for all $m, k > N$ and all $u \in L^2 \cap \mathfrak{S}$,

$$\|F^{-1}a_mFu - F^{-1}a_kFu\|_{\mathfrak{T}} \leq \varepsilon \|u\|_{\mathfrak{S}}.$$

Hence, for all $m > N$, all $s \in \mathbb{N}$ such that $k_s > N$, and all $u \in L^2 \cap \mathfrak{S}$,

$$\|F^{-1}a_mFu - F^{-1}a_{k_s}Fu\|_{\mathfrak{T}} \leq \varepsilon \|u\|_{\mathfrak{S}}. \tag{18}$$

Since for all $m > N$ and $u \in L^2 \cap \mathfrak{S}$,

$$F^{-1}a_mFu - F^{-1}a_{k_s}Fu \rightarrow F^{-1}a_mFu - F^{-1}aFu \quad \text{a.e. as } s \rightarrow \infty,$$

applying the Fatou lemma for \mathfrak{T} (Lemma 2.3 if \mathfrak{T} is one of the spaces $\Lambda_q(X)$ with $0 < q \leq \infty$ or [5, Chap. 1, Lemma 1(e)], [6, Chap. 1, Lemma 1.5] if $\mathfrak{T} = X$) to inequality (18), we get for all $m > N$ and $u \in L^2 \cap \mathfrak{S}$,

$$\|F^{-1}a_mFu - F^{-1}aFu\|_{\mathfrak{T}} \leq \liminf_{s \rightarrow \infty} \|F^{-1}a_mFu - F^{-1}a_{k_s}Fu\|_{\mathfrak{T}} \leq \varepsilon \|u\|_{\mathfrak{S}}.$$

Hence for $m > N$ we have $\|a_m - a\|_{\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0} \leq \varepsilon$, which completes the proof in the case of $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}}^0$.

The proof of the completeness of the space $\mathcal{M}_{\mathfrak{S}, \mathfrak{T}} \cap L^\infty$ is analogous.

4. Concluding remarks and open problems

Recall that if $1 \leq p < \infty$, then $\Lambda_1(L^p) = L^{p,1}$ and $\Lambda_\infty(L^p) = L^{p,\infty}$. The Marcinkiewicz space $L^{1,\infty}$ is not normable (see, e.g. [15, Chap. V, 5.12]). On the other hand, one can equip the Lorentz space $L^{p,\infty}$, $1 < p < \infty$, with an equivalent norm and turn it into a Banach function space (see, e.g. [6, Chap. 4, Theorem 4.6]). The problem of normability of more general Lorentz-type spaces was considered, for instance, in [16, Section 2.5].

Problem 4.1: Describe Banach function spaces X for which the abstract Lorentz spaces $\Lambda_q(X)$ with $1 \leq q \leq \infty$ can be equipped with equivalent Banach function norms.

The answer does not seem to be known even for Lebesgue spaces with variable exponents $L^{p(\cdot)}$ (see [9, Section 2.3]).

If the space $\Lambda_q(X)$ admits an equivalent Banach function norm, one can apply to $\Lambda_q(X)$ results from [3] (it follows from Lemma 2.4 that $\Lambda_q(X)$ satisfies the weak doubling property if and only if X does). This would provide results on Fourier multipliers acting from $S \cap \Lambda_q(X)$ to $\Lambda_q(X)$, while Theorem 1.2 is concerned with an *a priori* wider class of Fourier multipliers acting from $S \cap \Lambda_q(X)$ to $\Lambda_\infty(X)$ because, in view of Theorem 1.1(c), one has $\|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)}} \leq \|a\|_{\mathcal{M}_{\Lambda_q(X), \Lambda_r(X)}}$ for $0 < q, r \leq \infty$.

Note that for Fourier multipliers acting from $S \cap X$ to X , an analogue of Theorem 1.2 holds without the *a priori* assumption $a \in \mathcal{L}$ (see [3, Theorem 1.3 and Section 4.2]). Unfortunately, we don't know whether or not it can also be removed from Theorem 1.2.

Problem 4.2: Prove (or disprove) Theorem 1.2 for arbitrary

$$a \in \mathcal{M}_{\Lambda_q(X), \Lambda_\infty(X)} \subset \mathcal{S}'.$$

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Appendix. Abstract Lorentz spaces built upon rearrangement-invariant Banach function spaces

Following [6, Chap. 2, Definitions 1.1 and 1.2], the distribution function D_f of a function $f \in \mathcal{M}_0$ is given by

$$D_f(\lambda) := |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathcal{M}_0$ are said to be equimeasurable if $D_f(\lambda) = D_g(\lambda)$ for all $\lambda \geq 0$. The non-increasing rearrangement of a function $f \in \mathcal{M}_0$ is the function f^* defined by

$$f^*(t) := \inf\{\lambda \geq 0 : D_f(\lambda) \leq t\}, \quad t \geq 0,$$

with the convention that $\inf \emptyset = \infty$ (see, e.g. [6, Chap. 2, Definition 1.5]).

A Banach function norm $\rho : \mathcal{M} \rightarrow [0, \infty]$ is said to be rearrangement-invariant if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$. In that case, the Banach function space $X = X(\rho)$ generated by ρ is said to be a rearrangement-invariant Banach function space (see [6, Chap. 2, Definition 4.1]).

The fundamental function φ_X of a rearrangement-invariant Banach function space X is defined by

$$\varphi_X(t) := \|\chi_E\|_X, \quad t \in [0, \infty),$$

where $E \subset \mathbb{R}^n$ is a measurable set with $|E| = t$ (see, e.g. [6, Chap. 2, Definition 5.1]).

Lemma A.1. *Let $0 < q \leq \infty$ and X be a rearrangement-invariant Banach function space. For every function $f \in \Lambda_q(X)$, one has*

$$\|f\|_{\Lambda_q(X)} = \begin{cases} \left(\int_0^\infty (f^*(t))^q d\varphi_X^q(t) \right)^{1/q}, & q < \infty, \\ \sup_{t>0} (f^*(t)\varphi_X(t)), & q = \infty. \end{cases} \tag{A1}$$

Proof: It follows from the definition of φ_X that

$$\left\| \chi_{\{x \in \mathbb{R}^n: |f(x)| > \lambda\}} \right\|_X = \varphi_X(D_f(\lambda)), \quad \lambda \geq 0. \tag{A2}$$

Take any $t > 0$ and suppose that $f^*(t) > 0$. It follows from [6, Chap. 2, formula (1.10)] that for every $\varepsilon > 0$ there exists $\lambda > \frac{f^*(t)}{1+\varepsilon}$ such that $D_f(\lambda) > t$. Since φ_X is non-decreasing (see [6, Chap. 2, Corollary 5.3]), it follows from (A2) that

$$f^*(t)\varphi_X(t) \leq (1 + \varepsilon)\lambda \varphi_X(D_f(\lambda)) \leq (1 + \varepsilon) \sup_{\lambda>0} (\lambda \varphi_X(D_f(\lambda))) = (1 + \varepsilon)\|f\|_{\Lambda_\infty(X)}.$$

Since $\varepsilon > 0$ is arbitrary, one gets $f^*(t)\varphi_X(t) \leq \|f\|_{\Lambda_\infty(X)}$, and it is clear that this inequality holds in the case $f^*(t) = 0$ as well. So,

$$\sup_{t>0} (f^*(t)\varphi_X(t)) \leq \|f\|_{\Lambda_\infty(X)}. \tag{A3}$$

Now, fix any $\lambda > 0$ and suppose that $D_f(\lambda) > 0$. Take any $\varepsilon \in (0, D_f(\lambda)/2]$ and set $t := D_f(\lambda) - \varepsilon$. Since $D_f(\lambda) > t$, it follows from [6, Chap. 2, formula (1.10)] that $\lambda \leq f^*(t)$ and hence

$$\begin{aligned} \lambda \varphi_X(D_f(\lambda)) &\leq f^*(t)\varphi_X(t + \varepsilon) \leq f^*(t)(\varphi_X(t + \varepsilon) - \varphi_X(t)) + \sup_{t>0} (f^*(t)\varphi_X(t)) \\ &\leq f^*(D_f(\lambda)/2) \sup_{s \in [D_f(\lambda)/2, D_f(\lambda)]} (\varphi_X(s + \varepsilon) - \varphi_X(s)) + \sup_{t>0} (f^*(t)\varphi_X(t)). \end{aligned}$$

Sending ε to 0 and using the fact that φ_X is uniformly continuous on the segment $[D_f(\lambda)/2, 3D_f(\lambda)/2]$ (see [6, Chap. 2, Corollary 5.3]), one gets

$$\lambda \varphi_X(D_f(\lambda)) \leq \sup_{t>0} (f^*(t)\varphi_X(t)). \tag{A4}$$

If $D_f(\lambda) = 0$, then $\varphi_X(D_f(\lambda)) = \varphi_X(0) = 0$ (see [6, Chap. 2, Corollary 5.3]) and (A4) remains true. It follows from (A2) and (A4) that

$$\|f\|_{\Lambda_\infty(X)} = \sup_{\lambda>0} (\lambda \varphi_X(D_f(\lambda))) \leq \sup_{t>0} (f^*(t)\varphi_X(t)). \tag{A5}$$

Combining (A3) and (A5), we arrive at (A1) in the case $q = \infty$.

Now, suppose that $q < \infty$. Using (A2), the equality $D_{f^*} = D_f$ (see, e.g. [6, Chap. 2, formula (1.19)]), monotonicity of f^* , the equality $\varphi_X(0) = 0$, and continuity of φ_X (see [6, Chap. 2, Corollary 5.3]), one gets

$$\begin{aligned} \int_0^\infty (f^*(s))^q d\varphi_X^q(s) &= \int_0^\infty \left(\int_0^{(f^*(s))^q} 1 d\tau \right) d\varphi_X^q(s) \\ &= \int_0^\infty \left(\int_{0 \leq \tau < (f^*(s))^q} 1 d\tau \right) d\varphi_X^q(s) \\ &= \int_0^\infty \left(\int_{\{s \geq 0: f^*(s) > \tau^{1/q}\}} 1 d\varphi_X^q(s) \right) d\tau \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty \left(\int_0^{D_f(\tau^{1/q})} 1 \, d\varphi_X^q(s) \right) d\tau \\ &= \int_0^\infty \varphi_X^q(D_f(\tau^{1/q})) \, d\tau \\ &= \int_0^\infty \varphi_X^q(D_f(\lambda)) \, d\lambda^q \\ &= q \int_0^\infty (\lambda \varphi_X(D_f(\lambda)))^q \frac{d\lambda}{\lambda} = \|f\|_{\Lambda_q(X)}^q, \end{aligned}$$

which immediately implies (A1) for $q < \infty$. ■