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Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars

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Abstract

We characterize the possible lists of ordered multiplicities among matrices whose graph is a generalized star (a tree in which at most one vertex has degree greater than 2) or a double generalized star. Here, the inverse eigenvalue problem (IEP) for symmetric matrices whose graph is a generalized star is settled. The answer is consistent with a conjecture that determination of the possible ordered multiplicities is equivalent to the IEP for a given tree. Moreover, a key spectral feature of the IEP in the case of generalized stars is shown to characterize them among trees.

Keywords: Hermitian matrices; Eigenvalues; Multiplicities; Trees; Inverse eigenvalue problems

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1. Introduction

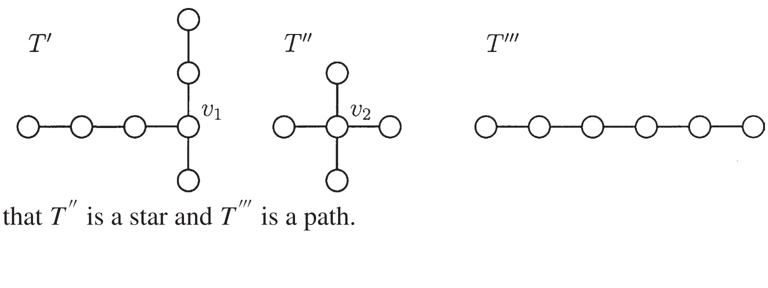
Given an n-by-n Hermitian matrix $A = (a_{ij})$, we denote by $\mathscr{G}(A)$ the (undirected) graph of A; it has vertex set $\{1, \ldots, n\}$ and an edge $\{i, j\}$, $i \neq j$, if and only if $a_{ij} \neq 0$. For an undirected graph G on vertices $1, \ldots, n$, we denote by $\mathscr{S}(G)$ the set of all Hermitian matrices whose graph is G. If $A = (a_{ij})$ and $\alpha \subseteq \{1, \ldots, n\}$ is an index set, we denote the principal submatrix of A resulting from deletion (retention) of the rows and columns α by $A(\alpha)$ $(A[\alpha])$. Note that the subgraph G' of G, induced by vertices in α corresponds, in a natural way, to a graph G'' whose vertex set is $\{1, \ldots, |\alpha|\}$. So we will often identify the two graphs, G' and G''; namely, we will refer to matrices with graph G', meaning matrices with graph G''. We also write $A(G')$ $(A[G'])$ instead of $A(\alpha)$ $(A[\alpha])$. When α consists of a single vertex i, we abbreviate $A(\{i\})$ $(G - \{i\})$ by $A(i)$ $(G - i)$. In particular, if G is a tree and A is a matrix in $\mathscr{S}(G)$, $A(v)$ is a direct sum whose summands correspond to components of $G - v$ (which we call *branches* of G at v), the number of summands or components being the degree of v (deg v) in G.

Here, we consider the case in which G is a tree T. If v is an identified vertex of T of degree k, we identify the neighbors of v in T as $u_1, \ldots, u_k$, and we denote the branch of T resulting from deletion of v and containing u_i by T_i, $i = 1, \ldots, k$. Special attention is given to a certain class of trees, the generalized stars and the double generalized stars.

Definition 1. A *generalized star* is a tree T having at most one vertex of degree greater than 2. We call *central vertex* of a generalized star T, to a vertex v of degree k, whose neighbors $u_1, \ldots, u_k$ are pendant vertices of their branches $T_1, \ldots, T_k$, respectively, and each of these branches is a path.

Note that, according to our definition of generalized stars, a *path* (tree with no vertex of degree greater than 2) is a (degenerate) generalized star; in this case, any vertex will be a central vertex. If T is a generalized star with a vertex of degree greater than 2, then it is the unique central vertex of T. The above definition also includes the case of stars; recall that a *star* on n vertices is a tree in which there is a vertex of degree $n - 1$.

The following trees T', T'' and T''', are examples of generalized stars. The central vertices of T' and T'' are, respectively, v_1 and v_2, while any vertex of T''' is a central vertex.

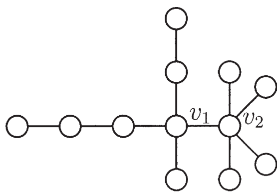


Note that T'' is a star and T''' is a path.

Definition 2. Given two generalized stars, T_1 and T_2, a *double generalized star* is the tree resulting from joining a central vertex of T_1 to a central vertex of T_2 by an edge. We denote the result as $D(T_1, T_2)$.

Observe that, if T_1 or T_2 is a path, the double generalized star resulting from joining a central vertex of T_1 with a central vertex of T_2, depends obviously on the selected central vertex in the path. When we write $D(T_1, T_2)$ we are supposing that the central vertices were previously fixed. We note that the paths and generalized stars are also (degenerate) double generalized stars, as well as the double paths studied in [4].

Considering, for example, the generalized stars T' and T'', the double generalized star $D(T', T'')$ is then



For an Hermitian matrix A, we denote the (algebraic) multiplicity of λ as an eigenvalue of A by $m_A(\lambda)$ and we denote the characteristic polynomial of A by $p_A(t)$.

Because of the interlacing theorem for Hermitian eigenvalues [10], there is a simple relation between $m_{A(i)}(\lambda)$ and $m_A(\lambda)$ when A is Hermitian:

$$m_{A(i)}(\lambda) = m_A(\lambda) + 1 \text{ or } m_{A(i)}(\lambda) = m_A(\lambda) \text{ or } m_{A(i)}(\lambda) = m_A(\lambda) - 1.$$

Later, in Section 4, we identify a unique class of trees (generalized stars) in which, considering any tree T, there is an identifiable vertex v of T such that, if A is any matrix in $\mathscr{S}(T)$ and λ is any eigenvalue of $A(v)$, then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.

For any generalized star T we determine the collection of lists of multiplicities, ordered by numerical order of the underlying eigenvalues, that occur among matrices in $\mathscr{S}(T)$.

We further solve the inverse eigenvalue problem (IEP) for matrices whose graph is a given generalized star T, and we note that the IEP is equivalent to determining which lists of ordered multiplicities occur in $\mathscr{S}(T)$; i.e., the only constraint on existence of a matrix in $\mathscr{S}(T)$ with a prescribed spectrum (real numbers, as matrices in $\mathscr{S}(T)$ are Hermitian) is the existence of the corresponding list of ordered multiplicities.

Finally, we turn our attention to the double generalized stars. For a double generalized star T we give a characterization of the lists of ordered multiplicities among matrices in $\mathscr{S}(T)$.

2. Prior results

We record here some known results that will be important for the present work. The following theorem was shown in [8].

Theorem 3. *Let T be a tree on n vertices and v be a vertex of T. Let $\lambda_1 < \cdots < \lambda_n$ and $\mu_1 < \cdots < \mu_{n-1}$ be real numbers. If*

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n,$$

then there exists a matrix A in $\mathscr{S}(T)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and such that, $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.

The key tool used in [8] to prove the above mentioned result was the decomposition of a real rational function into partial fractions. We recall here the following well known results, which will be useful for the present work.

Lemma 4. *Let $g(t)$ be a monic polynomial of degree n, $n > 1$, having all its roots real and distinct and let $h(t)$ be a monic polynomial with $\deg h(t) < \deg g(t)$. Then $h(t)$ has $n-1$ distinct real roots strictly interlacing the roots of $g(t)$ if and only if the coefficients of the partial fraction decomposition (pdf) of $\frac{h(t)}{g(t)}$ are positive real numbers.*

Remark 5. If $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-1}$ are real numbers such that

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n,$$

and, $g(t)$ and $h(t)$ are the monic polynomials

$$g(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n),$$
$$h(t) = (t - \mu_1)(t - \mu_2) \cdots (t - \mu_{n-1}),$$

then it is easy to show that $\frac{g(t)}{h(t)}$ can be represented in a unique way as

$$\frac{g(t)}{h(t)} = (t - a) - \sum_{i=1}^{n-1} \frac{x_i}{t - \mu_i}$$

in which $a = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i$ and x_i, $i = 1, \ldots, n-1$, are positive real numbers such that

$$x_i = -\frac{g(\mu_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n-1} (\mu_i - \mu_j)} = -\frac{\prod_{j=1}^{n} (\mu_i - \lambda_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{n-1} (\mu_i - \mu_j)}.$$

We will also need the characteristic polynomial of a matrix whose graph is a given tree T. In the following lemma we focus upon the expansion of the characteristic polynomial at a particular vertex v of T with neighbors $u_1, \ldots, u_k$ (see, e.g., [7]).

Lemma 6. *Let T be a tree on n vertices and $A = (a_{ij})$ be a matrix in $\mathscr{S}(T)$. If v is a vertex of T of degree k, whose neighbors in T are $u_1, \ldots, u_k$, then*

$$p_A(t) = (t - a_{vv})p_{A[T-v]}(t) - \sum_{i=1}^{k} |a_{vu_i}|^2 p_{A[T_i-u_i]}(t) \prod_{\substack{j=1\\ j\neq i}}^{k} p_{A[T_j]}(t) \qquad (1)$$

with the convention that $p_{A[T_i-u_i]}(t) = 1$ whenever the vertex set of T_i is $\{u_i\}$.

Since T is a tree, if A is a matrix in $\mathscr{S}(T)$ and v is a vertex of degree k, we have $A(v) = A[T_1] \oplus \cdots \oplus A[T_k]$ where T_i is the branch of $T - v$ containing the neighbor u_i of v in T. It was shown in [7] that the existence of a branch T_i of v, in whose branch the multiplicity of an eigenvalue λ of $A[T_i]$ goes down when u_i is removed from T_i, implies that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.

Lemma 7. *Let T be a tree and A be a matrix in $\mathscr{S}(T)$. Let v be a vertex of T and λ be an eigenvalue of $A(v)$. Let u_i be a neighbor of v in T and T_i be the branch of T at v containing u_i. If λ is an eigenvalue of $A[T_i]$ and*

$$m_{A[T_i-u_i]}(\lambda) = m_{A[T_i]}(\lambda) - 1, \qquad (2)$$

then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.

A branch T_i satisfying (2) for an eigenvalue λ of $A[T_i]$ is called a *downer branch* at v for the eigenvalue λ (downer branch, for short); the vertex u_i is called a *downer vertex*.

The following result is well known and may be easily checked considering the prior lemma and the interlacing theorem for Hermitian eigenvalues.

Lemma 8. *If T is a tree, the largest and smallest eigenvalues of each matrix A in $\mathscr{S}(T)$, have multiplicity 1. Moreover, the largest or smallest eigenvalue of a matrix A in $\mathscr{S}(T)$ cannot occur as an eigenvalue of a submatrix $A(v)$, for any vertex v of T.*

The paths play an important role in Section 4, so we record a long known fact that we shall use.

Lemma 9. *Let T be a path whose pendant vertices are the vertices u_i and u_j. If A is a matrix in $\mathscr{S}(T)$ then the eigenvalues of A are all of multiplicity 1 and the eigenvalues of $A[T - u_i]$ ($A[T - u_j]$) strictly interlace those of A.*

3. Inverse eigenvalue problems

One of the classical IEP is the following one.

General inverse eigenvalue problem (GIEP) for tridiagonal matrices: *Given real numbers $\lambda_1, \ldots, \lambda_n$, and $\mu_1, \ldots, \mu_{n-1}$, construct a symmetric irreducible* *tridiagonal, n-by-n matrix A such that A has eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and* $A(1)$ *has eigenvalues* $\mu_1, \ldots, \mu_{n-1}$.

Lemma 9 gives a necessary condition for this problem to have a solution and it is well known that this condition is also sufficient. For a survey of this and other IEPs see [2]; a physical interpretation of the above inverse problem is also presented in [2] (see specially [2, Section 3] and also [1]).

Note that the graph of a tridiagonal matrix is a path, so that it is natural to consider an analogous GIEP for the case in which A is a matrix in $\mathscr{S}(T)$, T being any particular tree.

> GIEP for $\mathscr{S}(T)$: *Given a tree* T *with vertex set* $\{1, \ldots, n\}$, *a vertex* v *of* T *of degree* k, $T_1, \ldots, T_k$ *being the connected components of* $T - v$ *and given real numbers* $\lambda_1, \ldots, \lambda_n$, *and monic polynomials* $g_1, \ldots, g_k$, *having only real roots,* $\deg g_i$ *equal to the number of vertices of* T_i, *construct a matrix A in* $\mathscr{S}(T)$ *such that A has eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and such that the eigenvalues of* $A[T_i]$ *are the roots of* g_i.

This problem was studied in [8] where it was shown that the strict interlacing between the λ's and the μ's (roots of $g_1 \cdots g_k$) was a sufficient condition for the problem to have a solution (see Theorem 3 above). Note that it follows immediately from Theorem 3 that for any tree T with n vertices and any given set of *distinct* real numbers there exists a matrix A in $\mathscr{S}(T)$ such that A has these numbers as eigenvalues.

The strict interlacing of Theorem 3 is not generally necessary for this IEP to have a solution. In fact, it is well known that a matrix A in $\mathscr{S}(T)$ can have multiple eigenvalues and recently much research has been done about the possible lists of multiplicities that may occur among the eigenvalues of matrices in $\mathscr{S}(T)$. (See [7], and references therein, for a survey of the subject.) The following IEP is related to this question.

> IEP for $\mathscr{S}(T)$: *Given a tree* T *with vertex set* $\{1, \ldots, n\}$ *and real numbers* $\lambda_1, \ldots, \lambda_n$, *construct a matrix A in* $\mathscr{S}(T)$ *such that A has eigenvalues* $\lambda_1, \ldots, \lambda_n$.

As mentioned before, this problem has a solution if the λ's are distinct, and, we believe, that the only restrictions on the λ's, for a solution to exist, are those on the ordered multiplicities. So, if this is the case, a description of all possible lists of ordered multiplicities for the eigenvalues of matrices in $\mathscr{S}(T)$ will give a necessary and sufficient condition for the IEP for $\mathscr{S}(T)$ to have a solution. We will see in the next section that, if T is a generalized star the two questions are in fact equivalent.

The following theorem gives a partial answer of the GIEP for $\mathscr{S}(T)$.

Theorem 10. *Let* T *be a tree on* n *vertices*, v *be a vertex of* T *of degree* k *whose neighbors are* $u_1, \ldots, u_k$, T_i *be the branch of* T *at* v *containing* u_i, *and* s_i *be the number of vertices in* T_i, $i = 1, \ldots, k$.

Let $g_1(t), \dots, g_k(t)$ be monic polynomials having only distinct real roots, with $\deg g_i(t) = s_i$, and $p_1, \dots, p_s$ be the distinct roots among polynomials $g_i(t)$ and m_i be the multiplicity of root p_i in $\prod_{i=1}^k g_i(t)$.

Let $g(t)$ be a monic polynomial of degree $s+1$.

There exists a matrix A in $\mathscr{S}(T)$ with characteristic polynomial $f(t) = g(t)\prod_{i=1}^s (t-p_i)^{m_i-1}$ and such that $A[T_i]$ has characteristic polynomial $g_i(t)$, $i = 1, \dots, k$, and, if $s_i > 1$, the eigenvalues of $A[T_i - u_i]$ strictly interlace those of $A[T_i]$ if and only if the roots of $g(t)$ strictly interlace those of $\prod_{i=1}^s (t-p_i)$.

Proof. Let us prove the necessity of the stated condition for the existence of the matrix A. Observe that the characteristic polynomial of $A(v) = A[T_1] \oplus \cdots \oplus A[T_k]$ is

$$\prod_{i=1}^k g_i(t) = \prod_{i=1}^s (t-p_i)^{m_i}.$$

By hypothesis the eigenvalues of $A[T_i - u_i]$ strictly interlace those of $A[T_i]$; this means that each T_i is a downer branch for every eigenvalue of $A[T_i]$ and so we can apply Lemma 7; by that lemma each root p_i of $p_{A(v)}(t)$ occurs as a root of $p_A(t)$, with multiplicity $m_i - 1$. Since $\sum_{i=1}^s m_i = n-1$, it follows that $\sum_{i=1}^s (m_i - 1) = n - 1 - s$. Thus, $p_A(t)$ must have $s+1$ more distinct roots, the roots of $g(t)$, each one different from each $p_1, \dots, p_s$. By the interlacing inequalities for Hermitian eigenvalues, the roots of $p_A(t)$ must interlace the roots of $p_{A(v)}(t)$. Since $g(t)$ has $s+1$ distinct roots, each of which is distinct from the s roots $p_1, \dots, p_s$, then the roots of $g(t)$ must strictly interlace those of $\prod_{i=1}^s (t-p_i)$.

Next, we prove the sufficiency of the stated conditions.

Because of the strict interlacing between the roots of $g(t)$ and those of $\prod_{i=1}^s (t-p_i)$, due to Remark 5, we conclude the existence of a real number a and positive real numbers $y_1, \dots, y_s$ such that

$$\frac{g(t)}{\prod_{i=1}^s (t-p_i)} = (t-a) - \sum_{i=1}^s \frac{y_i}{t-p_i},$$

i.e.,

$$g(t) = \left[(t-a) - \sum_{i=1}^s \frac{y_i}{t-p_i}\right] \prod_{i=1}^s (t-p_i). \tag{3}$$

We denote by m_{ij} the multiplicity of p_i as a root of $g_j(t)$. Observe that, by hypothesis $g_j(t)$ has distinct real roots, so $m_{ij} \in \{0, 1\}$. Note also that $\sum_{i=1}^s m_{ij} = s_j$ and $\prod_{i=1}^s (t-p_i)^{m_{ij}} = g_j(t)$.

Let $y_{i1}, \dots, y_{ik}$ be positive real numbers such that $m_{i1}y_{i1} + \cdots + m_{ik}y_{ik} = y_i$, $i = 1, \dots, s$. Now, (3) may be rewritten as

$$g(t) = \left[(t-a) - \sum_{i=1}^{s} \frac{m_{i1}y_{i1} + \cdots + m_{ik}y_{ik}}{t-p_i}\right]\prod_{i=1}^{s}(t-p_i)$$
$$= \left[(t-a) - \left(\sum_{i=1}^{s} \frac{m_{i1}y_{i1}}{t-p_i} + \cdots + \sum_{i=1}^{s} \frac{m_{ik}y_{ik}}{t-p_i}\right)\right]\prod_{i=1}^{s}(t-p_i). \quad (4)$$

Recall that $\prod_{i=1}^{s}(t-p_i)^{m_{ij}} = g_j(t)$ and observe that, when $\deg g_j(t) > 1$, $\sum_{i=1}^{s} \frac{m_{ij}y_{ij}}{t-p_i}$ is a pfd of $\frac{h_j(t)}{g_j(t)}$ for some polynomial $h_j(t)$ and, since the coefficients of this pfd are all positive, by Lemma 4, it means that $\deg h_j(t) = \deg g_j(t) - 1$ and $h_j(t)$ has only real roots, which strictly interlace those of $g_j(t)$. If $\deg g_j(t) = 1$, $\sum_{i=1}^{s} \frac{m_{ij}y_{ij}}{t-p_i} = \frac{m_{rj}y_{rj}}{g_j(t)}$, $m_{rj}y_{rj} > 0$, for some $r \in \{1, \ldots, s\}$. In this case, for convenience, we denote $m_{rj}y_{rj}$ by $h_j(t)$. We may rewrite (4) as

$$g(t) = \left[(t-a) - \left(\frac{h_1(t)}{g_1(t)} + \cdots + \frac{h_k(t)}{g_k(t)}\right)\right]\prod_{i=1}^{s}(t-p_i). \quad (5)$$

Observe that the leading coefficient of $h_j(t)$ is the positive real number $\sum_{i=1}^{s} m_{ij}y_{ij}$. Set x_j equal to the leading coefficient of $h_j(t)$ and let $\overline{h}_j(t)$ be the monic polynomial such that $h_j(t) = x_j\overline{h}_j(t)$. With this we obtain from (5)

$$g(t) = \left[(t-a) - \left(x_1\frac{\overline{h}_1(t)}{g_1(t)} + \cdots + x_k\frac{\overline{h}_k(t)}{g_k(t)}\right)\right]\prod_{i=1}^{s}(t-p_i). \quad (6)$$

Let T be a tree and v be a vertex of T of degree k, whose neighbors in T are $u_1, \ldots, u_k$. Let T_i, the branch of T at v containing u_i, be any tree on s_i vertices. By Theorem 3, there exist matrices $A_i \in \mathscr{S}(T_i)$ such that $p_{A_i}(t) = g_i(t)$ and $p_{A_i[T_i-u_i]}(t) = \overline{h}_i(t)$ (recall the convention that $p_{A_i[T_i-u_i]}(t) = 1$ whenever the vertex set of T_i is $\{u_i\}$).

Now define a matrix $A = (a_{ij}) \in \mathscr{S}(T)$, in the following way:

- $a_{vv} = a$;
- $a_{vu_i} = a_{u_iv} = \sqrt{x_i}$, for $i = 1, \ldots, k$;
- $A[T_i] = A_i$, for $i = 1, \ldots, k$;
- the remaining entries of A are 0.

According to (1), the characteristic polynomial of A may be written as

$$(t - a_{vv})p_{A[T-v]}(t) - \sum_{i=1}^{k} |a_{vu_i}|^2 p_{A[T_i-u_i]}(t) \prod_{\substack{j=1\\ j\neq i}}^{k} p_{A[T_j]}(t).$$

Note that $A[T-v] = A[T_1] \oplus \cdots \oplus A[T_k]$ so, $p_{A[T-v]}(t) = \prod_{i=1}^{k} p_{A[T_i]}(t)$. Moreover the characteristic polynomial of $A[T_i]$ is $g_i(t)$ and the characteristic polynomial of $A[T_i - u_i]$ is $\overline{h}_i(t)$ and the roots of these two polynomials strictly interlace.

Taking into account how we have defined the matrix A, it follows that

$$\begin{aligned}
p_A(t) &= (t-a_{vv})\prod_{i=1}^{k} p_{A[T_i]}(t) - \sum_{i=1}^{k} a_{vu_i}^2 p_{A[T_i-u_i]}(t) \prod_{\substack{j=1\\ j\neq i}}^{k} p_{A[T_j]}(t)\\
&= (t-a_{vv})\prod_{i=1}^{k} p_{A[T_i]}(t) - \sum_{i=1}^{k} a_{vu_i}^2 \frac{p_{A[T_i-u_i]}(t)}{p_{A[T_i]}(t)} \prod_{j=1}^{k} p_{A[T_j]}(t)\\
&= \left[(t-a_{vv}) - \sum_{i=1}^{k} a_{vu_i}^2 \frac{p_{A[T_i-u_i]}(t)}{p_{A[T_i]}(t)}\right] \prod_{j=1}^{k} p_{A[T_j]}(t)\\
&= \left[(t-a) - \sum_{i=1}^{k} x_i \frac{\overline{h}_i(t)}{g_i(t)}\right] \prod_{j=1}^{k} g_j(t).
\end{aligned}$$

Since $g_j(t)=\prod_{i=1}^{s}(t-p_i)^{m_{ij}}$ and $m_i=\sum_{j=1}^{k} m_{ij}$, it follows that $\prod_{j=1}^{k} g_j(t) = \prod_{j=1}^{s}(t-p_j)^{m_j}$. So, according to (6), we have

$$\begin{aligned}
p_A(t) &= \left[(t-a) - \sum_{i=1}^{k} x_i \frac{\overline{h}_i(t)}{g_i(t)}\right] \prod_{j=1}^{s}(t-p_j) \prod_{j=1}^{s}(t-p_j)^{m_j-1}\\
&= g(t)\prod_{j=1}^{s}(t-p_j)^{m_j-1},
\end{aligned}$$

i.e., $p_A(t)=f(t)$. □

The condition stated in Theorem 10 is not necessary in general; in fact, $A[T_i]$ may have multiple eigenvalues (and so we cannot apply Theorem 10), and even if the eigenvalues of $A[T_i]$ are simple, $A[T_i]$ and $A[T_i-u_i]$ may have common eigenvalues. Nevertheless, there is a class of trees for which Theorem 10 does give a necessary and sufficient condition for the solvability of the GIEP: the generalized stars. We state this in the next theorem.

Theorem 11. *Let T be a generalized star on n vertices with central vertex v, $T_1,\dots,T_k$ be the branches of T at v, and $l_1,\dots,l_k$ be the number of vertices of $T_1,\dots,T_k$ respectively $\left(n=1+\sum_{i=1}^{k} l_i\right)$.*

Let $g_1(t),\dots,g_k(t)$ be monic polynomials having only real roots and such that $\deg g_i(t)=l_i$, *$p_1,\dots,p_l$ be the distinct roots among polynomials $g_i(t)$ and m_i denote the multiplicity of root p_i in $\prod_{i=1}^{k} g_i(t)$, $(m_i \geqslant 1)$.*

Let $g(t)$ be a monic polynomial with $\deg g(t)=l+1$.

Then there exists a matrix A in $\mathscr{S}(T)$ such that A has characteristic polynomial $g(t)\prod_{i=1}^{l}(t-p_i)^{m_i-1}$, $g_i(t)$ is the characteristic polynomial of summand $A[T_i]$,

$i = 1, \dots, k$, *if and only if each* $g_i(t)$ *has only simple roots and the roots of* $g(t)$ *strictly interlace* $p_1, \dots, p_l$.

Proof. The "if" part is a particular case of Theorem 10. For the "only if" part just note that each T_i is a path and then apply Lemma 9 and the "only if" part of Theorem 10. □

4. Lists of multiplicities for the case of generalized stars

Throughout this section T will be a generalized star on n vertices and v a central vertex of T of degree k. Recall that, each branch, T_i, of T resulting from deletion of a central vertex v from T is a path which we call an *arm* of T. The *length* of an arm T_i of T is simply the number of vertices in the arm and is denoted by l_i. For convenience, we assume that $l_1 \geqslant \cdots \geqslant l_k$.

In [4, Theorem 9] the set of possible (unordered) multiplicities for matrices in $\mathscr{S}(T)$ was characterized; that is, given a generalized star T, a description was given for the set of lists $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k)$, $\tilde{p}_1 \geqslant \tilde{p}_2 \geqslant \cdots \geqslant \tilde{p}_k$, for which there exists a matrix A in $\mathscr{S}(T)$ having k distinct eigenvalues with multiplicities $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k$ respectively.

Here we will give a necessary and sufficient condition for the IEP for $\mathscr{S}(T)$ to have a solution; that is, we will describe the set of eigenvalues, counting multiplicities, that may occur for matrices in $\mathscr{S}(T)$ (Theorem 15 below). We will also consider the question of *ordered multiplicities*: If $\lambda_1 < \cdots < \lambda_r$ are the distinct eigenvalues of a matrix A in $\mathscr{S}(T)$, we associate with A the r-tuple, $q = q(A) = (q_1, \dots, q_r)$, in which $q_i = m_A(\lambda_i)$, $i = 1, \dots, r$. Such an r-tuple is the *list of ordered multiplicities* of A and we denote by $\mathscr{L}(T)$ the collection of lists q that occur, as A runs over $\mathscr{S}(T)$. We will give a complete description of this set.

First we state a lemma that we will use several times.

Lemma 12. *Let T be a generalized star with central vertex v. If A is a matrix in $\mathscr{S}(T)$ and λ is an eigenvalue of $A(v)$ then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.*

Proof. Observe that, if A is a matrix in $\mathscr{S}(T)$ then $A(v) = A[T_1] \oplus \cdots \oplus A[T_k]$, in which each T_i is a path. By Lemma 9, $A[T_i]$ has distinct eigenvalues and the eigenvalues of $A[T_i - u_i]$ strictly interlace those of $A[T_i]$. Thus, if λ is an eigenvalue of $A(v)$, then at least one arm T_i of T is a downer branch for λ and the result follows from Lemma 7. □

The characteristic polynomial of a matrix A in $\mathscr{S}(T)$ was characterized in Theorem 11. Moreover, if we prescribe the eigenvalues of each summand of $A(v)$, such characterization also gives the relative position of the eigenvalues of A, the eigenvalues of $A(v)$, and their multiplicities.

As the following lemma shows, the only constraint for the existence of a matrix in $\mathscr{S}(T-v)$ with a prescribed spectrum is the allocation of distinct eigenvalues to each arm of T (components of $T-v$). The Gale–Ryser Theorem (see, e.g., [11, p. 63]) characterizing the existence of a (0, 1)-matrix with given row-sums and column-sums is relevant to this allocation. Let $q_1 \geqslant \cdots \geqslant q_r$ be the multiplicities of the distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix B in $\mathscr{S}(T-v)$. Since each $B[T_j]$ has distinct eigenvalues, denoting by q_{ij} the multiplicity of the eigenvalue λ_i as an eigenvalue of $B[T_j]$, it follows that $q_{ij} \in \{0, 1\}$, $\sum_{j=1}^{k} q_{ij} = q_i$ and $\sum_{i=1}^{r} q_{ij} = l_j$. So, there must exist an r-by-k (0, 1)-matrix $Q = (q_{ij})$ with row-sum vector $q = (q_1, \ldots, q_r)$ and column-sum vector $l = (l_1, \ldots, l_k)$, each one being partitions of $n-1$. We denote by l^* the conjugate partition of l, where l_i^* is the number of j's such that $l_j \geqslant i$ so, $l^* = (l_1^*, \ldots, l_{l_1}^*)$ with $l_1^* \geqslant \cdots \geqslant l_{l_1}^* \geqslant 1$.

Let $u = (u_1, \ldots, u_b)$, $u_1 \geqslant \cdots \geqslant u_b$, and $v = (v_1, \ldots, v_c)$, $v_1 \geqslant \cdots \geqslant v_c$, be two partitions of integers M and N respectively, $M \leqslant N$, such that $u_1 + \cdots + u_s \leqslant v_1 + \cdots + v_s$ for all s, interpreting u_s or v_s as 0 when s exceeds b or c, respectively. If $M = N$, we say that v majorizes u and write $u \preceq v$. If $M < N$ we denote by u_e the partition of N obtained from u adding 1's to the partition u. It is easy to see that $u_e \preceq v$. Note that if $M = N$ then $u_e = u$.

By the Gale–Ryser Theorem, the matrix $Q = (q_{ij})$ mentioned above exists if and only if $q \preceq l^*$.

Lemma 13. *Let T be a generalized star on n vertices whose central vertex v has degree k and whose arm lengths are $l_1 \geqslant \cdots \geqslant l_k$. Then there is a matrix A in $\mathscr{S}(T-v)$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ such that $q_1 = m_A(\lambda_1) \geqslant \cdots \geqslant m_A(\lambda_r) = q_r$ if and only if $(q_1, \ldots, q_r) \preceq (l_1, \ldots, l_k)^*$.*

Proof. The above discussion justifies the necessity of the stated condition.

If $(q_1, \ldots, q_r) \preceq (l_1, \ldots, l_k)^*$, then there exists an r-by-k (0, 1)-matrix $Q = (q_{ij})$ with row-sum vector $(q_1, \ldots, q_r)$ and column-sum vector $(l_1, \ldots, l_k)$; i.e., it is possible to prescribe $\lambda_1, \ldots, \lambda_r$ as eigenvalues of A, counting multiplicities, in such way that each of the direct summands of A, $A[T_i]$, must have l_i distinct eigenvalues. The existence of such matrices is guaranteed by Theorem 3. □

The next step is to verify when a given sequence of real numbers can be the spectrum of a matrix in $\mathscr{S}(T)$. As we shall see, the only constraint to construct a matrix in $\mathscr{S}(T)$ with prescribed spectrum is the existence of a corresponding list of ordered multiplicities. We start by giving necessary conditions for the possible lists of ordered multiplicities that can occur for the distinct eigenvalues of A, as A runs over $\mathscr{S}(T)$, for a given generalized star T.

Note that conditions (a) and (b) of next theorem are essentially the same conditions as (a) and (d) of [4, Theorem 9] and, in fact, they follow from the necessity part of that theorem; for completeness we include a proof here.

Theorem 14. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_k$ $\left(n = 1 + \sum_{i=1}^{k} l_i\right)$. If $(q_1, q_2, \ldots, q_r) \in \mathscr{L}(T)$ then*:

(a) $\sum_{i=1}^{r} q_i = n$;
(b) *if* $q_i > 1$ *then* $1 < i < r$ *and* $q_{i-1} = 1 = q_{i+1}$;
(c) $(q_{i_1} + 1, q_{i_2} + 1, \ldots, q_{i_h} + 1)_e \preceq (l_1, l_2, \ldots, l_k)^*$, *in which* $q_{i_1} \geqslant q_{i_2} \geqslant \cdots \geqslant q_{i_h}$ *are the entries of the r-tuple* $(q_1, q_2, \ldots, q_r)$ *greater than* 1.

Proof. Suppose that A is a matrix in $\mathscr{S}(T)$ with distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ whose list of ordered multiplicities is $(q_1, q_2, \ldots, q_r)$. (a) Says that, since A is an n-by-n matrix, the number of eigenvalues, counting multiplicities, must be n. If $q_i > 1$ then λ_i is an eigenvalue of $A(v)$. By Theorem 11, there are two eigenvalues in A, $\lambda_{i-1} < \lambda_{i+1}$ but not in $A(v)$, strictly interlacing λ_i. Therefore, $1 < i < n$ and $q_{i-1} = 1 = q_{i+1}$, which proves (b). To prove (c) we must note that if $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_h}$ are eigenvalues of A with multiplicities $q_{i_1} \geqslant q_{i_2} \geqslant \cdots \geqslant q_{i_h} \geqslant 2$, by Lemma 12, such eigenvalues occur as eigenvalues of $A(v)$ with multiplicities $q_{i_1} + 1, q_{i_2} + 1, \ldots, q_{i_h} + 1$. By Lemma 13, if there is a matrix in $\mathscr{S}(T - v)$ with such multiple eigenvalues then $(q_{i_1} + 1, q_{i_2} + 1, \ldots, q_{i_h} + 1)_e \preceq (l_1, l_2, \ldots, l_k)^*$. □

The next theorem shows that the above necessary conditions of Theorem 14 for $(q_1, \ldots, q_r) \in \mathscr{L}(T)$ are also sufficient. For this purpose, given $q = (q_1, \ldots, q_r)$ satisfying the conditions (a)–(c) of Theorem 14, we need to construct a matrix in $\mathscr{S}(T)$ whose list of ordered multiplicities is q. Now Theorem 11 gives us a way to construct, in particular, a matrix A in $\mathscr{S}(T)$ with prescribed distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$, as soon as the corresponding list of ordered multiplicities satisfies conditions (a)–(c) in Theorem 14. So we may prove the sufficiency of the stated conditions (a)–(c) of Theorem 14.

Theorem 15. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_k$ $\left(n = 1 + \sum_{i=1}^{k} l_i\right)$. Let $\lambda_1 < \cdots < \lambda_r$ be any sequence of real numbers.*

Then there exists a matrix A in $\mathscr{S}(T)$ with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ and $q(A) = (q_1, \ldots, q_r)$ if and only if $(q_1, \ldots, q_r)$ satisfies conditions (a)–(c) *in Theorem* 14.

Proof. Since q satisfies condition (a) in Theorem 14, it means that the matrix A must have n eigenvalues, counting multiplicities. Let h, $h \geqslant 0$, be the number of q_i's greater than 1 in q.

If $h = 0$ then we have $q_1 = \cdots = q_r = 1$ and $r = n$. Then, by Theorem 3, considering any sequence of real numbers $\{\mu_i\}_{i=1}^{n-1}$, such that $\lambda_i < \mu_i < \lambda_{i+1}$, $i = 1, \ldots,$

$n-1$, there exists a matrix A in $\mathscr{S}(T)$ such that A and $A(v)$ have spectrum $\{\lambda_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n-1}$, respectively.

Suppose now that $h \geqslant 1$, and let $q_{j_1} \geqslant \cdots \geqslant q_{j_h}$ be the entries of q that are greater than 1. Since q satisfies condition (c) in Theorem 14, we have $(q_{j_1}+1, \ldots, q_{j_h}+1)_e \preceq (l_1, \ldots, l_k)^*$. It means that it is possible to construct matrices $A_i \in \mathscr{S}(T_i)$ such that $\lambda_{j_1}, \ldots, \lambda_{j_h}$ occur as eigenvalues of $A_1 \oplus \cdots \oplus A_k$ with total multiplicities, respectively $q_{j_1}+1, \ldots, q_{j_h}+1$ (each of these real numbers occurs as an eigenvalue of at most one of the A_i's). So, $(q_{j_1}+1)+\cdots+(q_{j_h}+1) \leqslant \sum_{i=1}^{k} l_i = n-1$. Let $z = n-1-[(q_{j_1}+1)+\cdots+(q_{j_h}+1)](\geqslant 0)$ be the number of remaining eigenvalues to prescribe for the construction of matrices A_i, $i = 1, \ldots, k$. Note that, since q satisfies condition (b) in Theorem 14, if $q_i > 1$ then $1 < i < r$ and $q_{i-1} = 1 = q_{i+1}$. So, there are, $h+1$ λ_i's strictly interlacing the real numbers $\lambda_{j_1}, \ldots, \lambda_{j_h}$.

Observe that $n = z+(h+1)+q_{j_1}+\cdots+q_{j_h}$ so, there are $z+h+1$ distinct λ_i's that must be (simple) eigenvalues of A but do not occur as eigenvalues of $A(v)$. If $z > 0$, choose the remaining z eigenvalues to prescribe for the construction of matrices A_i, all distinct and such that the $z+h$ distinct prescribed eigenvalues for $A_1 \oplus \cdots \oplus A_k$ strictly interlace the $z+h+1$ simple λ_i's (if $z = 0$, the $h+1$ simple prescribed eigenvalues for A strictly interlace the h real numbers $\lambda_{j_1}, \ldots, \lambda_{j_h}$).

From Theorem 11, there exists a real symmetric matrix A in $\mathscr{S}(T)$ with characteristic polynomial $g(t)\prod_{i=1}^{h}(t-\lambda_{j_i})^{q_{j_i}}$ in which $g(t)$ is a monic polynomial of degree $z+h+1$ whose roots are the λ_i's such that $q_i = 1$ and, $\prod_{i=1}^{h}(t-\lambda_{j_i})^{q_{j_i}}$ is a monic polynomial of degree $q_{j_1}+\cdots+q_{j_h} = n-(z+h+1)$. □

In the construction of a matrix A in $\mathscr{S}(T)$ with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ whose list of ordered multiplicities, $(q_1, \ldots, q_r)$, satisfies conditions (a)–(c) in Theorem 14, the simple eigenvalues (of multiplicity 1) of A do not occur as eigenvalues of $A(v)$. (Recall that, by Lemma 12, if λ is an eigenvalue of A and $A(v)$ then $m_{A(v)}(\lambda) = m_A(\lambda)+1$.) But under some constraints, a matrix A in $\mathscr{S}(T)$ can be constructed with a simple eigenvalue (or more than one) occurring as an eigenvalue of $A(v)$. For this purpose, if A is a matrix in $\mathscr{S}(T)$, we call an eigenvalue λ of A, satisfying $m_{A(v)}(\lambda) = m_A(\lambda)+1$, an *upward eigenvalue* of A at v. We call the multiplicity of λ in A, an *upward multiplicity* of A at v. If $q = q(A) = (q_1, \ldots, q_r)$, we define the *list of upward multiplicities* of A at v, which we denote by $\hat{q}$, the list with the same entries as q but in which any upward multiplicity of A at v, q_i, is marked as $\hat{q}_i$ in $\hat{q}$. Of course, when T is a generalized star and v is a central vertex of T, all the q_i's greater than 1 are marked in $\hat{q}$, and, if q_i is marked in $\hat{q}$, then $1 < i < r$, and neither q_{i-1} nor q_{i+1} can be marked in $\hat{q}$ (which implies that $q_{i-1} = 1 = q_{i+1}$). For a given vertex v of T, we denote by $\hat{\mathscr{L}}_v(T)$ the collection of lists of upward multiplicities at v that occur among matrices in $\mathscr{S}(T)$.

Now we can state the following theorem, whose proof is analogous to the proof of Theorem 15, so that we omit its proof.

Theorem 16. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_k$ $\left(n = 1 + \sum_{i=1}^{k} l_i\right)$. Let $\lambda_1 < \cdots < \lambda_r$ be any sequence of real numbers.*

Then there exists a matrix A in $\mathscr{S}(T)$ with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ and list of upward multiplicities $\hat{q} = (q_1, \widehat{\ldots}, q_r)$ if and only if $\hat{q}$ satisfies the following conditions:

(a) $\sum_{i=1}^{r} q_i = n$;
(b) *if q_i is an upward multiplicity in $\hat{q}$ then $1 < i < r$ and neither q_{i-1} nor q_{i-1} is an upward multiplicity in $\hat{q}$*;
(c) $(q_{i_1} + 1, q_{i_2} + 1, \ldots, q_{i_h} + 1)_e \preceq (l_1, l_2, \ldots, l_k)^*$, *where $q_{i_1} \geqslant q_{i_2} \geqslant \cdots \geqslant q_{i_h}$ are the upward multiplicities of $\hat{q}$.*

We have seen (Lemma 12) that, when T is a generalized star, there is a vertex v of T, a central vertex, such that, for any matrix A in $\mathscr{S}(T)$ and any eigenvalue λ of $A(v)$, we have $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. We close this section showing that the generalized stars are the only trees for which such a vertex v exists.

Theorem 17. *Let T be a tree and v be a vertex of T such that, for any matrix A in $\mathscr{S}(T)$ and any eigenvalue λ of $A(v)$, $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. Then T is a generalized star and v is a central vertex of T.*

Proof. Suppose that T is a tree but not a generalized star. Then T has at least two vertices of degree greater than 2. Let v be any vertex of T and choose a vertex u of degree $k \geqslant 3$ of T, $u \neq v$. We show that there is a matrix A in $\mathscr{S}(T)$ such that λ is an eigenvalue of $A(v)$ satisfying $m_{A(v)}(\lambda) = m_A(\lambda) - 1$. In order to construct A, consider the vertex u, whose removal leaves k components $T_1, \ldots, T_k$. For each of these components, construct a matrix A_i in $\mathscr{S}(T_i)$ whose smallest eigenvalue is λ. Let A be any matrix in $\mathscr{S}(T)$ with the submatrices A_i in appropriate positions. Recall that, by Lemma 8, the smallest eigenvalue of a matrix whose graph is a tree does not occur as an eigenvalue of any principal submatrix of size one smaller. It means that any T_i is a downer branch at u for λ. Thus $m_{A(u)}(\lambda) = k$ and, by Lemma 7, it follows that $m_A(\lambda) = k - 1$.

Let us see that $m_{A(v)}(\lambda) = m_A(\lambda) - 1$. Observe that λ occurs as an eigenvalue of only one of the direct summands of $A(v)$, corresponding to the component T' of $T - v$ containing the vertex u. Since now λ is an eigenvalue of $k - 1$ components of $A[T' - u]$ (in each one with multiplicity 1), again, by Lemma 7, it follows $m_{A[T'-u]}(\lambda) = m_{A[T']}(\lambda) + 1$ i.e., $m_{A[T']}(\lambda) = k - 2$. Since $m_{A(v)}(\lambda) = m_{A[T']}(\lambda)$, we have $m_{A(v)}(\lambda) = m_A(\lambda) - 1$.

If we assume that T is a generalized star and v is not a central vertex, the same argument holds to prove the claimed result. $\square$

5. Double generalized stars

Here, we give a characterization of the lists of ordered multiplicities among matrices whose graph is a double generalized star. As we shall see, any list of ordered multiplicities of a double generalized star $D(T_1, T_2)$, with prescribed central vertices for T_1 and T_2, may be obtained from the lists of upward multiplicities of T_1 and T_2.

Throughout this section, G will be a double generalized star $D(T_1, T_2)$. For convenience, we denote by v_i, $i = 1, 2$, the central vertex of T_i, in T_i and in G.

It is easy to see that, if A is a matrix in $\mathscr{S}(G)$, by permutation similarity, A is similar to a matrix

$$\begin{bmatrix} A_1 & e \\ \bar{e} & A_2 \end{bmatrix}, \tag{7}$$

in which A_i is a matrix in $\mathscr{S}(T_i)$, $i = 1, 2$, and e is the entry of A corresponding to the edge $\{v_1, v_2\}$ of G. Here, e and $\bar{e}$ lie in a particular entry of A, outside A_1 and A_2, depending upon the positions of v_i in A_i, $i = 1, 2$. All other entries outside A_1 and A_2 are 0. For convenience, if A is a matrix in $\mathscr{S}(G)$ we assume that it is written as in (7).

The lists of upward multiplicities of T_1 (T_2) at v_1 (v_2) play an important role in our results. Throughout, when we consider an upward eigenvalue (multiplicity) in A_1 or A_2 or A, the related vertices are v_1 or v_2. If λ is an upward eigenvalue of A and A_i, $i = 1$ or $i = 2$, we call such an eigenvalue of A a *doubly upward* eigenvalue.

Theorem 18. *Let A be a matrix in $\mathscr{S}(G)$ and λ be an eigenvalue of A_1 or A_2. Then λ is an eigenvalue of A if and only if λ is an eigenvalue of $A_1(v_1)$ or $A_2(v_2)$. In this event, we have $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$.*

Proof. To prove the necessity of the claimed result, we assume without loss of generality that λ is an eigenvalue of A and A_1. We start showing that λ must occur as an eigenvalue of $A_1(v_1)$ or $A_2(v_2)$. In order to get a contradiction, we suppose that λ does not occur as an eigenvalue of $A_1(v_1)$ and $A_2(v_2)$. Since $A(v_2) = A_1 \oplus A_2(v_2)$ and λ is an eigenvalue of A_1 ($A[T_1]$) but does not occur as an eigenvalue of $A_1(v_1)$ ($A[T_1 - v_1]$), this means that T_1 is a downer branch for λ at v_2, so, by Lemma 7, we have $m_{A(v_2)}(\lambda) = m_A(\lambda) + 1$. Thus, $m_A(\lambda) = m_{A_1}(\lambda) - 1$. Since λ is an eigenvalue of A_1 but does not occur as an eigenvalue of $A_1(v_1)$, by the interlacing inequalities for Hermitian eigenvalues, it follows that $m_{A_1}(\lambda) = 1$. But, then, $m_A(\lambda) = 0$, which gives a contradiction. Therefore, λ is an eigenvalue of $A_1(v_1)$ or $A_2(v_2)$.

It remains to prove that, when λ is an eigenvalue of $A_1(v_1)$ or $A_2(v_2)$, $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$. Suppose without loss of generality that λ is an eigenvalue of $A_2(v_2)$. Since T_2 is a generalized star, there is in T_2 a downer branch for λ at v_2. Such a downer branch of T_2 for λ is also a downer branch of G for λ at v_2 so, we have $m_{A(v_2)}(\lambda) = m_A(\lambda) + 1$ and $m_{A_2(v_2)}(\lambda) = m_{A_2}(\lambda) + 1$. Since $A(v_2) = A_1 \oplus A_2(v_2)$ it follows that $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$.

To prove the sufficiency it suffices to observe that if λ is an eigenvalue of $A_1(v_1)$ or $A_2(v_2)$ then $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$. $\square$

Corollary 19. *Let A be a matrix in $\mathscr{S}(G)$. If λ is an upward eigenvalue of A_1 or A_2 then λ is an upward eigenvalue of A and $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$.*

Corollary 20. *Let A be a matrix in $\mathscr{S}(G)$ and λ be an eigenvalue of A_1 or A_2. Then λ is a multiple eigenvalue of A if and only if λ is an upward eigenvalue of A_1 or A_2 and $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda) \geqslant 2$.*

Corollary 21. *Let A be a matrix in $\mathscr{S}(G)$ and λ be an eigenvalue of A_1 and A_2. Then λ is an eigenvalue of A if and only if λ is an upward eigenvalue of A_1 or A_2. In such a case, $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda) \geqslant 2$.*

Consider two lists of upward multiplicities for A_1 and A_2, respectively $(\widehat{b_1, \ldots, b_{s_1}})$ and $(\widehat{c_1, \ldots, c_{s_2}})$. If λ is an upward eigenvalue of A_1 with upward multiplicity b_i then, by Corollary 19, λ is an upward eigenvalue of A. If λ is also an eigenvalue of A_2 with multiplicity c_j, then $m_A(\lambda) = b_i + c_j$. If λ is not an eigenvalue of A_2 then $m_A(\lambda) = b_i$. In either case, λ is a doubly upward eigenvalue of A. (Observe that any multiple eigenvalue of A is a doubly upward eigenvalue of A.) It remains to consider what are the possible relative positions of λ (of $m_A(\lambda)$) in the ordered spectrum of A (in the list of ordered multiplicities of A). For this purpose, given a symmetric matrix B and a real number λ, we denote by $l_B(\lambda)$ ($r_B(\lambda)$) the number of eigenvalues (counting multiplicities) of B less (greater) than λ. Given two real numbers $\lambda < \lambda'$ we denote by $b_B(\lambda, \lambda')$ the number of eigenvalues of B strictly between λ and λ'.

Lemma 22. *Let A be a matrix in $\mathscr{S}(G)$ and λ be a doubly upward eigenvalue of A. Then $l_A(\lambda) = l_{A_1}(\lambda) + l_{A_2}(\lambda)$.*

Proof. Since λ is a doubly upward eigenvalue of A, we have $m_{A(v_i)}(\lambda) = m_A(\lambda) + 1 \geqslant 2$ and $m_{A_i(v_i)}(\lambda) = m_{A_i}(\lambda) + 1 \geqslant 2$, for $i = 1$ or $i = 2$. Suppose without loss of generality that $i = 1$. By the interlacing inequalities for Hermitian eigenvalues, $l_{A(v_1)}(\lambda) = l_A(\lambda) - 1$ and $l_{A_1(v_1)}(\lambda) = l_{A_1}(\lambda) - 1$. Since $A(v_1) = A_1(v_1) \oplus A_2$ it follows that $l_{A(v_1)}(\lambda) = l_{A_1(v_1)}(\lambda) + l_{A_2}(\lambda)$. Therefore, $l_A(\lambda) = l_{A_1}(\lambda) + l_{A_2}(\lambda)$. $\square$

In the same way, we may show that $r_A(\lambda) = r_{A_1}(\lambda) + r_{A_2}(\lambda)$. If $\lambda_{h_1} < \lambda_{h_2}$ are two doubly upward eigenvalues of A then, by Lemma 22, $b_A(\lambda_{h_1}, \lambda_{h_2}) = b_{A_1}(\lambda_{h_1}, \lambda_{h_2}) + b_{A_2}(\lambda_{h_1}, \lambda_{h_2})$.

Lemma 23. *Let A be a matrix in $\mathscr{S}(G)$ such that every common eigenvalue of A_1 and A_2 is an upward eigenvalue of one or the other. Then $q(A_1 \oplus A_2) = q(A)$; i.e., the ordered multiplicities of $A_1 \oplus A_2$ and A are the same.*

Proof. By hypothesis, if A_1 and A_2 have a common eigenvalue then it must be an upward eigenvalue of A_1 or A_2. This rules out the possibility that they have a common largest or smallest eigenvalue, because by Theorem 16 or Lemma 8, the smallest and largest eigenvalues of A_1 and A_2 cannot be upward. Thus, it follows that the smallest and largest eigenvalues of $A_1 \oplus A_2$ have multiplicity 1.

If there are no multiple eigenvalues of $A_1 \oplus A_2$, from Corollary 20, there are no multiple eigenvalues of A and, therefore, $q(A_1 \oplus A_2) = q(A)$. Suppose now that there is a multiple eigenvalue λ of $A_1 \oplus A_2$. Of course, $m_{A_1 \oplus A_2}(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$. If λ is an eigenvalue of multiplicity 1 of both A_1 and A_2, then, by hypothesis, λ is an upward eigenvalue of A_1 or A_2. If λ is multiple in either A_1 or A_2, then λ is upward in that one. By Corollary 20, there is a multiple eigenvalue λ of A if and only if λ is an upward eigenvalue of A_1 or A_2 and λ is a multiple eigenvalue of $A_1 \oplus A_2$ with the same multiplicity as in A. Thus, $A_1 \oplus A_2$ and A have the same multiple eigenvalues with the same multiplicities. Since $A_1 \oplus A_2$ and A have the same size, to complete the proof that $q(A_1 \oplus A_2) = q(A)$, it suffices to observe that, given any multiple eigenvalue λ of $A_1 \oplus A_2$ (of A), from Lemma 22, we have $l_{A_1 \oplus A_2}(\lambda) = l_A(\lambda)$. $\square$

The lists of ordered multiplicities for matrices A in $\mathscr{S}(G)$ whose A_1 and A_2 satisfy the assumption in Lemma 23 are easily determined. By Theorem 16, given any list of upward multiplicities $\hat{b} = (b_1, \widehat{\ldots}, b_{s_1})$ of T_1 and any list of upward multiplicities $\hat{c} = (c_1, \widehat{\ldots}, c_{s_2})$ of T_2, it is always possible to construct matrices A_1 in $\mathscr{S}(T_1)$ and A_2 in $\mathscr{S}(T_2)$ with prescribed spectrum, having such lists of upward multiplicities and, such that, λ occurs as an eigenvalue of A_1 and A_2 only when the multiplicity of λ is an upward multiplicity of $\hat{b}$ or $\hat{c}$. In this event, if

$$A = \begin{bmatrix} A_1 & e \\ \bar{e} & A_2 \end{bmatrix}$$

is a matrix in $\mathscr{S}(G)$ and λ is an eigenvalue of A_1 and A_2 then, by Corollary 21, it follows that $m_A(\lambda) = m_{A_1}(\lambda) + m_{A_2}(\lambda)$.

The following theorem, which we call the *Superposition Principle*, gives a way to generate all possible lists of ordered multiplicities for matrices A in $\mathscr{S}(G)$ whose A_1 and A_2 satisfy the assumption in Lemma 23.

Theorem 24 (Superposition Principle). *Let G be a double generalized star $D(T_1, T_2)$. Given two lists of upward multiplicities of T_1 and T_2, respectively $\hat{b} = (b_1, \widehat{\ldots}, b_{s_1})$ and $\hat{c} = (c_1, \widehat{\ldots}, c_{s_2})$, construct any $b^+ = (b_1^+, \ldots, b_{s_1+t_1}^+)$ and $c^+ = (c_1^+, \ldots, c_{s_2+t_2}^+)$ with $s_1 + t_1 = s_2 + t_2$, subject to the following conditions*:

1. *b^+ (c^+) is obtained from $\hat{b}$ ($\hat{c}$) by inserting t_1 (t_2) 0's, $t_1, t_2 \geqslant 0$;*
2. *b_i^+ and c_i^+ cannot both be* 0; *and*

3. *if* $b_i^+ > 0$ *and* $c_i^+ > 0$, *at least one of* b_i^+ *or* c_i^+ *must be an upward multiplicity of* $\hat{b}$ *or* $\hat{c}$.

Then the list $b^+ + c^+ = \left(b_1^+ + c_1^+, \ldots, b_{s_1+t_1}^+ + c_{s_2+t_2}^+\right)$ *is a list of ordered multiplicities of* G.

Proof. Let $b^+ + c^+ = (b_1^+ + c_1^+, \ldots, b_s^+ + c_s^+)$, $s = s_1 + t_1$, be any list obtained from $\hat{b}$ and $\hat{c}$ by the Superposition Principle. Choosing any s distinct real numbers $\lambda_1 < \cdots < \lambda_s$, by Theorem 16, there is a matrix A_1 in $\mathscr{S}(T_1)$ with list of upward multiplicities $\hat{b}$ such that $m_{A_1}(\lambda_i) = b_i^+$ and, there is a matrix A_2 in $\mathscr{S}(T_2)$ with list of upward multiplicities $\hat{c}$ such that $m_{A_2}(\lambda_i) = c_i^+$. Of course, $m_{A_1 \oplus A_2}(\lambda_i) = b_i^+ + c_i^+$ and, by construction of $b^+ + c^+$, the matrices A_1 and A_2 have a common eigenvalue λ only when λ is an upward eigenvalue of A_1 or A_2. Since $q(A_1 \oplus A_2) = b^+ + c^+$, by Lemma 23, it follows that $b^+ + c^+$ is a list of ordered multiplicities of G. □

Under the conditions and in the notation of Theorem 24, we say that the pair b^+ and c^+, obtained from the lists of upward multiplicities $\hat{b}$ and $\hat{c}$, is a valid pair. The Superposition Principle then says that the addition of any valid pair for T_1 and T_2 gives a possible list of ordered multiplicities for $D(T_1, T_2)$.

As we shall see, any list of ordered multiplicities for a double generalized star $D(T_1, T_2)$ may be obtained, by the Superposition Principle, from lists of upward multiplicities for T_1 and T_2.

Lemma 25. *Let*

$$A = \begin{bmatrix} A_1 & e \\ \bar{e} & A_2 \end{bmatrix}$$

be a matrix in $\mathscr{S}(G)$. *Then there is a matrix*

$$B = \begin{bmatrix} A_1' & e' \\ \bar{e}' & A_2 \end{bmatrix}$$

in $\mathscr{S}(G)$ *such that* $q(B) = q(A)$, $q(A_1') = q(A_1)$, *and*, A_1' *and* A_2 *have a common eigenvalue only when it is an upward eigenvalue of* A_1' *or* A_2. *Moreover*, $q(B) = q(A)$ *for any* $e' \in \mathbb{C}$.

Proof. Let $\lambda_1 < \cdots < \lambda_s$ be the distinct eigenvalues of $A_1 \oplus A_2$ and $\lambda_{i_1} < \cdots < \lambda_{i_{s_1}}$ be the distinct eigenvalues of A_1 with list of upward multiplicities $\hat{b}$. Let $\alpha_{i_1} < \cdots < \alpha_{i_{s_1}}$ be the distinct eigenvalues of a matrix A_1' in $\mathscr{S}(T_1)$ with list of upward multiplicities $\hat{b}$ and such that, for $i = i_1, \ldots, i_{s_1}$ we choose:

- $\alpha_i = r_i$, with $\begin{cases} r_i < \lambda_i & \text{for } i = 1 \\ \lambda_{i-1} < r_i < \lambda_i & \text{for } i > 1, \end{cases}$

if λ_i is an eigenvalue of multiplicity 1 of both A_1 and A_2 but is not an eigenvalue of A;

- $\alpha_i = \lambda_i$, otherwise.

Note that the existence of such a matrix A_1' is guaranteed by Theorem 16 because $\hat{b} \in \hat{\mathscr{L}}_{v_1}(T_1)$. By construction, $A_1' \oplus A_2$ and A have the same multiple eigenvalues with the same multiplicities and, for any multiple eigenvalue λ of $A_1' \oplus A_2$ (of A), $l_{A_1' \oplus A_2}(\lambda) = l_{A_1 \oplus A_2}(\lambda)$. Therefore, $q(A_1' \oplus A_2) = q(A)$. Again by construction, if A_1' and A_2 have a common eigenvalue it must be an upward eigenvalue of A_1' or A_2. By Lemma 23, if

$$B = \begin{bmatrix} A_1' & e' \\ \bar{e}' & A_2 \end{bmatrix}$$

then $q(B) = q(A)$, $e' \in \mathbb{C}$. □

Finally, we complete the characterization of the lists of ordered multiplicities for double generalized stars by showing that any list for $D(T_1, T_2)$ arises from a valid pair for T_1 and T_2.

Theorem 26. *Let G be a double generalized star $D(T_1, T_2)$. Then $a \in \mathscr{L}(G)$ if and only if there is a $\hat{b} \in \hat{\mathscr{L}}_{v_1}(T_1)$ and a $\hat{c} \in \hat{\mathscr{L}}_{v_2}(T_2)$ such that $a = b^+ + c^+$.*

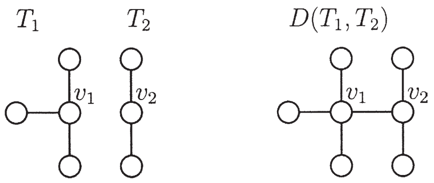
Proof. Since the sufficiency is a direct consequence of the Superposition Principle (Theorem 24), let us prove the necessity of the claimed result.

If $a \in \mathscr{L}(G)$ then, by Lemma 25, there is a matrix

$$A = \begin{bmatrix} A_1 & e \\ \bar{e} & A_2 \end{bmatrix}$$

in $\mathscr{S}(T)$ with $q(A) = a$, and such that A_1 and A_2 have a common eigenvalue only when it is an upward eigenvalue of A_1 or A_2. In this situation, by Lemma 23, we have $q(A_1 \oplus A_2) = a$. Let $\hat{b} = (b_1, \widehat{\ldots}, b_{s_1})$ and $\hat{c} = (c_1, \widehat{\ldots}, c_{s_2})$ be the lists of upward multiplicities of A_1 and A_2, respectively. Let us show that there is a valid pair b^+ and c^+, obtained from $\hat{b}$ and $\hat{c}$, such that $a = b^+ + c^+$. Let $\lambda_1 < \cdots < \lambda_s$ be the distinct eigenvalues of $A_1 \oplus A_2$ whose list of ordered multiplicities is $a = (a_1, \ldots, a_s)$. Observe that, for any eigenvalue λ_i of $A_1 \oplus A_2$, we have $m_{A_1 \oplus A_2}(\lambda_i) = m_{A_1}(\lambda_i) + m_{A_2}(\lambda_i)$. It allows us to construct $b^+ = (b_1^+, \ldots, b_s^+)$ and $c^+ = (c_1^+, \ldots, c_s^+)$ in which, $b_i^+ = m_{A_1}(\lambda_i)$ and $c_i^+ = m_{A_2}(\lambda_i)$. Observe that, if $b_i^+ > 0$ and $c_i^+ > 0$, this means that λ_i is an upward eigenvalue of A_1 or A_2. Thus, the pair b^+ and c^+ is validly obtained from $\hat{b}$ and $\hat{c}$ and the Superposition Principle verifies that $a = b^+ + c^+$. □

Example 27. Let T_1 and T_2 be the following stars with central vertices v_1 and v_2, respectively, and G be the double star $D(T_1, T_2)$.



By Theorem 16, we have that

$$\hat{\mathscr{L}}_{v_1}(T_1) = \{(1, \hat{2}, 1), (1, \hat{1}, 1, 1), (1, 1, \hat{1}, 1)\}$$

and

$$\hat{\mathscr{L}}_{v_2}(T_2) = \{(1, \hat{1}, 1)\}.$$

Applying the Superposition Principle to the lists of upward multiplicities of T_1 and T_2, it follows that

$$\begin{aligned}\mathscr{L}(G) = \{&(1, 3, 2, 1), (1, 2, 3, 1), (1, 3, 1, 1, 1), (1, 1, 3, 1, 1), (1, 1, 1, 3, 1),\\ &(1, 2, 2, 1, 1), (1, 2, 1, 2, 1), (1, 1, 2, 2, 1), (1, 2, 1, 1, 1, 1),\\ &(1, 1, 2, 1, 1, 1), (1, 1, 1, 2, 1, 1), (1, 1, 1, 1, 2, 1), (1, 1, 1, 1, 1, 1, 1)\}.\end{aligned}$$

References

[1] D. Boley, G.H. Golub, A survey of inverse eigenvalue problems, Inverse Probl. 3 (1987) 595–622.
[2] M.T. Chu, Inverse eigenvalue problems, SIAM Rev. 40 (1998) 1–39.
[3] C.R. Johnson, A. Leal-Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, Linear Multilinear Algebra 46 (1999) 139–144.
[4] C.R. Johnson, A. Leal-Duarte, On the possible multiplicities of the eigenvalues of an Hermitian matrix whose graph is a given tree, Linear Algebra Appl. 348 (2002) 7–21.
[5] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, B.D. Sutton, A.J. Witt, On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph, Linear Algebra Appl. 363 (2003) 147–159.
[6] C.R. Johnson, C.M. Saiago, Estimation of the maximum multiplicity of an eigenvalue in terms of the vertex degrees of the graph of a matrix, Electron. J. Linear Algebra 9 (2002) 27–31.
[7] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, The Parter–Wiener theorem: refinement and generalization, SIAM J. Matrix Anal. Appl., in press.
[8] A. Leal-Duarte, Construction of acyclic matrices from spectral data, Linear Algebra Appl. 113 (1989) 173–182.
[9] A. Leal-Duarte, C.R. Johnson, On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree, Math. Inequal. Appl. 5 (2) (2002) 175–180.
[10] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[11] H.J. Ryser, Combinatorial Mathematics, The Carus Mathematical Monographs, vol. 14, MAA, 1963.