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The trees for which maximum multiplicity implies the simplicity of other eigenvalues ☆

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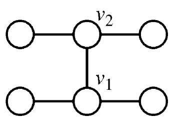
Abstract

Among those real symmetric matrices whose graph is a given tree *T*, the maximum multiplicity is known to be the path cover number of *T*. An explicit characterization is given for those trees for which whenever the maximum multiplicity is attained, all other multiplicities are 1.

Keywords: Real symmetric matrices; Eigenvalues; Multiplicities; Maximum multiplicity; Trees; NIM trees; Vertex degrees

Let G be an undirected graph on n vertices. By $\mathscr{S}(G)$ we mean the set of all real symmetric n-by-n matrices whose graph is precisely G. Note that no constraint, other than reality, is imposed upon the diagonal entries of $A \in \mathscr{S}(G)$. An overriding problem is to understand the possible lists of multiplicities for the eigenvalues among the matrices in $\mathscr{S}(G)$. As in prior work, we concentrate here upon the case in which $G = T$ is a tree.

For a tree T, there are several formulas for the maximum possible multiplicity $M(T)$ for a single eigenvalue of a matrix in $\mathscr{S}(T)$. It is, for example, $P(T)$, the *path cover number* of T [2]. For certain trees T, whenever a matrix $A \in \mathscr{S}(T)$ attains this maximum multiplicity, all other multiplicities are 1. This happens, for example, both for a path on n vertices and for a star on n vertices. Our purpose here is to characterize all such trees. We call such trees NIM trees (*n*o *i*ntermediate *m*ultiplicities). We note that not all trees are NIM. For example, the tree DP_3



has maximum multiplicity 2, but 2, 2, 1, 1 is a multiplicity list. This is the smallest tree (fewest vertices) for which a non-NIM tree exists.

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Our main result is a graph theoretic characterization of NIM trees. For a tree T, let $u(T)=\{v_1,\ldots,v_k\}$ be the set of vertices of T of degree at least 3 ("high" degree vertices) and let $H=H(T)$ denote the subgraph of T induced by $u(T)$. For a given vertex v, we denote its degree in the graph G by $\deg_G(v)$. By $G-v$ we mean the subgraph of G induced by the vertices of G other than v. For trees, $T-v$ has exactly $\deg_T(v)$ components, each one being a tree, which we call *branches* of T at v.

Theorem 1. *Let T be a tree on n vertices. Then T is* NIM *if and only if for each $v\in u(T)$:*

(i) *at most two components of $T-v$ have more than one vertex, and*
(ii) $\deg_T(v)\geqslant\deg_H(v)+3$.

In order to prove the claimed result, we need some auxiliary results and background. If G is an undirected graph on n vertices and $A\in\mathscr{S}(G)$, given $\gamma\subseteq\{1,\ldots,n\}$, we denote the principal submatrix of A resulting from retention (deletion) of the rows and columns γ by $A[\gamma]$ ($A(\gamma)$). If G' is the subgraph of G induced by γ, we may write $A[G']$ ($A(G')$) instead of $A[\gamma]$ ($A(\gamma)$). We often refer to the "eigenvalues" of G' meaning the eigenvalues of the principal submatrix $A[G']$ of A. We also denote by $\sigma(A)$ the spectrum of A and by $m_A(\lambda)$ the multiplicity of λ as an eigenvalue of A.

The following fact is easily verified, and we shall use its corollary to arrange a new common eigenvalue among the branches of T, without changing the multiplicity of another eigenvalue.

Lemma 2. *Let G be an undirected graph on n vertices, $A\in\mathscr{S}(G)$ and $\gamma\subseteq\{1,\ldots,n\}$. If $\alpha,\beta\in\mathbb{R}$, $\alpha\neq 0$, then $B=\alpha A+\beta I_n\in\mathscr{S}(G)$ and $m_{B[\gamma]}(\alpha\lambda+\beta)=m_{A[\gamma]}(\lambda)$.*

Corollary 3. *Let G be an undirected graph on n vertices, $A\in\mathscr{S}(G)$ and $\gamma\subseteq\{1,\ldots,n\}$. Suppose that $\lambda,\mu\in\sigma(A[\gamma])$, $\lambda\neq\mu$, and let $\lambda',\mu'\in\mathbb{R}$ be such that $\lambda'\neq\mu'$. Then, there is a $B\in\mathscr{S}(G)$ such that $m_{B[\gamma]}(\lambda')=m_{A[\gamma]}(\lambda)$ and $m_{B[\gamma]}(\mu')=m_{A[\gamma]}(\mu)$.*

Let T be a tree and $A\in\mathscr{S}(T)$. When λ is an eigenvalue of A of multiplicity m, by the interlacing inequalities (see, e.g., [1]), we have $m_{A(i)}(\lambda)\in\{m-1,m,m+1\}$. However, in [3] it was shown that if $\lambda\in\sigma(A)\cap\sigma(A(i))$ for a vertex i of T, then there is a vertex v of T such that $m_{A(v)}(\lambda)=m_A(\lambda)+1$. For historical reasons (see [6,7,3]) we call such a vertex v, a *Parter vertex* of T for λ relative to A (a Parter vertex, for short). Note that, when $m_A(\lambda)\geqslant 2$, there is always a Parter vertex for λ. Moreover, there must exist a Parter vertex v' of degree at least 3 and such that λ is an eigenvalue of at least 3 direct summands of $A(v')$. We call such a vertex v', a *strong Parter vertex* of T for λ relative to A (a strong Parter vertex, for short). When $m_A(\lambda)=1$ and λ is an eigenvalue of a principal submatrix $A(i)$ of A, there exists a Parter vertex v' of degree at least 2 such that λ is an eigenvalue of at least 2 direct summands of $A(v')$.

When $m_A(\lambda)\geqslant 1$ and $\{v_1,\ldots,v_k\}$ is a set of vertices of T such that $m_{A(\{v_1,\ldots,v_k\})}(\lambda)=m_A(\lambda)+k$, we call $\{v_1,\ldots,v_k\}$ a *Parter set* of vertices of T for λ relative to A (a Parter set, for short). Each vertex in a Parter set of vertices must be individually Parter [4]. However, a collection of Parter vertices does not necessarily form a Parter set [4].

In [2], the authors show that $M(T)$ not only is $P(T)$, but is also $\max[p-q]$, such that there exist q vertices of T whose removal from T leaves p paths. We call such a set of q vertices a *residual path maximizing* set (an RPM set, for short). In general, an RPM set of vertices is not unique, not even in the value of q.

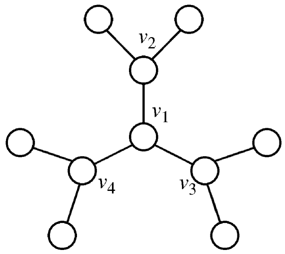
If the removal of q vertices $v_1,\ldots,v_q$ from T leaves p paths such that $p-q=M(T)$, i.e., $p=M(T)+q$, a matrix $A\in\mathscr{S}(T)$ having $\lambda\in\mathbb{R}$ as an eigenvalue of each summand corresponding to each of the p components (with multiplicity 1, as a real symmetric matrix whose graph is a path has only simple eigenvalues) satisfies $m_{A(\{v_1,\ldots,v_q\})}(\lambda)=M(T)+q$ and, therefore, $m_A(\lambda)=M(T)$. Since the removal of each v_i from T must have increased the multiplicity of λ by 1, we may conclude that each v_i is Parter for λ and, of course, $\{v_1,\ldots,v_q\}$ is a Parter set for λ.

In [3], it was also shown that a Parter vertex for an eigenvalue λ relative to a matrix $A\in\mathscr{S}(T)$ always belongs to a Parter set whose removal from T leaves components in which the corresponding summands of A have λ as an eigenvalue of multiplicity 1. So, we may now state the following facts.

Lemma 4. *Let T be a tree on n vertices and let λ be an eigenvalue of $A\in\mathscr{S}(T)$ of multiplicity $M(T)$. If a vertex v of T is Parter for λ then $\deg_T(v)\geqslant 2$, $m_{A(v)}(\lambda)=m_A(\lambda)+1$ and λ is an eigenvalue of each of the $\deg_T(v)$ components of $T-v$.*

Lemma 5. *Let T be a tree on n vertices, and let λ be an eigenvalue of $A \in \mathscr{S}(T)$ of multiplicity $M(T)$. A vertex v is Parter for the eigenvalue λ in A if and only if there is an* RPM *set Q containing v such that* $m_{A(Q)}(\lambda) = M(T) + |Q|$.

It may happen that several sets of q vertices achieve this maximum and that the maximum may be achieved for several values of q. For example, if we consider again the tree DP_3, we have $M(T) = 2$. In order to maximize $p - q$, we may remove $q = 2$ vertices (v_1 and v_2) or only $q = 1$ vertex (v_1 or v_2). This means that there are matrices in $\mathscr{S}(T)$ having an eigenvalue of multiplicity 2 for which, either v_1 and v_2 are Parter or only one of them is Parter. This is an example of a tree in which each of the high degree vertices may be removed in order to maximize $p - q$. However there are trees in which some high degree vertices cannot be part of an RPM set. For example, the following tree T'



has $M(T') = 4$. Observe that only $\{v_2, v_3, v_4\}$ is an RPM set of vertices for T'. So, there is no matrix in $\mathscr{S}(T')$ for which the vertex v_1 is Parter for an eigenvalue of multiplicity 4.

In [5], an algorithm was given to compute $M(T)$ for a general tree T. The strategy was to determine an RPM set of vertices. For this purpose, (1) and (2) of the following lemma was shown.

Lemma 6. *Let T be a tree and let $v \in u(T)$. Then we have the following*:

(1) *If* $\deg_T(v) \geqslant \deg_H(v) + 3$, *then v belongs to every* RPM *set of vertices.*
(2) *If* $\deg_T(v) = \deg_H(v) + 2$ *and if Q is an* RPM *set of vertices, then either* $v \in Q$, *or* $Q \cup \{v\}$ *is also an* RPM *set.*
(3) *If* $\max_{v \in u(T)} \deg_H(v) = 2$ *and there exists* $v \in u(T)$ *such that* $\deg_T(v) = \deg_H(v) + 1$, *then no* RPM *set of vertices contains all vertices v in* $u(T)$ *such that* $\deg_T(v) = \deg_H(v) + 1$.
(4) *If* $\max_{v \in u(T)} \deg_H(v) = 2$, $v \in u(T)$, $\deg_T(v) = \deg_H(v) + 1$ *and v does not belong to an* RPM *set Q, then at least one vertex adjacent to v in H must belong to the set Q. Moreover, there is an* RPM *set containing both the vertices adjacent to v in H.*

Proof. We only need to prove (3) and (4). By hypothesis in (3) and (4), the maximum degree of a vertex in H is 2; thus, it follows that any vertex in $u(T)$ satisfying $\deg_T(v) = \deg_H(v) + 1$ has degree 3. We may also conclude that there are no vertices satisfying $\deg_T(v) = \deg_H(v)$, so that $\deg_T(v) \geqslant \deg_H(v) + 1$ for all vertices in $u(T)$.

For (3), in order to obtain a contradiction, suppose that there is an RPM set Q containing all vertices v of $u(T)$ such that $\deg_T(v) = \deg_H(v) + 1$. By (1), Q contains all vertices v of $u(T)$ such that $\deg_T(v) \geqslant \deg_H(v) + 3$ and, by (2), we may assume without loss of generality that Q contains all vertices v of $u(T)$ such that $\deg_T(v) = \deg_H(v) + 2$. Since all vertices v in $u(T)$ satisfy $\deg_T(v) \geqslant \deg_H(v) + 1$, Q contains all high degree vertices of T. Now consider any particular v such $\deg_T(v) = \deg_H(v) + 1$ and $Q' = Q \backslash \{v\}$. Now, $\deg_{T-Q'}(v) = 1$ and removal of v from $T - Q'$ could not increase the number of components. This contradiction verifies claim (3).

For (4), let v be a vertex guaranteed by the hypothesis. By (3), v does not belong to some RPM set Q. As there are RPM sets contained in $u(T)$, we may assume without loss of generality that $Q \subseteq u(T)$. Because there are no vertices of degree greater than 2 among the p paths resulting from the removal of Q, we may conclude that at least one of the vertices adjacent to v in H belongs to Q (because v has one neighbor not in $u(T)$ and thus not in Q). Suppose that u is a vertex adjacent to v in H, and u does not belong to Q. Thus, u and v belong to one path T' of the remaining p paths of $T - Q$. Because $\deg_T(u) \geqslant \deg_H(u) + 1$, there is a neighbor of u in T that is not in $u(T)$. Since T' is a path, we may conclude that $\deg_{T'}(u) = 2$. Thus, if u is removed from T' the number of paths remaining increases by 1 (as well the number of removed vertices from T). Therefore, $Q \cup \{u\}$ is an RPM set. $\square$

We may now turn to the proof of Theorem 1.

Proof of Theorem 1. We start by showing that conditions (i) and (ii) are together sufficient. Suppose that T is a tree on n vertices satisfying conditions (i) and (ii) and let $A \in \mathscr{S}(T)$ having λ as an eigenvalue of multiplicity $M(T)$. By (1) of Lemmas 6 and 5, any vertex v of T satisfying (ii) must be a Parter vertex for λ, which implies that all vertices in $u(T)$ are Parter for λ. Because of Lemma 4, for any vertex $v \in u(T)$, λ is an eigenvalue of $\deg_T(v)$ direct summands of $A(v)$, and, since T satisfies (i), each vertex in $u(T)$ may be a strong Parter for at most one multiple eigenvalue. Therefore, (i) and (ii) together imply that each vertex in $u(T)$ must be Parter for exactly one multiple eigenvalue, the eigenvalue of multiplicity $M(T)$, which proves the sufficiency of the stated conditions.

For the necessity of the stated conditions, our strategy is to show that if either (i) or (ii) does not hold for a tree T, then a matrix in $\mathscr{S}(T)$ may be constructed with an eigenvalue of maximum multiplicity $M(T)$ and another multiple eigenvalue. We first show that not (i) implies not NIM, and then, when we show that not (ii) implies not NIM, we may and do assume that (i) holds.

First suppose that (i) is not satisfied. Then, there is a vertex v of degree at least 3 such that $T - v$ has at least 3 components of more than 1 vertex; we use only 3. We consider two cases: (a) v can be Parter for λ, the maximum multiplicity eigenvalue (in some $A \in \mathscr{S}(T)$); or (b) v is never Parter for λ.

In case (a), let $A \in \mathscr{S}(T)$ be such that $m_A(\lambda) = M(T)$, the maximum possible, and v is Parter for λ in A. Let T_1, T_2 and T_3 be 3 components of $T - v$ with at least 2 vertices and let A_1, A_2 and A_3 be the corresponding principal submatrices of A. Choose $\mu \in \mathbb{R}$ such that $\mu \neq \lambda$, and α_i, β_i, $i = 1, 2, 3$, according to Lemma 2, so that, by Corollary 3, $B_i = \alpha_i A_i + \beta_i I_n \in \mathscr{S}(T_i)$, $m_{B_i}(\lambda) = m_{A_i}(\lambda)$ and $m_{B_i}(\mu) \geqslant 1$, $i = 1, 2, 3$. Now, defining B by replacing A_1, A_2 and A_3 in A by B_1, B_2 and B_3, respectively (and no other changes), $B \in \mathscr{S}(T)$, $m_B(\lambda) = M(T)$ (since, by construction, we have $m_{B(v)}(\lambda) = m_{A(v)}(\lambda) = M(T) + 1$) and $m_B(\mu) \geqslant 2$ (by the interlacing inequalities for the eigenvalues of a symmetric matrix), so that T is not NIM.

In case (b), let $A \in \mathscr{S}(T)$ satisfy $m_A(\lambda) = M(T)$. Then, there is an RPM set Q, $|Q| = q$, of vertices whose removal from T leave $p = M(T) + q$ paths, in each of which λ occurs as an eigenvalue of multiplicity 1. By Lemma 5, vertex v is not in any RPM set, so that vertex v must remain as part of one of these p (possibly degenerate) paths. The path that contains v must have v as an endpoint or else it is possible to remove v (increasing q by 1) and increase p by 1, so that $Q \cup \{v\}$ would be an RPM set, contradicting Lemma 5. Now decompose T into the branches at v, and either make v a separate part of the decomposition (if it is a single vertex among the p paths) or include it in the unique branch that its path (among the p paths) intersects (otherwise). Now, each of the p paths lies fully within one of the parts of this decomposition: call them $T_1, \ldots, T_k$. Each part corresponds to a principal submatrix A_i of A, $i = 1, \ldots, k$. We may now apply Lemma 2 to each A_i, producing B_i, and then replace each A_i in A by B_i to produce $B \in \mathscr{S}(T)$. Choose, for each i such that T_i has more than 1 vertex (there are at least 3) $\alpha_i \neq 0$ and β_i so that $\alpha_i \lambda + \beta_i = \lambda$ and $\mu \in \sigma(B_i)$, except that if v is a vertex of T_i, choose α_i, β_i so as to attain $\mu \in \sigma(B_i(v))$, while applying the linear transformation to A_i to obtain B_i. Now in B, $m_B(\lambda) = m_A(\lambda) = M(T)$, as λ is still an eigenvalue of the principal submatrix corresponding to each of the p paths. But also $m_B(\mu) \geqslant 2$, as $m_{B(v)}(\mu) \geqslant 3$. Since $B \in \mathscr{S}(T)$, T is not NIM, completing this portion of the proof.

Suppose now that (ii) is not satisfied and assume that (i) holds. Let $v \in u(T)$ be such that $\deg_T(v) \leqslant \deg_H(v) + 2$. Thus, $\deg_T(v) = \deg_H(v) + 2$ or $\deg_T(v) = \deg_H(v) + 1$ or $\deg_T(v) = \deg_H(v)$. By (i), we have $\deg_H(v) \leqslant 2$, so that $\deg_T(v) = \deg_H(v)$ cannot occur. We consider the remaining two cases: (a′) there exists $v \in u(T)$ with $\deg_T(v) = \deg_H(v) + 1$; or (b′) there exists $v \in u(T)$ with $\deg_T(v) = \deg_H(v) + 2$.

In case (a′), let $v \in u(T)$ with $\deg_T(v) = \deg_H(v) + 1$. Observe that, since $\deg_H(v) \leqslant 2$ we have $\deg_T(v) = 3$ and $\deg_H(v) = 2$, so that there are exactly two high degree vertices adjacent to v. Because of (i), we conclude that there is a vertex pendant at v. By part (3) of Lemma 6, we may assume that v is a vertex such that $\deg_T(v) = \deg_H(v) + 1$ and that v does not belong to an RPM set of vertices. By part (4) of Lemma 6, there is an RPM set Q of q vertices containing both high degree neighbors of v and one of the $p = M(T) + q$ components resulting from deletion of Q is a path T_1 on 2 vertices including the vertex v. By Lemma 5, we may conclude that there is a matrix $A \in \mathscr{S}(T)$, having λ as an eigenvalue of multiplicity $M(T)$, such that v is not Parter for λ. Now let T_2 and T_3 be the two branches of T at v that do not include the vertex pendant at v. Consider the decomposition of T into the components T_1, T_2 and T_3. Using this decomposition and following the procedure used to prove case (b) above, we may obtain a matrix $B \in \mathscr{S}(T)$ such that $m_B(\lambda) = m_A(\lambda) = M(T)$ but with an additional multiple eigenvalue. Note that either $A_1 = A[T_1]$ need not be transformed (and then μ is the single entry of $A_1(v)$, which cannot be λ) or may be transformed as in the atypical case in which v is adjoined to one of its branches in the proof of (b) above. This proves that T is not NIM.

In case (b′), we may now assume that $\deg_T(v) \geqslant \deg_H(v)+2$ for all vertices in $u(T)$. Suppose that there is a particular vertex $v \in u(T)$ such that $\deg_T(v) = \deg_H(v) + 2$. Because $v \in u(T)$, (i) implies that $\deg_H(v) \in \{1, 2\}$ and there is at least one pendant vertex v' at v. Consider the set of all vertices u in $u(T)$ such that $\deg_T(u) \geqslant \deg_H(u) + 2$, except vertex v. By part (1) and (2) of Lemma 6, the removal from T of such a set of q vertices leaves p paths and maximizes $p - q$; i.e., this is an RPM set Q. Since $\deg_T(v) = \deg_H(v) + 2$, one of the p components is a path T_1 having v as an interior vertex. By Lemma 5, we may conclude that there is a matrix $A \in \mathscr{S}(T)$, having λ as an eigenvalue of multiplicity $M(T)$ and such that $m_{A[T_1]}(\lambda) = 1$. Since $A[T_1]$ may be chosen so that λ does not occur as an eigenvalue of $A[T_1 - v]$, we assume that v is not Parter for λ in $A[T_1]$.

Suppose first that $\deg_H(v) = 1$. Recall that one of the endpoints of T_1 is v' and denote by v'' the other endpoint of T_1. Observe that, when $\deg_H(v) = 1$ it may occur that v'' is adjacent, in T, to a vertex v''' of the RPM set Q. We shall use the following decomposition of T: if v'' is a pendant vertex in T we consider a decomposition of T into components T_1 and T_2, in which T_2 is the branch of T at v not containing vertices of T_1; if v'' is not a pendant vertex in T (i.e., there is a vertex v''' of the RPM set Q, adjacent to v'' in T) we consider a decomposition of T into components T_1, T_2 and T_3, in which T_2 is the branch of T at v not containing vertices of T_1, and T_3 is the branch of T at v'' not containing vertices of T_1 (i.e., containing the vertex v'''). Now, making the above described decomposition into 2 or 3 components, each of the p paths lies fully within one of the parts of this decomposition. Suppose that such a decomposition has parts T_i, each part corresponding to a principal submatrix A_i of A, in which $i = 1, 2, 3$ or $i = 1, 2$, depending on the number of components of the decomposition of T. As T_1 is a path, we have $m_{A_1}(\lambda) = 1$. Choose $\mu \in \mathbb{R}$, $\mu \neq \lambda$, and replace (in A) A_1 by a matrix $B_1 \in \mathscr{S}(T_1)$ having λ as an eigenvalue and such that $m_{B_1(v)}(\mu) = 2$. (Since T_1 is a path, every eigenvalue of a matrix in $\mathscr{S}(T_1)$ has multiplicity one. As v is an interior vertex of T_1, any matrix $A' \in \mathscr{S}(T_1)$ such that $m_{A'(v)}(\mu) = 2$ for a given $\mu \in \mathbb{R}$, we necessarily have $m_{A'}(\mu) = 1$. By choosing an eigenvalue γ of A', $\gamma \neq \mu$, we may use Lemma 2, and by a linear transformation to A' we obtain such a matrix B_1.) We may now apply Lemma 2 to A_2 producing B_2, and then replace A_2 in A by B_2 to produce $B \in \mathscr{S}(T)$. Choose $\alpha_2 \neq 0$ and β_2 so that $\alpha_2\lambda + \beta_2 = \lambda$ and $\mu \in \sigma(B_2)$, while applying the linear transformation to A_2 to obtain B_2. If the above described decomposition has only 2 components T_1 and T_2, we get a matrix $B \in \mathscr{S}(T)$ such that $m_B(\lambda) = m_A(\lambda) = M(T)$, as λ is still an eigenvalue of the principal submatrix corresponding to each of the p paths. But also with an additional multiple eigenvalue because, by construction, we have $m_{B(v)}(\mu) \geqslant 3$ and, by the interlacing inequalities, we have that $m_B(\mu) \geqslant 2$, which proves that T is not NIM. If the above described decomposition has 3 components, we also choose $\alpha_3 \neq 0$ and β_3 so that $\alpha_3\lambda + \beta_3 = \lambda$ and $\mu \in \sigma(B_3(v'''))$, while applying the linear transformation to A_3 to obtain B_3 (recall that v''' is the vertex of T_3 adjacent to v'' in T and that belongs to the RPM set Q). As in the case in which we have a decomposition with 2 components, we may conclude that we get a matrix $B \in \mathscr{S}(T)$ such that $m_B(\lambda) = m_A(\lambda) = M(T)$. By construction we have $m_{B(\{v,v'''\})}(\mu) \geqslant 4$ and, by the interlacing inequalities, we conclude that $m_B(\mu) \geqslant 2$, which proves that T is not NIM.

To finish the proof, we suppose now that $\deg_H(v) = 2$. Since $\deg_T(v) = \deg_H(v) + 2$, by (i), we conclude that v has exactly 2 pendant vertices. In this case, we shall use the following decomposition of T: let T_1 be the path on 3 vertices having v as the interior vertex and the 2 endpoints are the pendant vertices at v in T, and let T_i, $i = 2, 3$, be the 2 branches of T at v that do not contain vertices of T_1. This decomposition has 3 parts T_1, T_2 and T_3, each part corresponding to a principal submatrix A_i of A, $i = 1, 2, 3$. As T_1 is a path, we have $m_{A_1}(\lambda) = 1$. Choose $\mu \in \mathbb{R}$, $\mu \neq \lambda$, and replace (in A) A_1 by a matrix $B_1 \in \mathscr{S}(T_1)$ having λ as an eigenvalue and such that $m_{B_1(v)}(\mu) = 2$. We may now apply Lemma 2 to A_i, $i = 2, 3$, producing B_i, and then replace each A_i in A by B_i to produce $B \in \mathscr{S}(T)$. For each i, $i = 2, 3$, choose $\alpha_i \neq 0$ and β_i so that $\alpha_i\lambda + \beta_i = \lambda$ and $\mu \in \sigma(B_i)$, while applying the linear transformation to A_i to obtain B_i. Now in B, $m_B(\lambda) = m_A(\lambda) = M(T)$, as λ is still an eigenvalue of the principal submatrix corresponding to each of the p paths. But also $m_B(\mu) \geqslant 3$, because by construction we have $m_{B(v)}(\mu) \geqslant 4$ and, by the interlacing inequalities, we have that $m_B(\mu) \geqslant 3$, which proves that T is not NIM. □

We conclude by noting that a topological description of NIM trees may be deduced from the Theorem 1, and a list of "minimal" NIM trees (no pendant vertex may be removed from a high degree vertex and no NIM tree in the list is homeomorphic to a prior one in the list) could be given.

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