Synchronisation of weakly coupled oscillators

Rogério Martins

Abstract The synchronization phenomenon was reported for the first time by Christiaan Huygens, when he noticed the strange tendency of a couple of clocks to synchronise their movements. More recently this phenomena was shown to be ubiquitous in nature and it is broadly studied by its applications, for example in biological cycles. We consider the problem of synchronization of a general network of linearly coupled oscillators, not necessarily identical. In this case the existence of a linear synchronization space is not expected, so we present an approach based on the proof of the existence of a synchronization manifold, the so-called generalised synchronization. Based on some results developed by R. Smith and on Wazewski's principle, a general theory on the existence of invariant manifolds that attract the solutions of the system that are bounded in the future, is presented. Applications and estimates on parameters for the existence of synchronization are presented for several examples: systems of coupled pendulum type equations, coupled Lorenz systems of equations, and oscillators coupled through a medium, among many others.

1 Introduction

Christiaan Huygens was a central figure in the creation and development of mechanical clocks, he developed what we still know today as pendulum clocks. At the beginning of his work, clocks had measurement errors in the order of 15 minutes per day, at the end of his life that error was reduced to about 1 minute per day. One

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of Huygens' developments was the introduction of a restriction, in the form of a cycloid curve, at the base of the pendulum, in order to make it isochronous [2].

Interestingly, Huygens' great motivation was not to create wall clocks for everyday use. Nowadays we cannot imagine life without clocks, in order to know when to enter the next meeting or what time do we have to pick up kids from school. Apparently, in the seventeenth century, this was not a serious need in people's lives. One of the central problems at the time was the question of determining the longitude at the sea, the solution of which was believed to be in the creation of sufficiently accurate clocks that could be used on board. That was the problem that Huygens wanted to solve. In fact, this issue was so central to navigation that there were several cash prizes for those who manage to solve this problem, no matter the method. An excellent reference for this scientific achievement is the book [20], which tells the story of Huygens, but mainly of John Harrison, who dedicated 45 years of his life to the problem.

In 1665, Huygens was recovering from an illness when he noticed something curious: the two clocks he had at home, on top of a bookshelf, were oscillating synchronously. After doing some experiments he found that it was not a coincidence, no matter the position in which the pendula start the movement, after about half an hour they start to oscillate in unison. After doing several experiments in what he initially called ”an unusual kind of sympathy”, he realised that the phenomenon was due to a small oscillation on the shelf, which created the coupling between the two clocks.

Today we call this phenomenon *synchronization of coupled oscillators*. It is a well-known case of serendipity, something that was discovered by chance. These clocks were built in pairs in order to be used in the open ocean, in this way, even if one of them was malfunctioning, or some type of maintenance was needed, the other clock could keep the time. Note that, at sea, if the longitude is not known, there is no way to set a clock. So, it was fortunate that there were two identical clocks side by side on the same shelf.

Interestingly, Huygens was never rewarded for his contributions to the problem of longitude. One of the arguments against his clocks was precisely the fact that they could not be reliable as they were so easily driven by the clock on the side. However, he was the first person to report something that today is known to be present in so many different scenarios. Today it is known that certain species of fireflies tend to flash synchronously. The circadian cycle synchronises with the solar cycle, causing the jet lag in the absence of synchronization. Our neurons are subject to several synchronization phenomena. The financial markets exhibit synchronization phenomenon. Among many other cases.

Nowadays, the phenomenon of synchronization of oscillators is an extremely active field of research, largely due to its applications. The body of living organisms is full of biological cycles and nature seems to have used this phenomenon to make these systems more stable. If the function of a small organ depends on a synchronisation process, for example the synchronisation of several cells, this makes the functioning of that organ more stable, since an individual failure or imprecision of some of the cells does not compromise the functioning of the organ as a whole. For more examples of synchronisation phenomena see [22].

The concept of synchronization is very broad and encompasses many situations and systems of a very different nature, hence the definition of synchronization is usually given on a case-by-case basis. We say that there is synchronization when some type of adjustment of rhythms is observed. Another way to look at synchronization is to think about the possibility of making predictions about the state of one of the oscillators knowing the asymptotic behaviour of the other. Depending on the type of adjustment, several types of synchronization can be considered: complete/identical synchronisation, phase synchronisation, generalised synchronization, gap synchronisation, just to name a few. In the following, we will consider two or more weekly coupled oscillators and mainly consider the identical and the generalised case.

The best way to understand what we are talking about is to observe a concrete example. Consider a system of two coupled pendula

$$\begin{cases} x_1'' + \gamma x_1' + \sin(x_1) = f(t) + L(x_2 - x_1) \\ x_2'' + \gamma x_2' + \sin(x_2) = f(t) + L(x_1 - x_2) \end{cases},$$

where γ and L are two positive parameters. Each of the variables x_1 and x_2 measures the angular position of each pendulum. The terms $\gamma x_1'$ and $\gamma x_2'$ are a very rough representation of friction, with γ representing the magnitude of that friction. On the right hand side of the equation we have a continuous $f(t)$ that represents an external force, something that feeds the system with energy. In a very crude way we can consider that it represents the pendulum clock's spring. Finally, the second terms on the right hand side create a coupling, L represents the magnitude of this coupling. If $L = 0$ we would have two decoupled mathematical pendula equations with friction, in which case the solution of each equation is independent of the other. For $L \neq 0$ this system models the movement of two pendula coupled by a spring. This type of coupling is quite simple, is far from modelling the Huygens' case, yet it is interesting as a first example.

We can turn this into a first order system in the usual way

$$\begin{cases} x_1' = v_1 - \gamma x_1 \\ v_1' = -\sin(x_1) + f(t) + L(x_2 - x_1) \\ x_2' = v_2 - \gamma x_2 \\ v_2' = -\sin(x_2) + f(t) + L(x_1 - x_2) \end{cases}. \tag{1}$$

The first thing we notice is that the subspace $S = \{x_1 = x_2, v_1 = v_2\}$ is invariant for the solutions of this equation. By existence and uniqueness of solution, an orbit with initial conditions in S stays in S for all $t \in \mathbb{R}$. As we will see, under certain conditions, this subspace attracts all the orbits of the system, in this case we say that the two oscillators synchronise and that S is a synchronisation manifold.

We will now show in which situations this system synchronises. Given a solution $(x_1, v_1, x_2, v_2)^T$ of (1) we consider $(u, v)^T = (x_1 - x_2, v_1 - v_2)^T$ that is a solution of

$$\begin{cases} u' = v - \gamma u \\ v' = -\sin(x_1(t)) + \sin(x_2(t)) - 2Lu \end{cases}. \tag{2}$$

If we now consider $\xi = v/u$, for $u \neq 0$, that verifies

$$\xi' = -\frac{\sin(x_1(t)) - \sin(x_2(t))}{x_1(t) - x_2(t)} - 2L - \xi^2 + \gamma\xi,$$

which is essentially a Ricatti equation. Since the first term on the right side of the equation is bounded,

$$-1 < \frac{\sin(x_1(t)) - \sin(x_2(t))}{x_1(t) - x_2(t)} < 1,$$

we note that if $\xi = M$, with M large enough, we have $\xi' < 0$. On the other hand, if $\xi = \gamma/2$ then

$$\xi' = -\frac{\sin(x_1(t)) - \sin(x_2(t))}{x_1(t) - x_2(t)} - 2L + \frac{\gamma^2}{4} > -1 - 2L + \frac{\gamma^2}{4} > 0, \tag{3}$$

if

$$\frac{\gamma^2}{4} > 1 + 2L.$$

This shows that, if the previous equation is verified, the shaded area in the next graph is positively invariant.

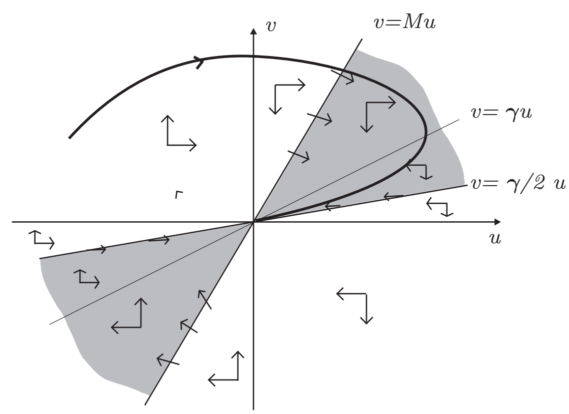

Fig. 1 Phase space of (2)

On the other hand, $u' > 0$ above the line $v = \gamma u$ and $u' < 0$ below. Finally, from the second equation in (2), if $L > 1/2$, we have

$$v' = -\left(\frac{\sin(x_1(t)) - \ sin(x_2(t))}{x_1(t) - x_2(t)} + 2L\right)u,$$

which shows that v' and u have opposite sign. We then obtain the flow suggested by the arrows in Figure 1.

We conclude that if

$$\frac{\gamma^2}{4} > 1 + 2L > 2,$$

the solutions will enter the shaded area and converge to the origin, as in the example of the orbit shown in the figure. That is, $x_1 - x_2$ and $v_1 - v_2$ both tend to zero and the solution $(x_1, v_1, x_2, v_2)^T$ converges to S. Asymptotically we have $x_1 = x_2$ and $v_1 = v_2$.

The last equation suggests that L must be large enough for the system to synchronise, which is normal, given that this parameter measures the strength of the coupling. In addition, the coefficient of friction, γ, also needs to be made large enough, which is also expected, a stronger dissipation is more likely to create attractors that "fit" in a subspace of smaller dimension.

The situation shown in the previous example is what we can call identical or complete synchronization. This type of synchronization appears more typically in systems where the two oscillators are identical, as in the previous case (see [3], [16], [24]). However, in practical applications, it is common to have different oscillators. Consider for example the case of the synchronization of the circadian cycle with the solar cycle, in which we have two oscillators with a completely different nature. On the other hand, we often have similar but not identical oscillators and it is important to take into account these differences. An example would be the case of the previous system if, for example, the friction coefficient or the natural frequency were distinct in the two pendula. In these cases, the symmetry is lost and the problem becomes considerably more complicated. In particular, it is not normal to expect to have a linear invariant subspace, as in the previous case. Therefore we will look for a generic synchronization manifold S that will assume the role taken by the synchronization subspace of the previous example. Although it is the kind of synchronization that is often not identifiable by the "naked eye", technically the orbits converge to a lower dimension manifold . Asymptotically, the behaviour of one oscillator depends of the behaviour of the other. This is the so-called generalised synchronization. In Section 3 we will develop a theory on the existence of invariant manifolds based on the work of R. Smith and the Wazewsky principle, which will be applied to several examples.

In fact, one of the simplest examples of synchronization, in which the two oscillators are definitely different, is the situation in which we have a system driven by a periodic force. In this case, we can try to determine when this force leads the system to, asymptotically, have a periodic behaviour. This is the theme of Section 2. In Section 4 the case of several coupled oscillators will be studied and in Section 5 the case of a generic equation of order n periodically driven.

Recently, it was discovered that it is possible to synchronise chaotic oscillators, which brought a wave of possible applications to information encryption schemes. In Section 7 we will see the case where two Lorenz oscillators synchronise.

In many applications, oscillators are not directly connected. Imagine a situation in which there are several cells interacting with a common medium where they are immersed. In this case, each oscillator interacts directly with its environment and the other oscillators are perturbed through this environment. This is, actually, the case of Huygens, each clocks interfere with a base, which has its own dynamics. This special case is treated in Section 8.

2 Forced oscillations

Unlike the example presented in the introduction, the coupling does not have to be bidirectional. We can consider a situation where we have two oscillators in which one of them follows its own rhythm without interference. One of the simplest situations occurs when we have an oscillator driven by a periodic force, for example:

$$x'' + h(x)x' + g(x) = f(t), \tag{4}$$

where $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, which verify the following periodicity conditions $h(x+1) = h(x)$ and $g(x) = g(x+1)$, for all $x \in \mathbb{R}$, and $f(t) = f(t+T)$ for some positive constant T. We can denote this fact by $h, g \in C(\mathbb{R}/\mathbb{Z})$ and $f \in C(\mathbb{R}/T\mathbb{Z})$. Suppose h is strictly positive, that is, $0 < \gamma = \min_{\mathbb{R}} h(x)$. We will also assume that the above equation has existence and uniqueness for each set of initial conditions.

Although we are writing rather general equation, we can think in the model case $g(x) = \sin(x)$, where we obtain the equation of a forced mathematical pendulum.

Considering, $H(x) = \int_0^x h(s)ds$, we see that x is a solution of (4) if and only if $(x, x' + H(x))$ is a solution of

$$\begin{cases} y_1' = y_2 - H(y_1) \\ y_2' = -g(y_1) + f(t) \end{cases} . \tag{5}$$

The above system has the form $y' = F(y,t)$ with $y = (y_1, y_2)^T$. Writing $h = \tilde{h} + \hat{h}$, where $\hat{h} = \int_0^1 h(s)ds$ is constant and $\int_0^1 \tilde{h}(s)ds = 0$ we get

$$H(x+1) = \int_0^x h(s)ds + \int_x^{x+1} [\tilde{h}(s) + \hat{h}]ds = H(x) + \hat{h}.$$

So, considering $R = (1, \hat{h})^T$ we obtain $F(y+R,t) = F(y,t)$, for all $(y,t) \in \mathbb{R}^3$, that is, the natural phase space for the system (5) is the quotient space $\mathscr{C} = \mathbb{R}^2/R\mathbb{Z}$. In practice, we are identifying each solution y with the solution $y + kR$, $k \in \mathbb{N}$. In this quotient space, which is a cylinder, we will denote by $\bar{y}$ the class of $y \in \mathbb{R}^2$.

Since we are in the presence of a periodic non-autonomous system, a natural thing to do is to consider the associated Poincaré map, also known as stroboscopic map, as it only records the state of the system for $t = kT$, $k \in \mathbb{Z}$. If $y(t;t_0,y_0)$ is the solution of (5) that verifies $y(t_0) = y_0$ then we consider the Poincaré map $P(y) =$

$y(T;0,y)$. Since $y(t;0,y_0+R) = y(t;0,y_0)+R$, we can consider the corresponding Poincaré map in the cylinder $\mathscr{C}$ defined by

$$\overline{P} : \mathscr{C} \to \mathscr{C}$$
$$\overline{y} \to \overline{y(T;0,y)}.$$

which is a homeomorphism.

In [10] we proved that the system (5) is dissipative, that is, there is a compact and non-empty set, called a window, $\overline{B} \subset \mathscr{C}$ such that $\overline{P}(\overline{B}) \subset \mathrm{int}\overline{B}$. We then obtain

$$\ldots \overline{P}^n(\overline{B}) \subset \overline{P}^{n-1}(\overline{B}) \subset \ldots \subset \overline{P}^2(\overline{B}) \subset \overline{P}(\overline{B}) \subset \overline{B}.$$

We can consider the intersection of all these compacts

$$\mathscr{A} = \bigcap_{n=0}^{\infty} \overline{P}^n(\overline{B}).$$

This type of construction is classic, and it is not difficult to prove that $\mathscr{A}$ is non-empty, compact, invariant for the Poincaré map and does not depend on $\overline{B}$. In addition, $\mathscr{A}$ projects over $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Finally, if g is continuously differentiable then $\mathscr{A}$ has measure zero (see [10]).

The set $\mathscr{A}$ is called an attractor for the Poincaré map in $\mathscr{C}$. In fact, it can be proved that for each $\overline{x} \in \mathscr{C}$, $d(\overline{P}^n(\overline{x}), \mathscr{A}) \to 0$, when $n \to \infty$ and convergence is uniform across each compact set $S \subset \mathscr{C}$ (see [10]).

The set $\mathscr{A}$ can be quite complex, this is what typically happens for small friction coefficients. However, as we will see, if γ is large enough then $\mathscr{A}$ is a curve homeomorphic to the circle. In this case, the dynamics become much simpler, all the orbits given by iterates of the Poincaré map converge to an invariant curve homeomorphic to a circle. The dynamics then can be described by the theory of the homeomorphisms of the circle, in particular we can consider an associated rotation number, which essentially determines the dynamics in the limit set. If the rotation number is integer or rational, we have situations where the orbits converge to a periodic orbit of period T or orbits whose period is a multiple of T. In some sense, we can say that the system synchronises with the periodic external force. We have the following:

Theorem 1. *If there is a constant γ_1 such that*

$$\gamma_1 < \frac{g(x)-g(y)}{x-y} < \frac{\gamma^2}{4}, \tag{6}$$

for all $(t,x,y) \in \mathbb{R}^3$, with $x \neq y$, then $\mathscr{A}$ is homeomorphic to $\mathbb{T}^1$.

The proof of this theorem is essentially technical and can be seen in [10]. It involves the fact that $\mathscr{A}$ is formed by the initial conditions of solutions that are bounded in the cylinder $\mathscr{C}$ along with an analysis of the phase space of (5).

Similar estimates were obtained in [8] and [14] for particular cases of equations of the type of (4), but with stronger regularity conditions on g.

It can also be shown that the previous theorem is optimal, in the sense that the second inequality in (6) cannot be improved. More specifically, we have the following theorem whose proof can be seen in [9].

Theorem 2. *Given $\mathscr{H} > \gamma^2/4$, there is $g \in C^\infty(\mathbb{R}/\mathbb{Z})$ with $g' < \mathscr{H}$ and there is $T \in \mathbb{R}$ and $f \in C(\mathbb{R}/T\mathbb{Z})$ such that attractor associated with the equation (5) (with $h \equiv \gamma$) is not homeomorphic to $\mathbb{T}^1$.*

Behind this result there is a topological concept: the idea of an inversely unstable solution. We say that y, a solution of (5), is (a,b)-periodic, for $a, b \in \mathbb{Z}$, $b \geq 1$, iff $y(t+bT) = y(t) + aR$, for all $t \in \mathbb{R}$. Intuitively, an (a,b)-periodic solution is a bT-periodic solution in $\mathscr{C}$ that goes around the cylinder a times in each period.

If y is a periodic (a,b)-solution of (5), then $y(0)$ is a fixed point of

$$P^b - aR : \mathbb{R}^2 \to \mathbb{R}^2,$$

where P is the Poincaré map in the plane. Assuming that $y(0)$ is an isolated fixed point, we can define the index of y as

$$\gamma_b(y) = \deg(I - [P^b - aR], B),$$

where deg designates the Brouwer degree and $B \subset \mathbb{R}^n$ is a sufficiently small disk so that $y(0)$ is the only fixed point of $P^b - aR$ in B.

If y is an (a,b)-periodic solution of (5) then this same solution is also $(2a,2b)$-periodic. We will say that y is inversely unstable if $y(0)$ is an isolated fixed point of $P^b - aR$ and $P^{2b} - 2aR$ and if

$$\gamma_b(y) = 1 \quad \text{and} \quad \gamma_{2b}(y) = -1.$$

The presence of inversely unstable periodic (a,b)-solutions and the flow characteristics associated with (4) requires the existence of a periodic $(2a,2b)$-solution that is not (a,b)-periodic which implies that the rotation number is not well defined and therefore $\mathscr{A}$ cannot topologically be a circle. The complete proof of the next theorem can be seen in [9].

Theorem 3. *Suppose for some $(a,b) \in \mathbb{Z} \times \mathbb{N}$, $b \geq 1$, the set of (a,b)-periodic solutions of (4) is finite and given by*

$$y_1,\ y_2, \ldots, y_p$$

(where we are assuming that y_i and $y_i + kR$, $k \in \mathbb{Z}$, is the same solution). If there is an inversely unstable (a,b)-periodic solution then $\mathscr{A}$ is not homeomorphic to $\mathbb{T}^1$.

Later on, the conditions of Theorem 1 were shown to be optimal using a class of equations studied by F. Tricomi in the 1930's (see [11]). It is essentially a class of pendulum-type oscillators forced by a constant torque. Considering a type of nonlinearity formed by a sectionally linear function, it was possible to find an alternative proof that the second inequality in Theorem 1 cannot be improved.

3 Invariant manifolds

As we have seen, the study of synchronization depends on the existence of invariant manifolds, which somehow attract the system's orbits. This was the case of the previous section. However, in the previous section we were essentially working in the plane, which allows the use of geometric and topological techniques that are not available in higher dimensions. In the next sections we will study systems of several coupled oscillators and we would like to prove the existence of synchronization manifolds, with this idea in mind we need a theory that is applicable in a higher dimension. In this section we will present some general results of invariant manifolds inspired by the work of R. Smith while we was studying periodic solutions of systems of differential equations (see [18], [19]). The proof of these results was presented in [12] and the application to several scenarios can be seen in [10], [12] and [13].

In general we will work with a system of the form

$$x' = F(x,t) + Cx \tag{7}$$

where $F : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is T-periodic and continuous function in t and locally Lipchitz in x. On the other hand, C is a real matrix. In general, in applications, F will contain the nonlinear part of each oscillator and C will include the coupling.

Let's start by briefly summarise the Wazewski topological principle [21], [23]. Given an equation of the type of (7) we will denote by $(\alpha(t_0,x_0),\omega(t_0,x_0)) \subset \mathbb{R}$ the maximal interval of definition of $x(t;t_0,x_0)$, the solution that verifies the initial condition $x(t_0) = x_0$. Given an open set $\Omega \subset \mathbb{R}^p \times \mathbb{R}$, let's say that the point $(x_0,t_0) \in \partial\Omega$ is an ingress point if there is a $\varepsilon > 0$ such that $(x(t;t_0,x_0),t) \in \Omega$ for each $t \in (t_0,t_0+\varepsilon)$. If in addition $(t,x(t;t_0,x_0)) \notin \bar{\Omega}$ for each $t \in (t_0-\varepsilon,t_0)$ and some $\varepsilon > 0$, we say that (x_0,t_0) is a strict ingress point. We will denote by Ω_i and Ω_{si}, the set of ingress points and the set of strict ingress points, respectively (see figure 2). Clearly $\Omega_{si} \subset \Omega_i \subset \partial\Omega$. Finally, if X is a topological space and $A \subset X$ is a subspace, we say that A is a retract of X if there is a continuous function $r : X \to A$ such that $r(x) = x$ for each $x \in A$. In that case, we say that r is a retraction.

We are now able to present the topological principle behind the results of this section. The Figure 2 illustrate a typical example of the associated setup.

Theorem 4 (Wazewski's principle). *Let's assume that $\Omega_i = \Omega_{si}$ and the existence of a set $S \subset \Omega \cup \Omega_i$ such that $S \cap \Omega_i$ is a retraction of Ω_i but $S \cap \Omega_i$ is not a retraction of S. In this case, there is a point $(x_0,t_0) \in S \cap \Omega$ such that $(x(t;t_0,x_0),t) \in \Omega$ for all $t \in (\alpha(t_0,x_0),t_0]$.*

In what follows we will outline the proof of the existence of an invariant manifold for (7), the details of the proof can be seen in [12]. The next hypothesis will be central.

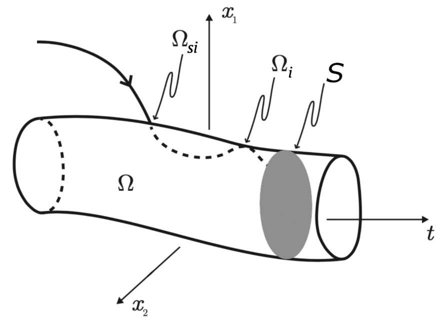

Fig. 2 Example of Wazewski topological configuration

(H) | There are constants $\lambda > 0$, $\varepsilon > 0$ and a symmetric real matrix P with precisely n negative eigenvalues, such what
$(x-y)^T P[F(x,t) - F(y,t) + (C+\lambda I)(x-y)] \leq -\varepsilon \|x-y\|^2$,
for all $x, y \in \mathbb{R}^p$ and $t \in \mathbb{R}$.

If $V(x) := x^T P x$, then it is easy to see that (H) is equivalent to

$$\frac{d}{dt}\left\{ e^{2\lambda t} V(x(t) - y(t)) \right\} \leq -e^{2\lambda t} \varepsilon ||x(t) - y(t)||^2 \tag{8}$$

for each pair $x(t)$, $y(t)$ of solutions of (7) and for each $t \in \mathbb{R}$. In particular, the function $t \to e^{2\lambda t} V(x(t) - y(t))$ is strictly decreasing. This means that if we consider the cone

$$B := \{x \in \mathbb{R}^p : V(x) < 0\},$$

then the set

$$x(t) + B,$$

(which depends on t) attracts in the future all other orbits of (7). In other words, (H) can be seen as a hypothesis of dissipation. In particular we note that if $x(t)$ and $y(t)$ are solutions of (7) such that $V(x(t_0) - y(t_0)) = 0$ for some $t_0 \in \mathbb{R}$, then $V(x(t) - y(t)) < 0$, $\forall t \in (t_0, +\infty)$ and $V(x(t) - y(t)) > 0$, $\forall t \in (-\infty, t_0)$. From the point of view of $x(t) + B$ this means that if $y(t_0) \in x(t_0) + \partial B$ then $y(t) \in x(t) + B$, $\forall t \in (t_0, +\infty)$ e $y(t) \notin x(t) + B$, $\forall t \in (-\infty, t_0)$ (see figure 3).

Let us now consider a special class of solutions. We will say that a solution $x(t)$ of (7) is amenable if the integral

$$\int_{-\infty}^{t_0} e^{2\lambda t} \|x(t)\|^2 dt \tag{9}$$

converges. For each $t \in \mathbb{R}$ we consider the amenable set

$$\mathscr{A}_t = \{x(t) : x(\cdot) \text{ is a amenable solution of } (7)\}.$$

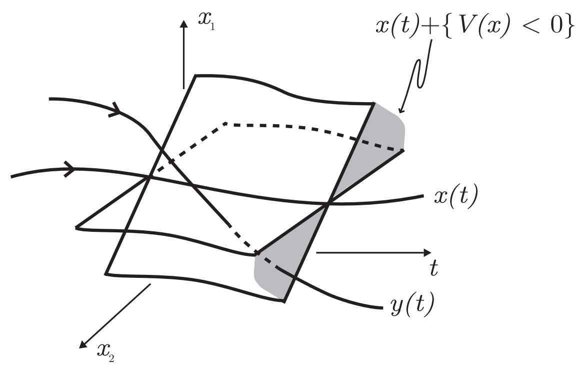

Fig. 3 Dynamics in the extended phase space, given (H).

Let's suppose as a hypothesis that we have an amenable solution $\bar{x}(t)$, it was proved in [19] that another solution $y(t)$ de (7) is also amenable if $V(\bar{x}(t)-y(t))<0$, for all $t\in\mathbb{R}$. Which implies, from the geometric point of view that for each $t\in\mathbb{R}$

$$\mathscr{A}_t\setminus\{\bar{x}(t)\}\subset\bar{x}(t)+B.$$

Let us now consider the subspaces V_- and V_+ of $\mathbb{R}^p$ generated respectively by the eigenvectors of P corresponding to the negative and positive eigenvalues. These subspaces of dimension n and $p-n$, are orthogonal and complementary, that is $\mathbb{R}^p=V_-\perp V_+$. Let $\mathscr{P}_-$ an orthogonal projection of $\mathbb{R}^p$ on V_-.

We are finally in a position to state the main theorem of this section.

Theorem 5. *If the system* (7) *verifies the hypothesis* (H) *and there is at least one amenable solution* $\bar{x}(t)$*, then for each* $t\in\mathbb{R}$*,* $\mathscr{P}_-$ *is a homeomorphism between* $\mathscr{A}_t$ *and* V_-*. Moreover,* $\mathscr{A}_t$ *is a graph of a globally Lipschitz function. In particular the amenable set* $\mathscr{A}_t$ *is a manifold of dimension* n*. Finally, the bounded solutions in the future converge to* $\mathscr{A}_t$*, that is, given a bounded* $x(t)$ *solution in the future,* $dist(\mathscr{A}_t,x(t))\to 0$ *when* $t\to+\infty$*.*

The basis of the proof is to consider $\Omega:=\{(\bar{x}(t)+B,t),\ t\in\mathbb{R}\}$, in this case $\Omega_i=\Omega_{si}=\partial\{\bar{x}(t)+B,t\in\mathbb{R}\}\setminus\{\bar{x}(t),\ t\in\mathbb{R}\}$. For each $\xi\in V_-$ we will consider $\Omega_r=\bar{x}(t_0)+B$ and

$$S:=\mathscr{P}_-^{-1}\xi\cap\bar{\Omega}_r$$

(see figure 4). It can be proved rigorously what is somehow intuitive in a geometric way, that $S\cap\Omega_i$ is a retract of Ω_i but $S\cap\Omega_i$ is not a retract of S. So, by the Wazewski, principle and assuming that the solutions are defined up to $-\infty$, there is a point $(x_0,t_0)\in S\cap\{(\bar{x}(t)+B,t),\ t\in\mathbb{R}\}$ such that $(x(t;t_0,x_0),t)\in\{\bar{x}(t)+B\}$ for all $t\in(-\infty,t_0]$. That is, we have $\mathscr{P}_-(x_0)=\xi$ and $x(t,t_0,x_0)$ is amenable, so $\mathscr{P}_-$ is onto. The convergence for $\mathscr{A}_t$, is a standard argument. For a complete proof see [12].

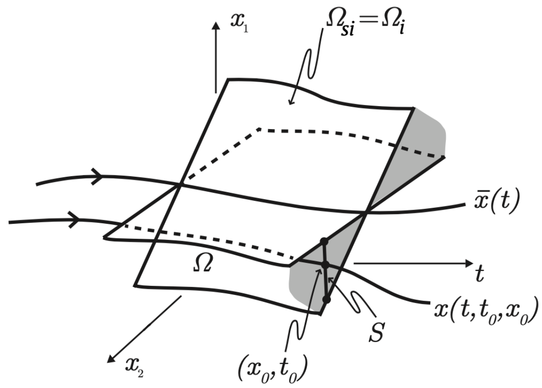

Fig. 4 Theorem proof scheme 5

We will end this section with a criterion for equation (7) to verify the condition (H). Let's suppose that there is a $\lambda \geq 0$ such that C has no eigenvalues with real part equal to $-\lambda$ and in such a way that C has precisely n eigenvalues with strictly real part greater than $-\lambda$. In this case $C+\lambda I$ will have precisely n eigenvalues with positive real part and the Lyapunov equation

$$(C+\lambda I)^T P + P(C+\lambda I) = -I \tag{10}$$

has a single P solution if and only if (see [5])

$$\sigma(C+\lambda I) \cap \overline{\sigma(-C-\lambda I)} = \emptyset. \tag{11}$$

Since the eigenvalues are in finite number, we can easily choose λ such that (11) is verified, let P be the solution of the Lyapunov equation for this λ. From (10) we get

$$(C+\lambda I)^T P^T + P^T (C+\lambda I) = -I^T = -I, \tag{12}$$

and from the uniqueness of the solution to this equation we conclude that P is symmetrical. Finally the General Inertia Theorem (see [5]) shows that P has n negative eigenvalues and $p-n$ positive.

The next theorem shows that under certain conditions (7) verifies (H) with this matrix P.

Theorem 6. *Given λ in the above conditions and P the solution to the Lyapunov equation* (10)*, if there is an $\varepsilon > 0$ such that*

$$(x-y)^T P[F(x,t) - F(y,t)] \leq (1/2-\varepsilon)\|x-y\|^2, \tag{13}$$

then the equation (7) *verifies* (H) *for that λ, ε and P.*

For the proof just observe that

$$(x-y)^T P[F(x,t)-F(y,t)+(C+\lambda I)(x-y)]$$

$$=\frac{1}{2}(x,y)^T[(C+\lambda I)^T P+P(C+\lambda I)](x,y)+(x,y)^T P[F(x,t)-F(y,t)]$$

$$\leq -\varepsilon\|x-y\|^2.$$

Often in applications the non-linearity F is globally K-Lipschitz in x, that is, there is a constant $K>0$ such that

$$\|F(x_1,t)-F(x_2,t)\| \leq K\|x_1-x_2\|,$$

for each $x_1,\ x_2 \in \mathbb{R}^p$ and $t \in \mathbb{R}$. We finally observe that the inequality in the previous theorem is obviously verified if F is globally K -Lipschitz and

$$K<\frac{1}{2\|P\|}.$$

4 Synchronization of several coupled forced oscillators

In this section we will present results similar to those in Section 2 but for a system of coupled differential equations. This systems also generalize the example given in the introduction. We consider a system in $\mathbb{R}^{p/2}$, for an even $p \geq 4$, as follows

$$u''+\gamma u'+Au+g(u)=f(t), \tag{14}$$

where $u \in \mathbb{R}^{p/2}$, γ is a positive constant and A is a matrix in $M_{p/2\times p/2}(\mathbb{R})$, symmetric, with an eigenvalue $\alpha_1=0$ and all other eigenvalues positive $\alpha_2 \leq \ldots \leq \alpha_{p/2}$ (written in ascending order and according to its multiplicity). Let's assume that $\eta \in \mathbb{R}^{p/2}$ is such that Ker$A=$ span$\{\eta\}$. For simplicity, let's consider $\|\eta\|=1$. In addition, $g:\mathbb{R}^{p/2}\to\mathbb{R}^{p/2}$ is locally Lipchitz and periodic in the direction of η, more precisely $g(u+\eta)=g(u)$ for all $u \in \mathbb{R}^{p/2}$. Finally $f:\mathbb{R}\to\mathbb{R}^{p/2}$ is continuous and T-periodic.

One of the most natural concretisations for this system of equations is the case

$$g(u_1,u_2,\ldots,u_{p/2})=(\ sin(u_1),\ sin(u_2),\ldots,\ sin(u_{p/2}))^T$$

with the matrix A

$$A = -\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}.$$

In this special case, the eigenvalues of A are:

$$4\, sin^2\left(\frac{(j-1)\; pi}{p}\right),\; j = 1,\; \dots,\; p/2.$$

This case is a model of a set of pendula arranged in a line, each of them connected by a spring to the pendula that are beside it. It is also the type of equation that arises naturally when the sine-Gordon partial differential equation is discretised. Furthermore, it is also a model for a set of Josephson junctions (see [4]). Although this concretisation is very natural, presenting the results for a generic equation, of the type (14), does not complicate the presentation at all, in fact it even makes it simpler, which is why we chose the more general formulation.

If we consider $u' = v$, the system (14) can be rewritten as a first order system of the type of (7),

$$x' = F(x,t) + Cx \tag{15}$$

where $x = (u, v)^T \in \mathbb{R}^p$,

$$C = \begin{pmatrix} 0 & I \\ -A & -\gamma I \end{pmatrix},\;\; F(x,t) = \begin{pmatrix} 0 \\ -g(u) + f(t) \end{pmatrix}.$$

Note that for $R = (\eta, 0)^T \in \mathbb{R}^p$ we have $F(y+R,t) = F(y,t)$ for all $(y,t) \in \mathbb{R}^p \times \mathbb{R}$. That is, as in Section 2, the natural phase space for this system is a cylinder.

Since A is a symmetric matrix, we can take an orthonormal basis $\{\eta, \eta_2, \eta_3, \dots, \eta_{p/2}\}$ of eigenvectors with associated eigenvalues $\alpha_1 = 0, \alpha_2, \alpha_3, \dots, \alpha_{p/2}$ respectively. The matrix whose columns are made up of the base vectors above

$$P_1 = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \eta & \eta_2 & \dots & \eta_{p/2} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

is orthogonal, that is $P_1^{-1} = P_1^T$, and $P_1^T A P_1 = \mathrm{diag}(0, \alpha_2, \dots, \alpha_{p/2})$. Considering

$$P_2 = \begin{pmatrix} P_1^T & 0 \\ 0 & P_1^T \end{pmatrix}$$

we get

$$P_2 C P_2^{-1} = \begin{pmatrix} 0 & I \\ -\mathrm{diag}(0, \alpha_2, \dots, \alpha_n) & -\gamma I \end{pmatrix}.$$

On the other hand, the matrix P_3 corresponding to the linear map

$$P_3 : \mathbb{R}^p \to \mathbb{R}^p$$

$$(u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p)^T \to (u_1, v_1, u_2, v_2, \ldots, u_p, v_p)^T$$

it is such that

$$P_3 P_2 C P_2^{-1} P_3^{-1} = \begin{pmatrix} 0 & 1 & & & \cdots & 0 & \\ 0 & -\gamma & & & & & \\ & & 0 & 1 & & & \\ & & -\alpha_2 & -\gamma & & & \\ \vdots & & & & \ddots & \vdots & \\ & & & & & 0 & 1 \\ 0 & & \cdots & & & -\alpha_p & -\gamma \end{pmatrix}.$$

For each $i = 1, \ldots, p/2$ consider the block

$$A_i = \begin{pmatrix} 0 & 1 \\ -\alpha_i & -\gamma \end{pmatrix}.$$

Let's now assume that $\gamma^2 - 4\alpha_i > 0$, for all $i = 1, \ldots, p/2$. In this case the matrix A_i has two real eigenvalues, $-\frac{\gamma}{2} + \frac{\sqrt{\gamma^2 - 4\alpha_i}}{2}$ e $-\frac{\gamma}{2} - \frac{\sqrt{\gamma^2 - 4\alpha_i}}{2}$ associated to the eigenvectors

$$\begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} \\ -\frac{\gamma}{2\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} + \frac{1}{2\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} \\ -\frac{\gamma}{2\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} - \frac{1}{2\sqrt{2}} \end{pmatrix}$$

respectively (these vectors were chosen in order to facilitate the computations later on). We conclude that the matrix

$$M_i = \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} & \frac{1}{\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} \\ -\frac{\gamma}{2\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} + \frac{1}{2\sqrt{2}} & -\frac{\gamma}{2\sqrt{2}\sqrt{\gamma^2-4\alpha_i}} - \frac{1}{2\sqrt{2}} \end{pmatrix},$$

with inverse

$$M_i^{-1} = \begin{pmatrix} \frac{\gamma}{\sqrt{2}} + \frac{\sqrt{\gamma^2-4\alpha_i}}{\sqrt{2}} & \sqrt{2} \\ -\frac{\gamma}{\sqrt{2}} + \frac{\sqrt{\gamma^2-4\alpha_i}}{\sqrt{2}} & -\sqrt{2} \end{pmatrix},$$

is such that

$$M_i^{-1} A_i M_i = \operatorname{diag}\left(-\frac{\gamma}{2} + \frac{\sqrt{\gamma^2 - 4\alpha_i}}{2}, -\frac{\gamma}{2} - \frac{\sqrt{\gamma^2 - 4\alpha_i}}{2} \right).$$

We can then consider the matrix

$$P_4 = \begin{pmatrix} M_1^{-1} & & 0 \\ & \ddots & \\ 0 & & M_{p/2}^{-1} \end{pmatrix}$$

such that $P_4P_3P_2CP_2^{-1}P_3^{-1}P_4^{-1} = D$ is a diagonal matrix, with the eigenvalues of C in the diagonal. In the next two lemmas, we will try to estimate the Lipschitz constant of

$$G(y,t) = P_4P_3P_2F(P_2^{-1}P_3^{-1}P_4^{-1}y,t)$$

in y.

Lemma 1. *If $y = (0,v)^T \in \mathbb{R}^p$ then $\|P_4P_3P_2y\| = 2\|v\|$.*

Proof. Given $y = (0,v)^T \in \mathbb{R}^p$, we have

$$\|P_4P_3P_2y\| = \|P_4(0,\eta^Tv,0,\eta_2^Tv,0,\ldots,0,\eta_n^Tv)\|$$

$$= \|(\sqrt{2}\eta^Tv,-\sqrt{2}\eta^Tv,\sqrt{2}\eta_2^Tv,\ldots,-\sqrt{2}\eta_s^Tv\|$$

$$= 2\|(\eta^Tv,\eta_2^Tv,\ldots,\eta_n^Tv)\| = 2\|P_1^Tv\| = 2\|v\|. \quad \square$$

Lemma 2. *For all $x = (u,v)^T \in \mathbb{R}^p$ we have*

$$\|P_4P_3P_2x\| \geq \sqrt{\gamma^2 - 4\alpha_{p/2}}\|u\|.$$

Proof. If $x = (u,v)^T \in \mathbb{R}^p$ then

$$\|P_4P_3P_2x\| = \|P_4(\eta^Tu,\eta^Tv,\eta_2^Tu,\eta_2^Tv,\ldots,\eta_{P/2}^Tu,\eta_{P/2}^Tv)\| =$$

$$\left\|\left(\left(\frac{\gamma}{\sqrt{2}} + \frac{\sqrt{\gamma^2-4\alpha_1}}{\sqrt{2}}\right)\eta^Tu + \sqrt{2}\eta^Tv, \left(-\frac{\gamma}{\sqrt{2}} + \frac{\sqrt{\gamma^2-4\alpha_1}}{\sqrt{2}}\right)\eta^Tu - \sqrt{2}\eta^Tv,\right.\right.$$

$$\left.\left.\ldots, \left(-\frac{\gamma}{\sqrt{2}} + \frac{\sqrt{\gamma^2-4\alpha_{P/2}}}{\sqrt{2}}\right)\eta_{P/2}^Tu - \sqrt{2}\eta_{P/2}^Tv\right)\right\|$$

$$= \sqrt{2\left(\frac{\sqrt{\gamma^2-4\alpha_1}}{\sqrt{2}}\eta^Tu\right)^2 + 4\left(\frac{\gamma}{2}\eta^Tu+\eta^Tv\right)^2 + \ldots + 4\left(\frac{\gamma}{2}\eta_{P/2}^Tu+\eta_{P/2}^Tv\right)^2}$$

$$\geq \sqrt{(\sqrt{\gamma^2-4\alpha_1}\eta^Tu)^2 + \ldots + (\sqrt{\gamma^2-4\alpha_{p/2}}\eta_{p/2}^Tu)^2}$$

$$\geq \sqrt{\gamma^2-4\alpha_{p/2}}\|P_1^Tu\| = \sqrt{\gamma^2-4\alpha_{p/2}}\|u\|.$$

Remember that $\alpha_{p/2}$ is the largest eigenvalue of A. $\square$

We are then in a position to state the main result of this section:

Theorem 7. *Suppose $\gamma^2-4\alpha_i>0$, for all $i=0,\dots,p/2$. We will also assume that there is at least one amenable solution for the corresponding first order equation* (15). *If g is K-Lipschitzian and*

$$K<\frac{1}{8}\left(\gamma-\sqrt{\gamma^2-4\alpha_2}\right)\sqrt{\gamma^2-4\alpha_{p/2}} \tag{16}$$

then the Poincaré map associated with this equation has an attractor homeomorphic to $\mathbb{T}^1$. Moreover, this manifold can be seen as a graph of a function with domain in the subspace spanned by $(\eta,0)$.

Proof. The change of variables $y=P_4P_3P_2x$ turns the equation (15) into

$$y'=G(y,t)+Dy.$$

On the other hand, for each $y_1=P_4P_3P_2(u_1,v_1)^T,\ y_2=P_4P_3P_2(u_2,v_2)^T$ and $t\in\mathbb{R}$, we get from the previous two lemmas

$$\begin{aligned}\sup_{y_1\neq y_2}\frac{\|G(y_1,t)-G(y_2,t)\|}{\|y_1-y_2\|}&=\sup_{y_1\neq y_2}\frac{\|P_4P_3P_2(0,-g(u_1)+g(u_2))^T\|}{\|y_1-y_2\|}\\ &=\sup_{y_1\neq y_2}\frac{2\|g(u_1)-g(u_2)\|}{\|y_1-y_2\|}\le\sup_{y_1\neq y_2}\frac{2K\|u_1-u_2\|}{\|y_1-y_2\|}\\ &\le\frac{2K}{\sqrt{\gamma^2-4\alpha_{p/2}}}\end{aligned}$$

We now consider $\lambda\in]0,\frac{\gamma}{2}-\frac{\sqrt{\gamma^2-4\alpha_2}}{2}[$. Note that $-(\frac{\gamma}{2}-\frac{\sqrt{\gamma^2-4\alpha_2}}{2})$ is the largest of the non-null eigenvalues of C. Since D is diagonal, the equation (10), for this λ, can be easily solved, obtaining

$$P=\operatorname{diag}\left(\frac{1}{-2\lambda+\gamma\pm\sqrt{\gamma^2-4\alpha_i}},\ i=1,\dots,p/2\right).$$

So for $\lambda=\frac{\gamma}{4}-\frac{\sqrt{\gamma^2-4\alpha_2}}{4}$ we get

$$\|P\|=\frac{2}{\gamma-\sqrt{\gamma^2-4\alpha_2}}.$$

Finally, by the remark following Theorem 6, the equation verifies (H) if

$$\frac{2K}{\sqrt{\gamma^2-4\alpha_{p/2}}}<\frac{1}{2\|P\|},$$

which is equivalent to (16). By theorem 5, we conclude that the system has an invariant manifold of dimension one. It is not difficult to see, reverting the change of variable, that the invariant manifold projects over the subspace generated by $R=(\eta,0)^T$. We conclude, by the periodicity of the equation in the direction of R, that if we consider this system in the cylinder, then this manifold is homeomorphic to $\mathbb{T}^1$. $\square$

5 An equation of order n

Something similar to what was done in the last sections can also be done with an ordinary equation of order n, periodically forced. So let's consider a n'th order equation with the form

$$x^{(n)}+a_{n-1}x^{(n-1)}+\ldots+a_2x''+a_1x'+g(x,x',\ldots,x^{(n-1)})=f(t) \tag{17}$$

where $g:\mathbb{R}^n\to\mathbb{R}$ is locally Lipchitz and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function. Let's also assume that

$$g(x,x',\ldots,x^{(n-1)})=g(x,x',\ldots,x^{(n-1)}+(1,0,\ldots,0))$$

and that f is T-periodic for some $T>0$. A function x is a solution of (17) if and only if $y=(x,x',x'',\ldots,x^{(n-1)})^T\in\mathbb{R}^n$ is a solution of the system

$$\begin{cases} y_1'=y_2 \\ y_2'=y_3 \\ \vdots \\ y_{n-1}'=y_n \\ y_n'=-a_{n-1}y_n-\ldots-a_1y_2-g(y)+f(t) \end{cases},$$

that is, solution of

$$y'=Cy+J(y,t), \tag{18}$$

where

$$C=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \quad \text{and} \quad J(y,t)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -g(y)+f(t) \end{pmatrix}$$

The C matrix has an eigenvalue $\lambda_1=0$ associated to the eigenvector $(1,0,\ldots,0)^T$. It is not difficult to prove by induction that the characteristic polynomial of C is

$$(-1)^n(x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x).$$

Let's suppose that all the roots of this polynomial are real, negative and with multiplicity one. That is, we will additionally assume that the remaining eigenvalues of C are distinct, real and negative. We will list them in order $\lambda_n < \lambda_{n-1} < \ldots < \lambda_3 < \lambda_2 < 0$. Each of these eigenvalues λ_i has its own eigenvector of the form $(1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{n-1})^T$. Therefore, the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ 0 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is a diagonalizing matrix, that is,

$$B^{-1}CB = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_n) = D.$$

The matrix B is well known in polynomial interpolation problems. In fact, if $y, z \in \mathbb{R}^n$, the equation $B^T y = z$ is equivalent to finding the coefficients $y_1, y_2, \ldots, y_n$ of a polynomial L of degree $n-1$ that verifies

$$L(0) = z_1,\ L(\lambda_2) = z_2, \ldots,\ L(\lambda_n) = z_n. \tag{19}$$

L is the so-called Lagrange interpolator polynomial. Obviously, the uniqueness of this polynomial depends of the determinant of B, which is well known (see [6], p.221) to be

$$\prod_{\substack{i,j=1 \\ i>j}}^{n} (\lambda_i - \lambda_j).$$

In this case the determinant of B is different from 0 and therefore the polynomial L is well defined. We can now follow the numerical analysis texts and, for each $i = 1, \ldots, n$, consider the polynomial

$$L_i(t) = \frac{(t-\lambda_1)(t-\lambda_2)\ldots(t-\lambda_{i-1})(t-\lambda_{i+1})\ldots(t-\lambda_n)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)}$$
$$= \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n} (t-\lambda_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} (\lambda_i-\lambda_j)}.$$

This polynomial has degree $n-1$ and

$$L_i(\lambda_j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases},$$

so

$$L(t) = z_1 L_1(t) + z_2 L_2(t) + \ldots + z_n L_n(t)$$

is a polynomial of degree $n-1$ and verifies (19). As there is only one polynomial in these conditions, this is the Lagrange interpolation polynomial.

We will now compute the Lipschitz constant of $G(y,t) = B^{-1}J(By,t)$. Consider the vector

$$\Omega = \left(\frac{1}{\prod_{\substack{j=1\\ j\neq 1}}^{n}(\lambda_1-\lambda_j)}, \frac{1}{\prod_{\substack{j=1\\ j\neq 2}}^{n}(\lambda_2-\lambda_j)}, \dots, \frac{1}{\prod_{\substack{j=1\\ j\neq n}}^{n}(\lambda_n-\lambda_j)} \right)^T .$$

Lemma 3. *Given $y = (0,0,\dots,0,y_n)^T \in \mathbb{R}^n$, we have $B^{-1}y = y_n\Omega$.*

Proof. If $y = (0,0,\dots,0,y_n)^T \in \mathbb{R}^n$, then

$$B^{-1}y = y_n\Omega \quad \text{iff} \quad (B^{-1}y)^T z = (y_n\Omega)^T z,\ \forall z \in \mathbb{R}^n.$$

But

$$(B^{-1}y)^T z = y^T(B^{-1})^T z = y^T(B^T)^{-1}z$$

is the product of y_n by the coefficient of t^{n-1} in the polynomial $L(t)$ and this coefficient (given the form of L) is

$$\sum_{i=1}^{n} \frac{z_i}{\prod_{\substack{j=1\\ j\neq i}}^{n}(\lambda_i-\lambda_j)}.$$

So

$$(B^{-1}y)^T z = y_n \sum_{i=1}^{n} \frac{z_i}{\prod_{\substack{j=1\\ j\neq i}}^{n}(\lambda_i-\lambda_j)} = y_n\Omega^T z = (y_n\Omega)^T z. \quad \square$$

We can finally state a sufficient condition for the existence of an attractor, that will be a manifold of dimension one.

Theorem 8. *Suppose the equation* (18) *has at least one amenable solution. If g is K-Lipschitz and*

$$K < -\frac{\lambda_2}{2\|\Omega\|\|B\|},$$

then the Poincaré map associated with the system (18) *has an attractor $\mathscr{A}$ homeomorph to $\mathbb{T}^1$.*

Proof. Since B diagonalizes C, we consider the change of variables $y = Bx$ that turns the equation (18) into

$$y' = G(y,t) + Dy.$$

Where $G(y,t) = B^{-1}J(By,t)$. Let's start by showing that the Lipschitz constant of $G(y,t) = B^{-1}J(By,t)$ in the second variable is less than $K\|\Omega\|\|B\|$. Given $z' = Bz,\ y' = By \in \mathbb{R}^n$, from the last lemma we have

$$\|G(z,t) - G(y,t)\| = \|B^{-1}(0,0,\dots,0,-g(z')+g(y'))^T\|$$

$$= \|\Omega\||g(z') - g(y')| \leq K\|\Omega\|\|z' - y'\|$$
$$= K\|\Omega\|\|BB^{-1}(z' - y')\| \leq K\|\Omega\|\|B\|\|z - y\|.$$

On the other hand, given $\lambda \in]0, -\lambda_2[$, the solution of the equation (12), for this λ is

$$P = \text{diag}\left(\frac{-1}{2(\lambda + \lambda_i)},\ i = 1, \ldots, n\right).$$

So for $\lambda = -\lambda_2/2$ we get $\|P\| = -1/\lambda_2$. Finally, by the observation that follows the Theorem 6, the equation (18) verifies (H) if

$$K\|\Omega\|\|B\| < \frac{1}{2\|P\|}.$$

By Theorem 5 we conclude that there is an invariant manifold of dimension one that attracts the orbits of the system. This manifold can be seen as a graph with domain in the subspace generated by $(1, 0, \ldots, 0)^T$. Therefore, due to the periodicity of the equation (18) in this direction, this manifold is homeomorphic to $\mathbb{T}^1$. $\square$

A concrete example of application of the results of this section would be to consider the system

$$\begin{cases} x'' + c_1 x' + \ sin(x) = p(t) \\ y'' + c_2 y' + y = x \end{cases},$$

which can seen as an harmonic damped oscillator forced by the movement of a pendulum, that is driven by a periodic function $p(t)$. In this case, y verifies the fourth order equation

$$y^{(4)} + (c_1 + c_2)y''' + (1 + c_1 c_2)y'' + c_1 y' = -\ sin(y'' + c_2 y' + y) + p(t),$$

which is of the type of (17). If $c_2 > 2,\ c_1 > 0$ then the roots of

$$\lambda^3 + (c_1 + c_2)\lambda^2 + (1 + c_1 c_2)\lambda + c_1$$

are $-c_1$ and $\frac{-c_2 \pm \sqrt{c_2^2 - 4}}{2}$ (real and negative), so we are in the conditions of the last theorem.

6 Two n dimensional coupled systems

Let's consider a system of two n dimension systems coupled in a bidirectional way

$$\begin{cases} x_1' = f_1(x_1, t) + L(x_2 - x_1) \\ x_2' = f_2(x_2, t) + L(x_1 - x_2) \end{cases}, \tag{20}$$

where $L>0$ is a parameter, called the coupling coefficient, and measures the coupling force. Let's assume that f_1 and f_2 are locally Lipchitz in the first variable and continuous T-periodic in t for some positive T.

In the example we saw in the introduction, the oscillators were identical, which allows us to consider a synchronization manifold that is the diagonal subspace. This type of synchronization is called identical synchronization. In this section the goal is to consider different oscillators, $f_1 \neq f_2$, even if the coupling is symmetric it is not expected to have an attractor contained in a subspace. We are going to prove the existence of invariant manifolds of inferior dimension that attract the orbits of the system, the so-called generalised synchronization. Anyway, as an exercise we start by considering the case where the oscillators are equal, $f_1 = f_2 := f$, later on we can compare the results with the general case. In this particular case the problem comes down to finding a suitable Lyapunov function. Note that the previous system can be written in the form of (7) with

$$C = \begin{pmatrix} -LI & LI \\ LI & -LI \end{pmatrix}.$$

Clearly, the diagonal $S = \{x_1 = x_2\}$ is invariant for the solutions of this system. Given a $(x_1, x_2)^T$ solution, we consider $u = x_1 - x_2$ which solves

$$u' = f(x_1,t) - f(x_2,t) - 2Lu.$$

Assuming that f is globally K-Lipschitz and $K < 2L$, then $E(u) = \|u\|^2$ is a Lyapunov function for the last equation. In fact, the derivative over a solution verifies

$$\dot{E}(u) = 2uu' = 2u(f(x_1,t) - f(x_2,t)) - 4L\|u\|^2 \leq 2(K-2L)\|u\|^2 < 0.$$

We conclude that $\|u\| = \|x_1(t) - x_2(t)\| \to 0$ when $t \to +\infty$. In other words, we have identical synchronisation.

Let's now consider the case where f_1 and f_2 are not necessarily identical. Since the eigenvalues of C are 0 and $-2L$, each with multiplicity n, we will choose $\lambda \in]0, 2L[$. We can now solve the equation (12) by blocks and get

$$P = \begin{pmatrix} -\frac{L-\lambda}{2(2L-\lambda)\lambda} I & -\frac{L}{2(2L-\lambda)\lambda} I \\ -\frac{L}{2(2L-\lambda)\lambda} & -\frac{L-\lambda}{2(2L-\lambda)\lambda} I \end{pmatrix}.$$

Since the eigenvalues of P are $\frac{1}{2(2L-\lambda)}$ and $-\frac{1}{2\lambda}$, we have $\|P\| = max\{\frac{1}{2(2L-\lambda)}, \frac{1}{2\lambda}\}$.

In view of the observations following the Theorem 6, (H) is verified if $F = (f_1, f_2)$ is K-Lipschitz in x and

$$K < \max_{\lambda \in]0,2L[} \frac{1}{2\|P\|} = \max_{\lambda \in]0,2L[} \min\{2L-\lambda, \lambda\} = L.$$

We then obtain the following result as a consequence of Theorem 5:

Theorem 9. *If $K < L$ and there is at least one amenable solution, then the system (20) has a synchronization manifold of dimension n, which can be seen as a graph with domain in the subspace generated by the eigenvectors associated to the negative eigenvalues of P.*

Note that the condition for the existence of generalised synchronization is stronger than the corresponding condition for obtaining identical synchronization. Not surprisingly, it is intuitively more difficult to have synchronization when the oscillators are not equal. In [12] there is a proof that the manifold given by the last theorem can be seen as a graph over the subspace spanned by $(x_1,0)^T$ or the subspace spanned by $(0,x_2)^T$.

7 Synchronization of two chaotic oscillators

In this section we will see an example of two chaotic oscillators that synchronise. This example is also useful to illustrate what can be done when we have a nonlinearity that is not globally Lipschitz. We will consider two chaotic oscillators, more specifically two Lorenz systems. We will also consider a unidirectional coupling. Thus, choosing parameters in an interval in which we have chaotic behaviour and leaving one of the systems free, we guarantee that when the coupled system synchronises it follows a chaotic orbit.

More precisely, let's consider the system

$$\begin{cases} x_1' = \sigma_1(y_1 - x_1) \\ y_1' = -y_1 - x_1 z_1 + \rho_1 x_1 \\ z_1' = -\beta_1 z_1 + x_1 y_1 \\ x_2' = \sigma_2(y_2 - x_2) + L(x_1 - x_2) \\ y_2' = -y_2 - x_2 z_2 + \rho_2 x_2 + L(y_1 - y_2) \\ z_2' = -\beta_2 z_2 + x_2 y_2 + L(z_1 - z_2) \end{cases},$$

where $\sigma_1,\ \sigma_2,\ \rho_1,\ \rho_2,\ \beta_1,\ \beta_2$ are positive parameters of the Lorenz system and L is the coupling parameter. As the origin is an amenable solution, according to Theorem 5 the system synchronises if (H) is verified. We will also see that this system is dissipative, that is, there will be a positively invariant set.

Similar to the last example, let's consider

$$C = \begin{pmatrix} 0 & 0 \\ LI & -LI \end{pmatrix},$$

with eigenvalues $0, -L$ and choose $\lambda = L/4$. With these values the equation (12) can be solved, obtaining

$$P = \frac{2}{3L} \begin{pmatrix} -11I & 2I \\ 2I & I \end{pmatrix}.$$

This matrix has eigenvalues $\frac{2(-5\pm 2\sqrt{10})}{3L}$, and so $\|P\| = \frac{2(5+2\sqrt{10})}{3L}$.

In this case F is not globally Lipschitz. However, we can prove that there is a compact set so that all orbits enter and do not leave it, this way we can truncate F out of this compact and apply the results of Section 3 to the truncated system.

Since the first three variables are decoupled from the remaining three, let's consider the Lyapunov function

$$E_1(x_1,y_1,z_1) = x_1^2 + y_1^2 + (z_1 - \sigma_1 - \rho_1)^2.$$

The derivative along a solution of the first three equations is

$$\dot{E}_1 = -2\left(\sigma_1 x_1^2 + y_1^2 + \beta_1\left(z_1 - \frac{\sigma_1+\rho_1}{2}\right)^2 - \beta_1\frac{(\sigma_1+\rho_1)^2}{4}\right),$$

so we conclude that there is a compact set (dependent on $\sigma_1,\ \rho_1,\ \beta_1$) that attracts the solutions and is positively invariant for $x_1,\ y_1,\ z_1$.

Now let's consider the last three equations as a system forced by (x_1,y_1,z_1) and consider a second Lyapunov function

$$E_2(x_2,y_2,z_2) = x_2^2 + y_2^2 + (z_2 - \sigma_2 - \rho_2)^2.$$

The derivative along the solutions is

$$\begin{aligned}\dot{E}_2 = -2\Bigg[&\left(\sqrt{\sigma_2+L}\,x_2 - \frac{Lx_1}{2\sqrt{\sigma_2+L}}\right)^2 - \frac{L^2x_1^2}{4(\sigma_2+L)} + \left(\sqrt{1+L}\,y_2 - \frac{Ly_1}{2\sqrt{1+L}}\right)^2\\ &- \frac{L^2y_1^2}{4(1+L)} + \left(\sqrt{\beta_2+L}\,z_2 - \frac{(\sigma_2+\rho_2)(\beta_2+L)+Lz_1}{2\sqrt{\beta_2+L}}\right)^2\\ &- \left(\frac{(\sigma_2+\rho_2)(\beta_2+L)+Lz_1}{2\sqrt{\beta_2+L}}\right)^2 + L(\sigma_2+\rho_2)z_1\Bigg],\end{aligned}$$

in particular,

$$\begin{aligned}\dot{E}_2 < -2L\Bigg[&\left(x_2 - \frac{L}{2(\sigma_2+L)}x_1\right)^2 - \frac{L}{4(\sigma_2+L)}x_1^2 + \left(y_2 - \frac{L}{2(1+L)}y_1\right)^2\\ &- \frac{L}{4(1+L)}y_1^2 + \left(z_2 - \frac{\sigma_2+\rho_2}{2} - \frac{L}{2(\beta_2+L)}z_1\right)^2\\ &- \left(\frac{\sigma_2+\rho_2}{2}\sqrt{\frac{\beta_2+L}{L}} + \frac{1}{2}\sqrt{\frac{L}{\beta_2+L}}z_1\right)^2 + (\sigma_2+\rho_2)z_1\Bigg].\end{aligned}$$

We conclude that there is a convex set B in $\mathbb{R}^6$, that contains the origin, is positively invariant, and attracts the orbits of the system. This set depends on $\sigma_1,\ \rho_1,\ \beta_1,\ \sigma_2,\ \rho_2,\ \beta_2$.

We note however that given a positively invariant set for $L = L_0$, then the same set it is also positively invariant for each $L > L_0$.

If $K = \sup_{x\in\tilde{B}} \|D_x F\|$, then F is K-Lipschitz in x in B. We now consider the truncated function

$$\tilde{F}(x,t) = \begin{cases} F(x,t), \text{ if } x \in B \\ F(g(x),t), \text{ if } x \notin B \end{cases},$$

where $g(x)$ is the only point in the boundary of B and in the line connecting the origin to x. Clearly that $\tilde{F}$ is also K -Lipschitz in x in all $\mathbb{R}^6$.

Now using the observation that follows the Theorem 6, we conclude that we have a synchronization manifold for the equation $x' = \tilde{F}(x,t) + Dx$ if

$$K \leq \frac{1}{2\|P\|} = \frac{3L}{4(5+2\sqrt{10})}.$$

Each orbit of the original system enters and never leaves B, on the other hand, within B coincides with one of the solutions of the truncated equation, so converges to the synchronization manifold. We conclude that we have generalised synchronization whenever the last inequality is verified for the original equation.

8 Oscillators coupled by medium

In the previous section, we considered situations where the oscillators are somehow directly connected, even though the coupling could be unidirectional or bidirectional. In this section we will consider the situation in which we have several oscillators, all connected to a medium that will have its own dynamics. This situation is very natural, imagine for example the case of a group of cells immersed in a common environment, each of them exchanges a certain chemical substance with the environment where it is immersed and in this way interacts indirectly with all the others. In the examples we have seen, typically each oscillator interacts with its neighbour oscillators, in this case, with a coupled through a medium, each oscillator interfere with all the others simultaneously (see [7], [15] for more examples).

Let's start with a very simple example, a linear model, yet with an interesting interpretation. Imagine two reservoirs with a chemical substance. Each of these reservoirs is connected by a semi-permeable membrane to a third common reservoir. If the concentration of this substance is measured by the variables x_1, x_2 and y respectively, then the evolution of the concentrations can be described by the following linear model

$$\begin{cases} x_1' = L(y - x_1) \\ x_2' = L(y - x_2) \\ y' = L(x_1 - y) + L(x_2 - y), \end{cases} \tag{21}$$

where L is a constant that depends on the permeability of the membrane.

This system can also be written in the matrix form

$$\begin{pmatrix} x_1' \\ x_2' \\ y' \end{pmatrix} = LA \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}$$

where

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrix A has eigenvalues -3, -1 and 0 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this way the structure of the phase space is very clear, there is a stable central manifold which is the subspace generated by $(1,1,1)$. Asymptotically, the orbits converge to a state where $x_1 = x_2 = y$, we can say that the system synchronises, in fact it is a case of identical synchronization.

Now let's see what happens when we perturb this model, for that we consider the following system

$$\begin{cases} x_1' = L(y - x_1) + f_1(x_1, t) \\ x_2' = L(y - x_2) + f_2(x_2, t) \\ y' = L(x_1 - y) + L(x_2 - y) + h(y, t). \end{cases} \tag{22}$$

Let's assume that f_1, f_2 and h are locally Lipchitz in the first variable and continuous and T-periodic in t.

Let's start by looking at what happens when $f_1 = f_2 = f$, that is, when the perturbation is the same in each oscillator. This is the case where the oscillators are equal. This case is relatively simple because we are able to write a Lyapunov function. Let us start by noting that in this case the subspace generated by $(1,1,1)^T$ is no longer invariant, however it is contained in an invariant two-dimensional subspace: the subspace $\{x_1 = x_2\}$. In addition, we can find conditions for this subspace to attract all the other solutions.

Let $z = x_1 - x_2$, if $x_1 \neq x_2$ then

$$\begin{aligned} \dot{z} &= -L(x_1 - x_2) + \frac{f(x_1,t) - f(x_2,t)}{x_1 - x_2}(x_1 - x_2) \\ &= -(L - a(x_1, x_2, t))z_1, \end{aligned}$$

where $a(x_1,x_2,t) = (f(x_1,t) - f(x_2,t))/(x_1 - x_2)$. So, if $a(x_1,x_2,t) < L_1 < L$, for some $L_1 \in \mathbb{R}$ and for all x_1, x_2, t, $x_1 \neq x_2$, then $z(t) \to 0$ when $t \to +\infty$. This shows that in this case $\{x_1 = x_2\}$ is a synchronization manifold. We then have the following result.

Theorem 10. *If $f_1 = f_2 = f$ e for some $L_1 \in \mathbb{R}$ we have*

$$\frac{f(x_1,t)-f(x_2,t)}{x_1-x_2} < L_1 < L,$$

for all x_1, x_2, t, $x_1 \neq x_2$, *then the system (22) synchronizes with* $\{x_1 = x_2\}$ *as the synchronization manifold .*

What we have just seen is a case of identical synchronization. Now let's see what happens when the oscillators are not necessarily equal. Note that we can also write the system (22) in a matrix form similar to (7)

$$\begin{pmatrix} x_1' \\ x_2' \\ y' \end{pmatrix} = \begin{pmatrix} f_1(x_1,t) \\ f_2(x_2,t) \\ h(y,t) \end{pmatrix} + L \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}$$

In order to apply the results of Section 3, we will solve (12) for $\lambda \in]0, L[$ and $D = LA + \lambda I$ obtaining

$$P = \begin{pmatrix} -\frac{L^2-3L\lambda+\lambda^2}{2\lambda(\lambda-3L)(\lambda-L)} & -\frac{L^2}{2\lambda(\lambda-3L)(\lambda-L)} & \frac{L}{2\lambda(\lambda-3L)} \\ -\frac{L^2}{2\lambda(\lambda-3L)(\lambda-L)} & -\frac{L^2-3L\lambda+\lambda^2}{2\lambda(\lambda-3L)(\lambda-L)} & \frac{L}{2\lambda(\lambda-3L)} \\ \frac{L}{2\lambda(\lambda-3L)} & \frac{L}{2\lambda(\lambda-3L)} & \frac{L-\lambda}{2\lambda(\lambda-3L)} \end{pmatrix}.$$

Although this matrix does not look nice, the eigenvalues of P are

$$-\frac{1}{2(\lambda-3L)}, -\frac{1}{2(\lambda-L)}, -\frac{1}{2\lambda},$$

with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

On the other hand, for $x_1 \neq q_1$, $x_2 \neq q_2$, and $y \neq w$, defining

$$\alpha = \alpha(x_1, q_2, t) = \frac{f_1(x_1,t) - f_1(q_1,t)}{x_1 - q_1},$$

$$\beta = \beta(x_2, q_2, t) = \frac{f_2(x_2,t) - f_2(q_2,t)}{x_2 - q_2},$$

$$\delta = \delta(y, w, t) = \frac{h(y,t) - h(w,t)}{y - w},$$

we can rewrite inequality (13) as

$$\begin{pmatrix} x_1 - q_1 \\ x_2 - q_2 \\ y - w \end{pmatrix}^T \left[\left(\frac{1}{2} - \varepsilon\right) I - P \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right] \begin{pmatrix} x_1 - q_1 \\ x_2 - q_2 \\ y - w \end{pmatrix} \geq 0.$$

Let us now consider the symmetric matrix of the associated quadratic form

$$\Omega = \frac{1}{2}\left(\left[\left(\frac{1}{2}-\varepsilon\right)I - P\left(\begin{smallmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \delta\end{smallmatrix}\right)\right]^T + \left(\frac{1}{2}-\varepsilon\right)I - P\left(\begin{smallmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \delta\end{smallmatrix}\right)\right).$$

Which can be written explicitly as

$$\Omega = \begin{pmatrix} \frac{1}{2}-\varepsilon+\frac{\alpha(L^2-3L\lambda+\lambda^2)}{2\lambda(\lambda-3L)(\lambda-L)} & \frac{(\alpha+\beta)L^2}{4\lambda(\lambda-3L)(\lambda-L)} & \frac{(\alpha+\delta)L}{4\lambda(3L-\lambda)}\\ \frac{(\alpha+\beta)L^2}{4\lambda(\lambda-3L)(\lambda-L)} & \frac{1}{2}-\varepsilon+\frac{\beta(L^2-3L\lambda+\lambda^2)}{2\lambda(\lambda-3L)(\lambda-L)} & \frac{(\beta+\delta)L}{4\lambda(3L-\lambda)}\\ \frac{(\alpha+\delta)L}{4\lambda(3L-\lambda)} & \frac{(\beta+\delta)L}{4\lambda(3L-\lambda)} & \frac{1}{2}-\varepsilon+\frac{\delta(L-\lambda)}{2\lambda(3L-\lambda)} \end{pmatrix}.$$

In this way, the inequality (13) in Theorem 6 is verified if the matrix Ω is positive defined. We then have the following result.

Theorem 11. *Suppose there is a $\lambda \in (0,L)$ and ε such that Ω is positive definite for all x_1, x_2, y, q_1, q_2, w, $x_1 \neq x_2$, $q_1 \neq q_2$, $y \neq w$. Assuming that there is at least one amenable solution, then there is a synchronization manifold that can be seen as a graph of the subspace generated by $(1,1,1)^T$.*

We can try to see under what conditions Ω is positive definite, which can be done in two ways, by calculating eigenvalues or studying the smallest ones. We will follow the second option.

Our intuition says that if the α, β and δ coefficients are bounded and if L is large empty, then the system synchronise, that's what the next theorem says.

Theorem 12. *Let's assume that the system (22) has at least one amenable solution and the α, β and δ coefficients are bounded. So, if L is large enough, there is a synchronization manifold that can be seen as a graph with domain in the subpace spanned by $(1,1,1)^T$.*

Proof. If we choose a concrete value for λ, the previous expressions are much simpler. Let's choose $\lambda = L/2$. For this choice of λ, the minors of Ω are

$$\begin{aligned} m_1(\alpha) &= \frac{1}{2}-\varepsilon+\alpha\frac{2}{5L};\\ m_2(\alpha,\beta) &= \left(\frac{1}{2}-\varepsilon+\alpha\frac{2}{5L}\right)\left(\frac{1}{2}-\varepsilon+\beta\frac{2}{5L}\right)-\frac{4}{25L^2}(\alpha+\beta)^2;\\ m_3(\alpha,\beta,\delta) &= \left(\frac{1}{2}-\varepsilon+\frac{\delta}{5L}\right)m_2(\alpha,\beta)+\frac{4}{125L^2}(\alpha+\beta)(\alpha+\delta)(\beta+\delta)\\ &\quad -\frac{1}{25L^2}\left(m_1(\alpha)(\beta+\delta)^2+m_1(\beta)(\alpha+\delta)^2\right). \end{aligned}$$

As we can see, if α, β and δ are bounded, we can choose a sufficiently large L and an ε small enough so that the three minors are all positive. The result is then a consequence of the previous theorem. □

In [13] we can see a study on the geometry of the values of α, β and δ that make the matrix Ω positive definite, and a numerical study on this same set as well.

9 Conclusion

We studied several examples of oscillators and coupling schemes that lead to the so-called generalised synchronisation. The results were mainly obtained through the use of a general result on the existence of invariant manifolds that attract the bounded solutions of the system. Even if we considered, as an example, some cases where the oscillators are equal, and where the synchronization manifold is a diagonal, the full potential of our methods is shown when we consider systems of oscillators which are not equal, or coupling schemes without symmetry, where the synchronization manifold is no longer a diagonal.

It would be interesting to consider similar but not equal oscillators and to obtain convergence results of the synchronization manifold to the diagonal when a parameter, which measures the difference between the oscillators, tends to zero.

It would be interesting to obtain results like those obtained in Section 8 but for more natural oscillators. For example second order equations modelling mechanical oscillators. Huygens' experiments were done with relatively heavy clocks on top of a wooden shelf. There are some works trying to reproduce Huygens' conditions in some way, building replicas of the clocks developed in the 17th century [17] or experimenting with two metronomes on top of a board, suspended in two empty cans [15]. The model of two coupled pendula that we presented in the introduction does not capture the dynamics of the situation described by Huygens, the case we considered is a simple model of two pendula connected directly by a spring. Huygens' setup is much more the type of systems studied in Section 8, each pendulum interferes with a medium, which in this case is the common support. When we leave just one metronome on the board it is very clear that the moment of the pendulum in motion creates an oscillation in the board, it is this oscillation that interferes with the movement of the second pendulum. It is not difficult to write equations for this situation, we can see this deduction for example in [15]. Where its concluded that two pendula, whose position is described by the variables x_1, x_2, on top of a base that moves in a direction parallel to the pendula, are described by the following system of equations,

$$\begin{cases} x_1'' + \gamma\left(\left(\frac{x_1}{x_0}\right)^2 - 1\right)x_1' + sin(x_1) - L\,cos(x_1)(\,sin(x_1) + \,sin(x_2))'' = 0 \\ x_2'' + \gamma\left(\left(\frac{x_2}{x_0}\right)^2 - 1\right)x_2' + sin(x_2) - L\,cos(x_2)(\,sin(x_1) + \,sin(x_2))'' = 0 \end{cases}$$

It would be nice to gain a deeper understanding of this system from an analytical point of view and eventually present intervals for the parameters γ and L so that we have synchronization.

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