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Banach algebras of Fourier multipliers equivalent at infinity to nice Fourier multipliers

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Abstract Let $\mathcal{M}_{X(\mathbb{R})}$ be the Banach algebra of all Fourier multipliers on a Banach function space $X(\mathbb{R})$ such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. For two sets $\Psi, \Omega \subset \mathcal{M}_{X(\mathbb{R})}$, let Ψ_Ω be the set of those $c \in \Psi$ for which there exist $d \in \Omega$ such that the multiplier norm of $\chi_{\mathbb{R}\setminus[-N,N]}(c-d)$ tends to zero as $N \to \infty$. In this case we say that the Fourier multiplier c is equivalent at infinity to the Fourier multiplier d. We show that if Ω is a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ consisting of nice Fourier multipliers (for instance, continuous or slowly oscillating in certain sense) and Ψ is an arbitrary unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$, then Ψ_Ω is a also a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$.

Keywords Fourier convolution operator · Fourier multiplier · slowly oscillating function · equivalence at infinity · Banach algebra · C^*-algebra.

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1 Introduction

We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of all infinitely differentiable and rapidly decaying functions (see, e.g., [13, Section 2.2.1]). Let $\mathcal{F}$ denote the Fourier transform, defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}f)(x) := \widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{itx}\,dt, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1}$ be the inverse of $\mathcal{F}$ defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}\,dx, \quad t \in \mathbb{R}.$$

It is well known that these operators extend uniquely to the space $L^2(\mathbb{R})$. As usual, we will use symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the direct and inverse Fourier transform on $L^2(\mathbb{R})$. It is well known (see, e.g., [13, Theorem 2.5.10]) that the Fourier convolution operator

$$W^0(a) := \mathcal{F}^{-1}a\mathcal{F} \tag{1.1}$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$.

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. Their technical definitions are postponed to Section 2.1. The class of Banach function spaces is very large. It includes Lebesgue, Orlicz, Lorentz spaces, variable Lebesgue spaces and their weighted analogues (see, e.g., [3,6]). Let $\mathcal{B}(X(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators acting on $X(\mathbb{R})$.

Recall that the (non-centered) Hardy-Littlewood maximal function $\mathcal{M}f$ of a function $f \in L^1_{\rm loc}(\mathbb{R})$ is defined by

$$(\mathcal{M}f)(x) := \sup_{I \ni x} \frac{1}{|I|}\int_I |f(y)|\,dy,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length containing x. The Hardy-Littlewood maximal operator $\mathcal{M}$ defined by the rule $f \mapsto \mathcal{M}f$ is a sublinear operator.

If $X(\mathbb{R})$ is separable, then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see, e.g., [9, Lemma 2.2]). A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator $W^0(a)$ defined by (1.1) maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the symbol of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

If, in addition, the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ or on its associate space $X'(\mathbb{R})$, then for all $a \in \mathcal{M}_{X(\mathbb{R})}$,

$$\|a\|_{L^\infty(\mathbb{R})} \le \|a\|_{\mathcal{M}_{X(\mathbb{R})}}. \tag{1.2}$$

The constant 1 on the right-hand side of (1.2) is best possible (see [17, Corollary 4.2] and [18, Theorem 2.2]). Since (1.2) is available, one can show that $\mathcal{M}_{X(\mathbb{R})}$ is a Banach algebra (see [18, Corollary 2.3] and also [15, Corollary 1]).

Let us define the notion of equivalence at infinity. Following [7, p. 140], functions $a, b \in L^\infty(\mathbb{R})$ are called equivalent at ∞ if

$$\lim_{N\to\infty} \left\|\chi_{\mathbb{R}\setminus[-N,N]}(a-b)\right\|_{L^\infty(\mathbb{R})} = 0. \tag{1.3}$$

In this case we will write $a \overset{L^\infty(\mathbb{R})}{\sim} b$. For subsets Φ and Θ of $L^\infty(\mathbb{R})$, let Φ^*_Θ denote the set of all functions $a \in \Phi$ for which there exist functions $b \in \Theta$ such that $a \overset{L^\infty(\mathbb{R})}{\sim} b$.

Suppose that $X(\mathbb{R})$ is a separable Banach function space such that the Hardy-Littlewood maximal operator is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. It follows from the Stechkin-type inequality (see Theorem 2 below) that the characteristic function of every segment $[\alpha, \beta] \subset \mathbb{R}$ is a Fourier multiplier on the space $X(\mathbb{R})$ and

$$\sup_{-\infty<\alpha<\beta<\infty} \left\|\chi_{[\alpha,\beta]}\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \infty. \tag{1.4}$$

Inequality (1.4) implies that that $\chi_{\mathbb{R}\setminus[-N,N]} \in \mathcal{M}_{X(\mathbb{R})}$ for every $N > 0$. Two Fourier multipliers $c, d \in \mathcal{M}_{X(\mathbb{R})}$ are called equivalent at ∞ if

$$\lim_{N\to\infty} \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c-d)\right\|_{\mathcal{M}_{X(\mathbb{R})}} = 0. \tag{1.5}$$

In this case we will write $c \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} d$. It follows from [13, Theorem 2.5.10] that for the space $X(\mathbb{R}) = L^2(\mathbb{R})$, definitions (1.3) and (1.5) coincide. For subsets Ψ and Ω of the Banach algebra $\mathcal{M}_{X(\mathbb{R})}$, let Ψ_Ω denote the set of all Fourier multipliers $c \in \Psi$ for which there exist Fourier multipliers $d \in \Omega$ such that $c \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} d$.

For a Banach algebra $\mathfrak{A}$ and a subset $\mathfrak{S} \subset \mathfrak{A}$, let $\operatorname{alg}_{\mathfrak{A}} \mathfrak{S}$ denote the smallest closed subalgebra of $\mathfrak{A}$ that contains the set $\mathfrak{S}$. The notion of equivalence at infinity was used in [7, Theorem 4.4] to formulate results on the compactness of commutators $aW^0(b) - W^0(b)aI$ on Lebesgue spaces $L^p(\mathbb{R}, w)$ with Muckenhoupt weights. Those results extended previous results on such commutators from [8, Lemmas 7.1–7.4], [1, Theorem 4.2, Corollary 4.3], [22, Lemma 5.3], [21, Theorem 4.6]. Compactness of such commutators is important in the Fredholm study of the Banach algebras

$$\mathcal{A}(\Phi, \Psi) = \operatorname{alg}_{\mathcal{B}(X(\mathbb{R}))}\{aI, W^0(b) \ : \ a \in \Phi,\ b \in \Psi\}$$

generated by all operators aI and $W^0(b)$, where all functions a belong to a certain C^*-subalgebra Φ of $L^\infty(\mathbb{R})$ and all Fourier multipliers b belong to a certain Banach subalgebra Ψ of $\mathcal{M}_{X(\mathbb{R})}$ (see, e.g., [8,5,1,19]).

The aim of this paper is to show that if Ψ is an arbitrary unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ and Ω is a certain unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ that consists of "nice" Fourier multipliers, then Ψ_Ω is a unital Banach subalgebra of

$\mathcal{M}_{X(\mathbb{R})}$. Similarly, we will show that if Φ is an arbitrary unital C^*-subalgebra of $L^\infty(\mathbb{R})$ and Θ is a certain unital C^*-subalgebra of $L^\infty(\mathbb{R})$ that consists of "nice" functions, then Φ^*_Θ is a C^*-subalgebra of $L^\infty(\mathbb{R})$. We plan to use these results in the study of algebras $\mathcal{A}(\Phi^*_\Theta, \Psi_\Omega)$.

Let $C_0^\infty(\mathbb{R})$ be the set of all infinitely differentiable complex-valued functions on $\mathbb{R}$ with compact support. We say that a Banach subalgebra Ω of $\mathcal{M}_{X(\mathbb{R})}$ consists of "nice" Fourier multipliers if $u\Omega \subset \Omega$ for every $u \in C_0^\infty(\mathbb{R})$, where $u\Omega := \{ua : a \in \Omega\}$. Analogously, we say that a C^*-subalgebra Θ of $L^\infty(\mathbb{R})$ consists of "nice" functions if $u\Theta \subset \Theta$ for every $u \in C_0^\infty(\mathbb{R})$.

Theorem 1 (Main result) *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If Ψ and Ω are unital Banach subalgebras of $\mathcal{M}_{X(\mathbb{R})}$ such that $u\Omega \subset \Omega$ for every $u \in C_0^\infty(\mathbb{R})$, then Ψ_Ω is also a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$.*

Let $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{R}} = [-\infty, +\infty]$ be the one-point and the two-point compactifications of the real line. Let $C_X(\dot{\mathbb{R}})$ and $C_X(\overline{\mathbb{R}})$ denote the algebras of continuous Fourier multipliers and let $SO^\diamond_{X(\mathbb{R})}$ and $\widetilde{SO}^\diamond_{X(\mathbb{R})}$ be the algebras of slowly oscillating Fourier multipliers. It follows from their definitions given in Section 2 that they consist of "nice" Fourier multipliers. As a corollary of the above result, we immediately get the following.

Corollary 1 *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If Ψ is a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ and*

$$\Omega \in \left\{ C_X(\dot{\mathbb{R}}), C_X(\overline{\mathbb{R}}), SO^\diamond_{X(\mathbb{R})}, \mathrm{alg}_{\mathcal{M}_{X(\mathbb{R})}}\{SO^\diamond_{X(\mathbb{R})}, C_X(\overline{\mathbb{R}})\}, \widetilde{SO}^\diamond_{X(\mathbb{R})} \right\},$$

then Ψ_Ω is also a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$.

We will also derive the following corollary from the above result.

Corollary 2 *If Φ is a unital C^*-subalgebra of the algebra $L^\infty(\mathbb{R})$, and*

$$\Omega \in \left\{ C(\dot{\mathbb{R}}), C(\overline{\mathbb{R}}), SO^\diamond, \mathrm{alg}_{L^\infty(\mathbb{R})}\{SO^\diamond, C(\overline{\mathbb{R}})\} \right\},$$

then Φ^*_Ω is a unital C^*-subalgebra of $L^\infty(\mathbb{R})$.

Let QC and PQC be the C^*-algebras of quasicontinuous and piecewise quasicontinuous functions on $\mathbb{R}$, respectively (see, e.g., [25]). The above corollary is implicit in [7, Section 5] in the particular cases of the algebras $PQC^*_{SO^\diamond}$ and $QC^*_{\mathrm{alg}_{L^\infty(\mathbb{R})}\{SO^\diamond, C(\overline{\mathbb{R}})\}}$.

The paper is organized as follows. In Section 2, we recall the definitions of a Banach function space and its associate spaces, the Stechkin-type inequality for Banach function spaces, which permits to define the algebras of continuous Fourier multipliers $C_X(\dot{\mathbb{R}})$ and $C_X(\overline{\mathbb{R}})$. Further we recall the definitions of algebras of slowly oscillating functions SO_λ with $\lambda \in \dot{\mathbb{R}}$ and $SO^\diamond$. To define the Fourier multipliers counterpart of SO_λ and $SO^\diamond$, we need to impose more regularity conditions and consider three times continuously differentiable functions admitting slowly oscillating behavior at $\lambda \in \dot{\mathbb{R}}$. These algebras are denoted by $\widetilde{SO}_\lambda^3$ and $SO_\lambda^3 := SO_\lambda \cap \widetilde{SO}_\lambda^3$. It turns out that each function in $\widetilde{SO}_\lambda^3$ is a Fourier multiplier on $X(\mathbb{R})$ and we can define the algebras $SO_{\lambda,X(\mathbb{R})}$ and $\widetilde{SO}_{\lambda,X(\mathbb{R})}$ of Fourier multipliers slowly oscillating at $\lambda \in \dot{\mathbb{R}}$ as the closure of SO_λ^3 and $\widetilde{SO}_\lambda^3$, respectively. Finally, we define $SO^\diamond_{X(\mathbb{R})}$ and $\widetilde{SO}^\diamond_{X(\mathbb{R})}$ by analogy with $SO^\diamond$.

In Section 3, we prove Theorem 1. For a given Cauchy sequence $\{c_n\}$ in Φ_Ω, we consider a corresponding associated sequence $\{d_n\}$ in Ω such that $d_n \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} c_n$ for all $n \in \mathbb{N}$. Although $\{d_n\}$ might not be a Cauchy sequence in Ω, we can choose a convenient subsequence $\{d_{n_k}\}$ and then consider a sequence $h_k := v_k(d_{n_{k+1}} - d_{n_k})$ such that $v_k - 1 \in C_0^\infty(\mathbb{R})$ and $\|h_k\|_{\mathcal{M}_{X(\mathbb{R})}} = O(2^{-k})$ as $k \to \infty$. This allows us to conclude that the sequence defined by $\varphi_1 := d_{n_1}$ and $\varphi_{k+1} := \varphi_1 + \sum_{j=1}^k h_j$ is a Cauchy sequence in Ω. Since Ω is complete, its limit φ belongs to Ω. Finally we show that $\varphi \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} c$, where c is the limit of $\{c_n\}$ in Ψ. Thus $c \in \Psi_\Omega$ and Ψ_Ω is complete. We are indebted to one of the referees, whose suggestions allowed us to obtain a stronger main result and considerably simplify our initial arguments of its proof. We conclude this section with the proof of Corollary 2.

Finally, in Section 4, for a unital C^*-algebra Φ of $L^\infty(\mathbb{R})$ and a unital Banach subalgebra Ψ of $\mathcal{M}_{X(\mathbb{R})}$, we study the subalgebras

$$\mathcal{MO}^\pi(\Phi) = \{aI + \mathcal{K}(X(\mathbb{R})) : a \in \Phi\},\ \mathcal{CO}^\pi(\Psi) := \{W^0(b) + \mathcal{K}(X(\mathbb{R})) : b \in \Psi\}$$

of the Calkin algebra $\mathcal{B}^\pi(X(\mathbb{R})) := \mathcal{B}(X(\mathbb{R}))/\mathcal{K}(X(\mathbb{R}))$, where $\mathcal{K}(X(\mathbb{R}))$ is the ideal of compact operators in the algebra $\mathcal{B}(X(\mathbb{R}))$. We show that the intersection of $\mathcal{MO}^\pi(\Phi)$ and $\mathcal{CO}^\pi(\Psi_{SO^\diamond_{X(\mathbb{R})}})$ is equal to $\{cI + \mathcal{K}(X(\mathbb{R})) : c \in \mathbb{C}\}$. This result generalizes the main result of our recent paper [12].

2 Preliminaries

2.1 Banach function spaces

The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}$ is denoted by $\mathfrak{M}(\mathbb{R})$. Let $\mathfrak{M}^+(\mathbb{R})$ be the subset of functions in $\mathfrak{M}(\mathbb{R})$ whose values lie in $[0,\infty]$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}$ is denoted by $|E|$ and its characteristic function is denoted by χ_E. Following [3, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\mathbb{R}) \to [0,\infty]$ is called a Banach function norm if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{R})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{R}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad \rho(af) = a\rho(f), \quad \rho(f+g) \leq \rho(f) + \rho(g)$,
(A2) $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),
(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$,

(A5) $|E| < \infty \Rightarrow \int_E f(x)\,dx \le C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of all functions $f \in \mathfrak{M}(\mathbb{R})$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{R})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R})} := \rho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathbb{R})$ becomes a Banach space (see [3, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{R})$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}} f(x)g(x)\,dx \ : \ f \in \mathfrak{M}^+(\mathbb{R}),\ \rho(f) \le 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{R}).$$

The associate norm ρ' is itself a Banach function norm [3, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R})$. The associate space $X'(\mathbb{R})$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathbb{R})]^*$.

2.2 Stechkin-type inequality and Banach algebras $C_X(\dot{\mathbb{R}})$ and $C_X(\overline{\mathbb{R}})$ of continuous Fourier multipliers

Let $V(\mathbb{R})$ be the Banach algebra of all functions $a : \mathbb{R} \to \mathbb{C}$ with finite total variation

$$V(a) := \sup \sum_{i=1}^{n} |a(t_i) - a(t_{i-1})|,$$

where the supremum is taken over all finite partitions

$$-\infty < t_0 < t_1 < \cdots < t_n < +\infty$$

of the real line $\mathbb{R}$ and the norm in $V(\mathbb{R})$ is given by

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a).$$

Theorem 2 *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $a \in V(\mathbb{R})$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and*

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X \|a\|_V \tag{2.1}$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

This result follows from [14, Theorem 4.3].

For Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, inequality (2.1) is usually called Stechkin's inequality (see, e.g., [8, Theorem 2.11]).

Let $C_X(\dot{\mathbb{R}})$ (resp., $C_X(\overline{\mathbb{R}})$) be the closure of the set $C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ (resp., $C(\overline{\mathbb{R}}) \cap V(\mathbb{R})$) with respect to the norm of $\mathcal{M}_{X(\mathbb{R})}$. Fourier multipliers in $C_X(\dot{\mathbb{R}})$ and in $C_X(\overline{\mathbb{R}})$ will be called continuous Fourier multipliers.

2.3 Slowly oscillating functions

For a set $E \subset \dot{\mathbb{R}}$ and a function $f : \dot{\mathbb{R}} \to \mathbb{C}$ in $L^\infty(\mathbb{R})$, let the oscillation of f over E be defined by

$$\operatorname{osc}(f, E) := \operatorname*{ess\,sup}_{s,t \in E} |f(s) - f(t)|.$$

Following [2, Section 4] and [21, Section 2.1], we say that a function $f \in L^\infty(\mathbb{R})$ is slowly oscillating at a point $\lambda \in \dot{\mathbb{R}}$ if for every $r \in (0, 1)$ or, equivalently, for some $r \in (0, 1)$, one has

$$\begin{aligned} &\lim_{x \to 0+} \operatorname{osc}\big(f, \lambda + ([-x, -rx] \cup [rx, x])\big) = 0 \text{ if } \lambda \in \mathbb{R}, \\ &\lim_{x \to +\infty} \operatorname{osc}\big(f, [-x, -rx] \cup [rx, x]\big) = 0 \qquad \text{if } \lambda = \infty. \end{aligned}$$

For every $\lambda \in \dot{\mathbb{R}}$, let SO_λ denote the C^*-subalgebra of $L^\infty(\mathbb{R})$ defined by

$$SO_\lambda := \Big\{ f \in C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) \ : \ f \text{ slowly oscillates at } \lambda \Big\},$$

where $C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) := C(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbb{R})$.

Let $SO^\diamond$ be the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbb{R}}$. The functions in $SO^\diamond$ are called slowly oscillating functions.

2.4 Banach algebra SO^3_λ and $\widetilde{SO}^3_\lambda$ of three times continuously differentiable slowly oscillating functions

For a point $\lambda \in \dot{\mathbb{R}}$, let $C^3(\mathbb{R} \setminus \{\lambda\})$ be the set of all three times continuously differentiable functions $a : \mathbb{R} \setminus \{\lambda\} \to \mathbb{C}$. Slightly extending definitions in [22, Section 3] and [21, Section 2.3], consider the commutative Banach algebras $\widetilde{SO}^3_\lambda$ of functions $a \in C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap C^3(\mathbb{R} \setminus \{\lambda\})$ such that

$$\lim_{x \to \lambda} (D^k_\lambda a)(x) = 0, \quad k = 1, 2, 3,$$

and

$$\|a\|_{SO^3_\lambda} := \sum_{j=0}^{3} \frac{1}{j!} \big\| D^j_\lambda a \big\|_{L^\infty(\mathbb{R})} < \infty,$$

where $(D_\lambda a)(x) = (x-\lambda)a'(x)$ for $\lambda \in \mathbb{R}$ and $(D_\lambda a)(x) = xa'(x)$ for $\lambda = \infty$. The following Banach algebras were introduced for $\lambda = \infty$ in [22, Section 3] and for arbitrary $\lambda \in \dot{\mathbb{R}}$ in [21, Section 2.3]:

$$SO_\lambda^3 := SO_\lambda \cap \widetilde{SO}_\lambda^3.$$

Lemma 1 ([12, Lemma 2.6]) *For every $\lambda \in \dot{\mathbb{R}}$, the set SO_λ^3 is dense in the C^*-algebra SO_λ.*

2.5 Slowly oscillating Fourier multipliers

The following result leads us to the definition of slowly oscillating Fourier multipliers.

Theorem 3 *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $\lambda \in \dot{\mathbb{R}}$ and $a \in \widetilde{SO}_\lambda^3$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and*

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le s_X \|a\|_{SO_\lambda^3},$$

where s_X is a positive constant depending only on $X(\mathbb{R})$.

This theorem is proved [16, Theorem 2.5] for the smaller algebra SO_λ^3. In fact, for the algebra $\widetilde{SO}_\lambda^3$ the proof is the same. It is based on [16, Theorem 3.4], which in turn essentially uses [22, Lemma 2.2, Corollary 2.8].

Let $SO_{\lambda,X(\mathbb{R})}$ (resp., $\widetilde{SO}_{\lambda,X(\mathbb{R})}$) denote the closure of SO_λ^3 (resp., $\widetilde{SO}_\lambda^3$) in the norm of $\mathcal{M}_{X(\mathbb{R})}$. Further, let $SO^\diamond_{X(\mathbb{R})}$ (resp., $\widetilde{SO}^\diamond_{X(\mathbb{R})}$) be the smallest Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$ that contains all the Banach algebras $SO_{\lambda,X(\mathbb{R})}$ (resp., $\widetilde{SO}_{\lambda,X(\mathbb{R})}$) for $\lambda \in \dot{\mathbb{R}}$. The functions in $SO^\diamond_{X(\mathbb{R})}$ and $\widetilde{SO}^\diamond_{X(\mathbb{R})}$ will be called slowly oscillating Fourier multipliers.

3 Proofs of the main results

3.1 Proof of Theorem 1

It is easy to check that the set Ψ_Ω is a unital normed subalgebra of the Banach algebra $\mathcal{M}_{X(\mathbb{R})}$. It remains to show its completeness.

Take a Cauchy sequence $\{c_n\}_{n=1}^\infty$ in Ψ_Ω and an associated sequence $\{d_n\}_{n=1}^\infty$ in Ω such that

$$\lim_{N\to\infty} \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_n - d_n)\right\|_{\mathcal{M}_{X(\mathbb{R})}} = 0 \tag{3.1}$$

for all $n \in \mathbb{N}$. It follows from (1.4) that

$$D_X := \sup_{N>0} \left\|\chi_{\mathbb{R}\setminus[-N,N]}\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \infty. \tag{3.2}$$

Since $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence, one can find a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$\left\|c_{n_{k+1}} - c_{n_k}\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \frac{1}{3D_X} \cdot \frac{1}{2^{k+1}}. \tag{3.3}$$

In view of (3.2) and (3.3), we obtain for all $N > 0$ and all $k \in \mathbb{N}$,

$$\left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_{k+1}} - c_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} \le D_X \left\|c_{n_{k+1}} - c_{n_k}\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \frac{1}{3 \cdot 2^{k+1}}. \tag{3.4}$$

By (3.1), there exists a strictly increasing sequence $\{M_k\}_{k=1}^{\infty}$ of real positive numbers such that for all $k \in \mathbb{N}$ and all $N > M_k$,

$$\left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_{k+1}} - d_{n_{k+1}})\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \frac{1}{3 \cdot 2^{k+1}}, \tag{3.5}$$

$$\left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_k} - d_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} < \frac{1}{3 \cdot 2^{k+1}}. \tag{3.6}$$

It follows from (3.4)–(3.6) that for all $k \in \mathbb{N}$ and all $N > M_k$,

$$\begin{aligned}\left\|\chi_{\mathbb{R}\setminus[-N,N]}(d_{n_{k+1}} - d_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} &\le \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_{k+1}} - d_{n_{k+1}})\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\ &\quad + \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_{k+1}} - c_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\ &\quad + \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_k} - d_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\ &< \frac{1}{2^{k+1}}. \end{aligned} \tag{3.7}$$

Take

$$g_k := d_{n_{k+1}} - d_{n_k} \in \Omega. \tag{3.8}$$

Set $N_0 = 0$. For every $k \in \mathbb{N}$ take a number $N_k > \max\{M_k, N_{k-1} + 1\}$. Let $\varrho_k \in C_0^{\infty}(\mathbb{R})$ be such that $\varrho_k(t) \ge 0$ for $t \in \mathbb{R}$, $\varrho_k(t) = 0$ for $t \in \mathbb{R} \setminus (N_k, N_k + 1)$ and

$$\int_{\mathbb{R}} \varrho_k(t)\, dt = 1.$$

Set

$$v_k(x) := \int_0^x \varrho_k(t)\, dt, \quad x \ge 0,$$

and then consider its even extension:

$$v_k(x) := v_k(-x), \quad x < 0.$$

It is easy to see that the function v_k is infinitely differentiable, $v_k(x) = 1$ for $x \in \mathbb{R} \setminus [-N_k - 1, N_k + 1]$, $v_k(x) = 0$ for $x \in [-N_k, N_k]$ and v_k is monotone on $[-N_k - 1, -N_k]$ and on $[N_k, N_k + 1]$. Therefore $V(v_k) = 2$ and $\|v_k\|_V = 3$. Set

$$h_k := v_k g_k. \tag{3.9}$$

Since $g_k \in \Omega$ and $v_k - 1 \in C_0^{\infty}(\mathbb{R})$, we have

$$h_k = (v_k - 1)g_k + g_k \in \Omega.$$

Taking into account the fact that $v_k(x) = 0$ for $x \in [-N_k, N_k]$, we deduce from Theorem 2 that

$$\|h_k\|_{\mathcal{M}_{X(\mathbb{R})}} = \left\|v_k \chi_{\mathbb{R}\setminus[-N_k,N_k]} g_k\right\|_{\mathcal{M}_{X(\mathbb{R})}}$$

$$\begin{aligned}
&\leq \|v_k\|_{\mathcal{M}_{X(\mathbb{R})}} \left\|\chi_{\mathbb{R}\setminus[-N_k,N_k]} g_k\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\
&\leq c_X \|v_k\|_V \left\|\chi_{\mathbb{R}\setminus[-N_k,N_k]} g_k\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\
&= 3c_X \left\|\chi_{\mathbb{R}\setminus[-N_k,N_k]} g_k\right\|_{\mathcal{M}_{X(\mathbb{R})}} .
\end{aligned} \tag{3.10}$$

Let

$$\varphi_1 := d_{n_1}, \quad \varphi_{k+1} := \varphi_1 + \sum_{j=1}^{k} h_j, \quad k \in \mathbb{N}. \tag{3.11}$$

Now let $n, m \in \mathbb{N}$. Without loss of generality we can assume that $n > m$. It follows from (3.7)–(3.8) and (3.10)–(3.11) that

$$\begin{aligned}
\|\varphi_n - \varphi_m\|_{\mathcal{M}_{X(\mathbb{R})}} &= \left\|\sum_{k=m}^{n-1} h_k\right\|_{\mathcal{M}_{X(\mathbb{R})}} \leq \sum_{k=m}^{n-1} \|h_k\|_{\mathcal{M}_{X(\mathbb{R})}} \leq 3c_X \sum_{k=m}^{n-1} \frac{1}{2^{k+1}} \\
&= \frac{3c_X}{2^{m+1}} \cdot \frac{1-(1/2)^{n-m}}{1-1/2} < \frac{3c_X}{2^m},
\end{aligned}$$

which implies that $\{\varphi_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_{X(\mathbb{R})}$.

Hence, there exists the limit

$$\varphi := \lim_{k\to\infty} \varphi_k \in \Omega \subset \mathcal{M}_{X(\mathbb{R})}.$$

Let c be the limit of the Cauchy sequence $\{c_n\}_{n=1}^\infty$ in the Banach subalgebra Ψ of $\mathcal{M}_{X(\mathbb{R})}$. Hence

$$c = \lim_{k\to\infty} c_{n_k} \in \Psi \subset \mathcal{M}_{X(\mathbb{R})}.$$

Let

$$\delta_k := \|(c-\varphi) - (c_{n_k} - \varphi_k)\|_{\mathcal{M}_{X(\mathbb{R})}}, \quad k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} \delta_k = 0 \tag{3.12}$$

and for all $k \in \mathbb{N}$ and $N \geq N_k$,

$$\begin{aligned}
\left\|\chi_{\mathbb{R}\setminus[-N,N]}(c-\varphi)\right\|_{\mathcal{M}_{X(\mathbb{R})}} &\leq \left\|\chi_{\mathbb{R}\setminus[-N,N]}(c_{n_k} - d_{n_k})\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\
&\quad + \left\|\chi_{\mathbb{R}\setminus[-N,N]}(d_{n_k} - \varphi_k)\right\|_{\mathcal{M}_{X(\mathbb{R})}} \\
&\quad + \delta_k \left\|\chi_{\mathbb{R}\setminus[-N,N]}\right\|_{\mathcal{M}_{X(\mathbb{R})}} .
\end{aligned} \tag{3.13}$$

If $k \geq 2$ and $N \geq N_k > N_{k-1} + 1$, then

$$\chi_{\mathbb{R}\setminus[-N,N]} v_j = \chi_{\mathbb{R}\setminus[-N,N]} \quad \text{for} \quad j \in \{1, \dots, k-1\}. \tag{3.14}$$

It follows from (3.8)–(3.9), (3.11), and (3.14) that

$$\chi_{\mathbb{R}\setminus[-N,N]}(d_{n_k} - \varphi_k) = \chi_{\mathbb{R}\setminus[-N,N]} \left(d_{n_k} - \varphi_1 - \sum_{j=1}^{k-1} h_j\right)$$

$$
\begin{aligned}
&= \chi_{\mathbb{R}\setminus[-N,N]} \left(d_{n_k} - d_{n_1} - \sum_{j=1}^{k-1} v_j \left(d_{n_{j+1}} - d_{n_j} \right) \right) \\
&= \chi_{\mathbb{R}\setminus[-N,N]} \left(d_{n_k} - d_{n_1} - \sum_{j=1}^{k-1} \left(d_{n_{j+1}} - d_{n_j} \right) \right) \\
&= 0. \qquad (3.15)
\end{aligned}
$$

Combining (3.2), (3.6), (3.13) and (3.15), we have for $k \ge 2$,

$$
\left\| \chi_{\mathbb{R}\setminus[-N_k,N_k]}(c-\varphi) \right\|_{\mathcal{M}_{X(\mathbb{R})}} \le \frac{1}{3\cdot 2^{k+1}} + D_X \delta_k .
$$

This inequality, (3.2) and (3.12) yield that for every $\varepsilon > 0$ there exists a number $s \in \mathbb{N}$ such that for all $N > N_s$,

$$
\left\| \chi_{\mathbb{R}\setminus[-N,N]}(c-\varphi) \right\|_{\mathcal{M}_{X(\mathbb{R})}} \le D_X \left\| \chi_{\mathbb{R}\setminus[-N_s,N_s]}(c-\varphi) \right\|_{\mathcal{M}_{X(\mathbb{R})}} < \varepsilon,
$$

which implies that

$$
\lim_{N\to\infty} \left\| \chi_{\mathbb{R}\setminus[-N,N]}(c-\varphi) \right\|_{\mathcal{M}_{X(\mathbb{R})}} = 0,
$$

that is, $c \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} \varphi$. Thus $c \in \Psi_\Omega$ and the algebra Ψ_Ω is complete. □

3.2 Proof of Corollary 2

By [13, Theorem 2.5.10],

$$
\mathcal{M}_{L^2(\mathbb{R})} = L^\infty(\mathbb{R}) \qquad (3.16)
$$

with equal norms. Hence, by Corollary 1, for $X(\mathbb{R}) = L^2(\mathbb{R})$, we conclude that $\Phi^*_{C_{L^2}(\mathbb{I})}$ for $\mathbb{I} \in \{\dot{\mathbb{R}}, \overline{\mathbb{R}}\}$, $\Phi^*_{SO^\diamond_{L^2(\mathbb{R})}}$ and $\Phi^*_{\mathrm{alg}_{L^\infty(\mathbb{R})}\{SO^\diamond_{L^2(\mathbb{R})}, C_{L^2}(\overline{\mathbb{R}})\}}$ are unital Banach subalgebras of $L^\infty(\mathbb{R})$. It is clear that, in fact, they are unital C^*-subalgebras of $L^\infty(\mathbb{R})$.

It is well known that

$$
C_{L^2}(\mathbb{I}) = C(\mathbb{I}) \quad \text{for} \quad \mathbb{I} \in \{\dot{\mathbb{R}}, \overline{\mathbb{R}}\}. \qquad (3.17)
$$

Thus $\Phi^*_{C(\mathbb{I})}$ are unital C^*-subalgebras of $L^\infty(\mathbb{R})$ for $\mathbb{I} \in \{\dot{\mathbb{R}}, \overline{\mathbb{R}}\}$.

Equality (3.16), inequality (1.2) and Lemma 1 imply that $SO_{\lambda, L^2(\mathbb{R})} = SO_\lambda$ for all $\lambda \in \dot{\mathbb{R}}$. Then

$$
SO^\diamond_{L^2(\mathbb{R})} = SO^\diamond. \qquad (3.18)
$$

Thus $\Phi^*_{SO^\diamond_{L^2(\mathbb{R})}} = \Phi^*_{SO^\diamond}$ is a unital C^*-subalgebra of $L^\infty(\mathbb{R})$. It follows from (3.17)–(3.18) that

$$
\mathrm{alg}_{L^\infty(\mathbb{R})}\{SO^\diamond_{L^2(\mathbb{R})}, C_{L^2}(\overline{\mathbb{R}})\} = \mathrm{alg}_{L^\infty(\mathbb{R})}\{SO^\diamond, C(\overline{\mathbb{R}})\}.
$$

This observation implies that $\Phi^*_{\mathrm{alg}\{SO^\diamond, C(\overline{\mathbb{R}})\}}$ is a unital C^*-subalgebra of the algebra $L^\infty(\mathbb{R})$. □

4 Calkin images of Fourier convolution operators with symbols equivalent at infinity to slowly oscillating Fourier multipliers

4.1 Calkin images of multiplication operators and Fourier convolution operators

Let $X(\mathbb{R})$ be a separable Banach function space and $\mathcal{K}(X(\mathbb{R}))$ be the closed two-sided ideal of compact operators in the Banach algebra $\mathcal{B}(X(\mathbb{R}))$. For a unital C^*-subalgebra Φ of the algebra $L^\infty(\mathbb{R})$, we consider the quotient algebra $\mathcal{MO}^\pi(\Phi)$ consisting of the cosets

$$[aI]^\pi := aI + \mathcal{K}(X(\mathbb{R}))$$

of multiplication operators by functions in Φ:

$$\mathcal{MO}^\pi(\Phi) := \{[aI]^\pi : \ a \in \Phi\} = \{aI + \mathcal{K}(X(\mathbb{R})) : \ a \in \Phi\}.$$

For a unital Banach subalgebra Ψ of the algebra $\mathcal{M}_{X(\mathbb{R})}$, we also consider the quotient algebra $\mathcal{CO}^\pi(\Psi)$ consisting of the cosets

$$[W^0(b)]^\pi := W^0(b) + \mathcal{K}(X(\mathbb{R}))$$

of convolution operators with symbols in the algebra Ψ:

$$\mathcal{CO}^\pi(\Psi) := \{[W^0(b)]^\pi : \ b \in \Psi\} = \{W^0(b) + \mathcal{K}(X(\mathbb{R})) : \ b \in \Psi\}.$$

It is easy to see that $\mathcal{MO}^\pi(\Phi)$ and $\mathcal{CO}^\pi(\Psi)$ are commutative unital Banach subalgebras of the Calkin algebra $\mathcal{B}^\pi(X(\mathbb{R})) := \mathcal{B}(X(\mathbb{R}))/\mathcal{K}(X(\mathbb{R}))$. It is natural to refer to the algebras $\mathcal{MO}^\pi(\Phi)$ and $\mathcal{CO}^\pi(\Psi)$ as the Calkin images of the algebras

$$\mathcal{MO}(\Phi) = \{aI : a \in \Phi\} \subset \mathcal{B}(X(\mathbb{R})), \ \mathcal{CO}(\Psi) = \{W^0(b) : b \in \Psi\} \subset \mathcal{B}(X(\mathbb{R})),$$

respectively. The algebras $\mathcal{MO}(\Phi)$ and $\mathcal{CO}(\Psi)$ are building blocks of the algebra of convolution type operators

$$\mathrm{alg}_{\mathcal{B}(X(\mathbb{R}))} \left\{aI, W^0(b) : \ a \in \Phi, \ b \in \Psi\right\},$$

the smallest closed subalgebra of $\mathcal{B}(X(\mathbb{R}))$ that contains the algebras $\mathcal{MO}(\Phi)$ and $\mathcal{CO}(\Psi)$.

We proved in [12, Theorem 1.1] that if $X(\mathbb{R})$ is a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$, then for an arbitrary unital C^*-subalgebra Φ of $L^\infty(\mathbb{R})$,

$$\mathcal{MO}^\pi(\Phi) \cap \mathcal{CO}^\pi(SO^\diamond_{X(\mathbb{R})}) = \mathcal{MO}^\pi(\mathbb{C}), \tag{4.1}$$

where

$$\mathcal{MO}^\pi(\mathbb{C}) := \{[cI]^\pi : \ c \in \mathbb{C}\}. \tag{4.2}$$

The aim of this section is to extend formula (4.1), substituting the algebra $SO^\diamond_{X(\mathbb{R})}$ by the algebra $\Psi_{SO^\diamond_{X(\mathbb{R})}}$ with an arbitrary unital Banach subalgebra Ψ of $\mathcal{M}_{X(\mathbb{R})}$.

4.2 Known results about limit operators on Banach function spaces

Let $X(\mathbb{R})$ be a Banach function space. For a sequence $\{A_n\}_{n\in\mathbb{N}}$ of bounded linear operators on $X(\mathbb{R})$, let

$$\operatorname*{s-lim}_{n\to\infty} A_n$$

denote the strong limit of this sequence, if it exists. For $\lambda, x \in \mathbb{R}$, consider the function

$$e_\lambda(x) := e^{i\lambda x}.$$

Let $T \in \mathcal{B}(X(\mathbb{R}))$ and let $h = \{h_n\}_{n\in\mathbb{N}}$ be a sequence of numbers $h_n > 0$ such that $h_n \to +\infty$ as $n \to \infty$. The strong limit

$$T_h := \operatorname*{s-lim}_{n\to\infty} e_{h_n} T e_{h_n}^{-1} I$$

is called the limit operator of T related to the sequence $h = \{h_n\}_{n\in\mathbb{N}}$, if it exists.

We refer to [23] and [24] for the method of limit operators and its various applications in operator theory. This method was successfully applied in many works devoted to the study of convolution type operators with slowly oscillating data (see, e.g., [4, 20, 21]).

In our recent works [10, 11, 12] we calculated the limit operators for all compact operators, all Fourier convolution operators with symbols equivalent to zero at infinity, and all Fourier convolution operators with slowly oscillating symbols, respectively, which act on Banach function spaces.

Lemma 2 ([10, Lemma 3.2]) *Let $X(\mathbb{R})$ be a separable Banach function space and K be a compact operator on $X(\mathbb{R})$. Then for every sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive numbers satisfying $h_n \to +\infty$ as $n \to \infty$, one has*

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} K e_{h_n}^{-1} I = 0$$

on the space $X(\mathbb{R})$.

Lemma 3 ([11, Theorem 2]) *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $b \in \mathcal{M}_{X(\mathbb{R})}$ is such that $b \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} 0$, then for every sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive numbers satisfying $h_n \to +\infty$ as $n \to \infty$, one has*

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(b) e_{h_n}^{-1} I = 0$$

on the space $X(\mathbb{R})$.

To formulate the last auxiliary result, we have to recall the notion of fibers of maximal ideal spaces. For a commutative C^*-algebra $\mathfrak{A}$, let $M(\mathfrak{A})$ denote its maximal ideal space. If $\mathfrak{B}$ is a C^*-subalgebra of $\mathfrak{A}$ and $\lambda \in M(\mathfrak{B})$, then the set

$$M_\lambda(\mathfrak{A}) := \{\xi \in M(\mathfrak{A}) : \xi|_{\mathfrak{B}} = \lambda\}$$

is called the fiber of $M(\mathfrak{A})$ over $\lambda \in M(\mathfrak{B})$. Hence for every C^*-algebra $\Phi \subset L^\infty(\mathbb{R})$ with $M(C(\dot{\mathbb{R}}) \cap \Phi) = \dot{\mathbb{R}}$ and every $t \in \dot{\mathbb{R}}$, the fiber $M_t(\Phi)$ is the set of all multiplicative linear functionals (characters) on Φ that annihilate the set $\{f \in C(\dot{\mathbb{R}}) \cap \Phi : f(t) = 0\}$. As usual, for all $a \in \Phi$ and all $\xi \in M(\Phi)$, we put $a(\xi) := \xi(a)$.

Theorem 4 ([12, Theorem 5.2]) *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $b \in SO^\diamond_{X(\mathbb{R})}$, then for every $\xi \in M_\infty(SO^\diamond)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive numbers such that $h_n \to +\infty$ as $n \to \infty$ and*

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(b) e_{h_n}^{-1} I = b(\xi) I$$

on the space $X(\mathbb{R})$.

4.3 Application of the method of limit operators to the study of Calkin images

Now we are in a position to prove the main result of this section, which extends [12, Theorem 1.1] (see also [19, Lemma 4.3]).

Theorem 5 *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If Φ is a unital C^*-subalgebra of $L^\infty(\mathbb{R})$ and Ψ is a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$, then*

$$\mathcal{MO}^\pi(\Phi) \cap \mathcal{CO}^\pi\left(\Psi_{SO^\diamond_{X(\mathbb{R})}}\right) = \mathcal{MO}^\pi(\mathbb{C}), \tag{4.3}$$

where $\mathcal{MO}^\pi(\mathbb{C})$ is defined by (4.2).

Proof By Corollary 1, $\Psi_{SO^\diamond_{X(\mathbb{R})}}$ is a unital Banach subalgebra of $\mathcal{M}_{X(\mathbb{R})}$. Since the function $e_0 \equiv 1$ belongs to Φ and $\Psi_{SO^\diamond_{X(\mathbb{R})}}$, we see that the set of all constant functions is contained in Φ and in $\Psi_{SO^\diamond_{X(\mathbb{R})}}$. Therefore

$$\mathcal{MO}^\pi(\mathbb{C}) \subset \mathcal{MO}^\pi(\Phi) \cap \mathcal{CO}^\pi\left(\Psi_{SO^\diamond_{X(\mathbb{R})}}\right). \tag{4.4}$$

Let $A^\pi \in \mathcal{MO}^\pi(\Phi) \cap \mathcal{CO}^\pi\left(\Psi_{SO^\diamond_{X(\mathbb{R})}}\right)$. Then $A^\pi = [aI]^\pi = [W^0(b)]^\pi$, where $a \in \Phi$, $b \in \Psi$ and there exists $d \in SO^\diamond_{X(\mathbb{R})}$ satisfying $b \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} d$. Therefore, there is an operator $K \in \mathcal{K}(X(\mathbb{R}))$ such that

$$aI = W^0(b) + K. \tag{4.5}$$

By Theorem 4, for every $\xi \in M_\infty(SO^\diamond)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive numbers such that $h_n \to +\infty$ as $n \to \infty$ and

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(d) e_{h_n}^{-1} I = d(\xi) I. \tag{4.6}$$

Since $b \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} d$, we have $b - d \overset{\mathcal{M}_{X(\mathbb{R})}}{\sim} 0$. Then, by Lemma 3,

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(b-d) e_{h_n}^{-1} I = 0. \tag{4.7}$$

Equalities (4.5)–(4.7) and Lemma 2 imply that

$$\begin{aligned} aI &= \operatorname*{s-lim}_{n\to\infty} e_{h_n}(aI)e_{h_n}^{-1}I = \operatorname*{s-lim}_{n\to\infty} e_{h_n}(W^0(b)+K)e_{h_n}^{-1}I \\ &= \operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(d) e_{h_n}^{-1} I + \operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(b-d) e_{h_n}^{-1} I + \operatorname*{s-lim}_{n\to\infty} e_{h_n} K e_{h_n}^{-1} I \\ &= d(\xi) I. \end{aligned}$$

Hence $[aI]^\pi = [d(\xi)I]^\pi \in \mathcal{MO}^\pi(\mathbb{C})$ and

$$\mathcal{MO}^\pi(\varPhi) \cap \mathcal{CO}^\pi\left(\varPsi_{SO^\diamond_{X(\mathbb{R})}}\right) \subset \mathcal{MO}^\pi(\mathbb{C}). \tag{4.8}$$

Combining (4.4) and (4.8), we arrive at (4.3). □

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