

Rebalancing Frequency and the Welfare Cost of Inflation[†]

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Cash-in-advance models usually require agents to reallocate money and bonds in fixed periods. Every month or quarter, for example. I show that fixed periods underestimate the welfare cost of inflation. I use a model in which agents choose how often they exchange bonds for money. In the benchmark specification, the welfare cost of 10 percent instead of 0 inflation increases from 0.1 percent of income with fixed periods to 1 percent with optimal periods. The results are robust to different preferences, to different compositions of income in bonds or money, and to the introduction of capital and labor. (JEL: E30, E40, E50)

I calculate the welfare cost of inflation in a cash-in-advance model in which agents choose how often to trade bonds for money. The agents make periodic trades of bonds for money as in Baumol (1952) and Tobin (1956). I show that endogenizing the timing of the bond market trades increases the welfare cost of inflation. When the timing of bond market trades is fixed, the elasticity of the demand for money with respect to the interest rate is small, which implies a small welfare cost of inflation. When the timing of bond market trades is endogenous, an increase in inflation makes agents trade more frequently. The demand for money becomes sensitive to the interest rate, the fit to the data improves, and the welfare cost of inflation increases. I show that an increase in inflation from 0 to 10 percent per year implies a welfare cost of 1 percent of GDP when the timing of bond market trades is allowed to respond to inflation. The welfare cost is zero when the timing is fixed. With US GDP in 2000, a welfare cost of 1 percent of GDP corresponds to \$100 billion per year, or \$900 given to every household every year forever (data on GDP and on the number of households is from the Bureau of Economic Analysis (BEA) and the US Census Bureau).

To study whether fixed periods affect the welfare cost of inflation, I compare the welfare cost from two cash-in-advance models: one in which agents choose how often to trade bonds for money, and another in which agents trade bonds for money

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in fixed periods. It is optimal for each agent to increase the trading frequency when inflation increases. However, increasing the trading frequency affects equilibrium through a market clearing condition. The increase in the trading frequency increases transactions costs, which reduces welfare. Taking into account that agents change their trading frequency substantially increases the welfare cost estimates. A similar effect occurs with distortionary taxation. It is optimal for each agent to decrease labor when the income tax increases, but the aggregate decrease in labor implies a decrease in welfare.

The results on the welfare cost are related to the ability of cash-in-advance models to match data on interest rates and money. According to the data, an increase in the interest rate makes the real demand for money decrease. With fixed periods, the demand for money is approximately constant, it is inelastic with respect to the interest rate. Without further frictions, the welfare cost is approximately equal to the area under the demand for money above equilibrium money holdings. This area increases with the elasticity of the demand for money. An inelastic demand for money, therefore, implies zero welfare cost of inflation. Optimal periods imply more elasticity, a better match of interest rates and money, and a higher welfare cost of inflation.

Cash-in-advance models, in which agents buy goods with all balances from the previous period, with a constraint such as $P_1 c_1 \leq M_0$, imply money velocity equal to one, and so an inelastic demand for money by construction. Allowing cash and credit goods or changing the trading sequence (trading bonds before or after trading goods) implies variation in velocity. But Hodrick, Kocherlakota, and Lucas (1991) show that the variation of velocity is small.

Velocity varies more when agents hold money for many periods, with $P_1 c_1 + \dots + P_N c_N \leq M_0$. In this case, different groups of agents trade bonds for money every $N > 1$ periods. This is the tradition of market segmentation initiated by Grossman and Weiss (1983) and Rotemberg (1984). Alvarez, Atkeson, and Edmond (2009) show that this type of model matches the data on the short-run variation in velocity. However, as pointed out by Romer (1986) and Grossman (1987), the models with fixed N generate no variation in long-run velocity. Even with large N , the demand for money is inelastic.

Here, I allow N to vary according to the interest rate. For realistic parameter values, this change implies a long-run demand for money with -0.5 elasticity (semi-elasticity of -12.5) and a better match with the data. As in the literature of market segmentation, agents trade in different periods. The difference is that I let agents choose how often to trade bonds for money. This choice made by the agents complicates the problem, but it still allows the analytical characterization of the problem. I obtain formulas for the demand for money and for the welfare cost of inflation, which facilitates the analysis of the results.

A known result of market segmentation is that N should be large to match data on velocity (Edmond and Weill 2008). Alvarez, Atkeson, and Edmond (2009), for example, require intervals of 24 to 36 months. This paper is not an exception. I find intervals of 6 to 16 months. (Smaller intervals because I use M1 instead of M2 as monetary aggregate. I use M1 to make my results comparable to Lucas 2000, Lagos and Wright 2005, and others.)

TABLE 1—WELFARE COST OF 10 PERCENT INFLATION INSTEAD OF 0 INFLATION
(in percentage of income)

	Optimal periods		Fixed periods	
	$a = 0.6$	$a = 0$	$a = 0.6$	$a = 0$
Logarithmic utility	0.97	0.95	0.12	0.02

Notes: a : fraction of income received in money. Values from Table 2 in Section III.

There are two potential problems with a large N :

- High intertemporal substitution facilitates the variation of consumption within holding periods and might change results.
- Large trading intervals imply that agents accumulate interest in bonds for long periods and, therefore, make large transfers of money.

I show that neither of the two problems is important. The welfare cost with fixed periods of 10 percent instead of 0 inflation is always small. And optimal trading periods always yield a welfare cost about 1 percentage point higher.

To study if other means to react to inflation could change results beyond the decision on consumption and rebalancing frequency, I make an extension of the model for capital and labor. The results are robust. I find that the welfare costs in terms of income of 10 percent instead of 0 inflation increase in parallel, about 0.3 percentage points in terms of income for fixed and endogenous periods. The welfare cost with endogenous periods is still 1 percentage point higher.

Table 1 shows the main message of the paper—the estimates of the welfare cost increase from approximately 0 to 1 percent with optimal periods. The table shows the case with logarithmic utility and when agents receive zero or a fraction of income in money, a , promptly available for consumption. I consider 0 or a large fraction, 60 percent (the same value used by Alvarez, Atkeson, and Edmond 2009; and Khan and Thomas 2010, interpreted as the fraction of labor income in total income).

I. The Model

Agents manage money holdings by solving a Baumol (1952) and Tobin (1956) problem of money as inventory. They have to use money to buy goods, only bonds receive interest, and there is a cost to transfer the proceeds from bond sales to the goods market. I use the term agents instead of households or consumers because more than 62 percent of M1 in the United States is held by firms (Bover and Watson 2005). The money holdings of consumers and firms in the economy are represented by the money holdings of agents in the model. The model has elements of Jovanovic (1982), Romer (1986), and Grossman (1987).

There is a continuum of agents with measure one. Each agent has a brokerage account and a bank account, as in Alvarez, Atkeson, and Edmond (2009). The brokerage account is used to hold bonds, and the bank account is used to hold money for goods purchases. Time is continuous, $t \geq 0$. Let M_0 denote money in the bank

account at time zero, and B_0 denote bonds in the brokerage account at time zero. Index agents by $s = (M_0, B_0)$.

The agents pay a cost Γ in goods to transfer resources between the brokerage account and the bank account. Γ represents a fixed cost of portfolio adjustment. Let $T_j(s)$, $j = 1, 2, \dots$, denote the times of the transfers of agent s . Let $P(t)$ denote the price level. At $T_j(s)$, agent s pays $P(T_j(s))\Gamma$ to make a transfer between the brokerage account and the bank account. The agents choose the times of the transfers.

The consumption good is produced by firms. The firms produce Y goods with one unit of labor. Let the transfer cost be given by $\Gamma = \gamma Y$, linear in income. With this, the budget constraint of the agents and the demand for money will be linear in income. The income elasticity of the demand for money will be equal to one, which matches the evidence as stated in Lucas (2000) and others.

The agent is a composition of a shopper, a trader, and a worker, as in Lucas (1990). The shopper uses money in the bank account to buy goods, the trader manages the brokerage account, and the worker supplies one unit of labor to the firms. The firms keep a fraction a of the sales proceeds in money, transfer the remaining fraction $1 - a$ to their brokerage accounts, and convert this portion into bonds.

The firms pay $aP(t)Y$ in money and $(1 - a)P(t)Y$ in bonds to the worker for the unit of labor supplied. The firms make the payments in money with the money kept in the firm and the bond payments with a transfer from the brokerage account of the firm to the brokerage account of the agent. With the payments of the firm, the bank account of the agent is credited by $aP(t)Y$ and the brokerage account is credited by $(1 - a)P(t)Y$. These credits can be used at the same date for the purchases of goods and bonds.

Money holdings at time t of agent s are denoted by $M(t, s)$. Money holdings just after a transfer are denoted by $M^+(T_j(s), s)$, and they are equal to $\lim_{t \rightarrow T_j, t > T_j} M(t, s)$. Analogously, $M^-(T_j(s), s) = \lim_{t \rightarrow T_j, t < T_j} M(t, s)$ denotes money just before a transfer. The net transfer from the brokerage account to the bank account is given by $M^+ - M^-$. If $M^+ < M^-$, the agent makes a negative net transfer, a transfer from the bank account to the brokerage account, immediately converted into bonds. Money holdings in the brokerage account are zero, as bonds receive interest, and it is not possible to buy goods directly with money in the brokerage account. All money holdings are in the bank account. To have M^+ just after a transfer at $T_j(s)$, agent s needs to transfer $M^+ - M^- + P(T_j(s))\Gamma$ to the bank account. $P(T_j(s))\Gamma$ is used to buy goods to pay the transfer cost.¹

The agents choose consumption $c(t, s)$, money in the bank account $M(t, s)$, and the transfer times $T_j(s)$, $j = 1, 2, \dots$. They make this decision at time zero given the paths of the interest rate and of the price level. Let the price of a bond at time zero be given by $Q(t)$, with $Q(0) = 1$. The nominal interest rate is $r(t) \equiv -d \log Q(t)/dt$. The maximization problem of agent s is then given by

$$(1) \quad \max_{c, T_j, M} \sum_{j=0}^{\infty} \int_{T_j(s)}^{T_{j+1}(s)} e^{-\rho t} u(c(t, s)) dt$$

¹ There are many other ways of defining the flows of money and bonds in the model. For example, the agent at T_j could sell $M^+ - M^- + P\Gamma$ bonds and pay $P\Gamma$ to the bank that holds the brokerage account. The bank would then buy goods from firms and authorize the transfer of $M^+ - M^-$ to the bank account of the agent. This and other similar changes make no difference to the conclusions.

subject to

$$(2) \quad \sum_{j=1}^{\infty} Q(T_j(s)) [M^+(T_j(s), s) + P(T_j(s)) \gamma Y] \\ \leq \sum_{j=1}^{\infty} Q(T_j(s)) M^-(T_j(s), s) + W_0(s),$$

$$(3) \quad \dot{M}(t, s) = -P(t) c(t, s) + aP(t) Y, \quad t \geq 0, \quad t \neq T_1(s), T_2(s), \dots,$$

$M(t, s) \geq 0$, $T_{j+1}(s) \geq T_j(s)$, given $M_0 \geq 0$, where $\rho > 0$ is the intertemporal rate of discount, and $W_0(s) \equiv B_0 + \int_0^{\infty} Q(t)(1-a)P(t)Y dt$. To simplify the exposition, $T_0(s) \equiv 0$, but there is not a transfer at $t = 0$, unless $T_1(s) = 0$. At $t = T_1(s)$, $T_2(s), \dots$, constraint (3) is replaced by $\dot{M}(T_j(s), s)^+ = -P(T_j(s)) c^+(T_j(s), s) + aP(T_j(s)) Y$, where $\dot{M}(T_j(s), s)^+$ is the right derivative of $M(t, s)$ with respect to time at $t = T_j(s)$, and $c^+(T_j(s), s)$ is consumption just after the transfer.

The utility function is $u(c(t, s)) = c(t, s)^{1-1/\eta}/(1-1/\eta)$, for $\eta \neq 1$, $\eta > 0$; and $u(c(t, s)) = \log c(t, s)$, for $\eta = 1$. Preferences are a function of goods only. The transfer cost does not enter the utility function. η is the elasticity of intertemporal substitution.

The constraint (2) states that the present value of money transfers and transfer fees is equal to the present value of deposits in the brokerage account, including initial bond holdings. Constraint (3) states that money holdings decrease with goods purchases and increase with money receipts. This constraint shows the transactions role of money—agents need money to buy goods. As bonds receive interest and money does not, the agents transfer the exact amount of money to consume until the next transfer. That is, the agents adjust $M^+(T_j)$, T_j , and T_{j+1} to obtain $M^-(T_{j+1}) = 0$, $j \geq 1$. We can still have $M^-(T_1) > 0$ as M_0 is given rather than being a choice. Using (3), as $M^-(T_{j+1}) = 0$ for $j \geq 1$, money just after the transfer at T_j is

$$(4) \quad M^+(T_j(s), s) = \int_{T_j}^{T_{j+1}} P(t) c(t, s) dt - \int_{T_j}^{T_{j+1}} aP(t) Y dt, \quad j = 1, 2, \dots$$

The government collects seigniorage and redistributes it to agents as initial bonds. The government budget constraint is $B_0^G = \int_0^{\infty} Q(t)P(t) (\dot{M}(t)/P(t)) dt$, where B_0^G is the aggregate quantity of bonds and $M(t)$ is the aggregate money supply. The government controls the aggregate money supply at each time t .

The market clearing conditions for money and bonds are $M(t) = \int M(t, s) dF(s)$ and $B_0^G = \int B_0(s) dF(s)$, where F is a given distribution of s . The market clearing condition for goods takes into account the goods used to pay the transfer cost. Let $A(t, \delta) \equiv \{s : T_j(s) \in [t, t + \delta]\}$ represent the set of agents that make a transfer during $[t, t + \delta]$. The number of goods used, on average, during $[t, t + \delta]$ to pay the transfer cost is then given by $\int_{A(t, \delta)} (1/\delta) \Gamma dF(s)$. Taking the limit to obtain the number of goods used at time t yields that the market clearing condition for goods is given by $\int c(t, s) dF(s) + \lim_{\delta \rightarrow 0} \int_{A(t, \delta)} (1/\delta) \Gamma dF(s) = Y$.

An equilibrium is defined as prices $P(t)$, $Q(t)$, allocations $c(t, s)$, $M(t, s)$, transfer times $T_j(s)$, $j = 1, 2, \dots$, and a distribution of agents F , such that $c(t, s)$, $M(t, s)$, and

$T_j(s)$ solve the maximization problem (1)–(3) given $P(t)$ and $Q(t)$ for all $t \geq 0$ and s in the support of F ; the government budget constraint holds; and the market clearing conditions for money, bonds, and goods hold.

The advantage of this version of the Baumol-Tobin model is being a standard cash-in-advance model with the additional decision on the time to trade bonds for money. The model has intertemporal discounting, infinitely-lived agents, and optimization with consumption smoothing. These assumptions are common in cash-in-advance models but have not been considered simultaneously in a Baumol-Tobin model. (Jovanovic 1982, for example, assumes constant consumption; Romer 1986 assumes overlapping generations and zero intertemporal discounting. Other related models are in Heathcote 1998, Chiu 2007, and Rodriguez Mendizabal 2006.) By comparing the cases with fixed and optimal periods, we can use the model to evaluate how much fixed periods change estimations of the welfare cost of inflation.

II. The Demand for Money

Focus on an equilibrium in the steady state, an equilibrium such that the nominal interest rate is constant at r and inflation is constant at π . The transfer cost implies that it is optimal to rebalance the portfolio of bonds and money infrequently. Therefore, I look for an equilibrium in which agents follow (S, s) policies on consumption and money. Moreover, I look for an equilibrium in which all agents have the same consumption pattern within holding periods and choose the same interval between transfers N .

The demand for money depends on N and on the consumption pattern followed by the agents. The consumption pattern, in turn, depends on the distribution of agents as the aggregation of individual consumptions must satisfy the market clearing condition for goods. Therefore, we have to obtain the distribution of agents, the consumption pattern, and N to characterize the equilibrium and, especially, the demand for money.

Let $n \in [0, N)$ denote the position of an agent in a holding period in a steady-state equilibrium. Agent n makes transfers at $T_1(n) = n$, $T_2(n) = n + N$, and so on. Consider the distribution of agents along $[0, N)$ compatible with the steady state (with the properties of the steady state, we can then obtain the values of M_0 and B_0 of each agent n , such that the economy starts in the steady state). As the market clearing condition requires constant aggregate consumption and the agents follow the same consumption pattern, the number of agents that make a transfer at each time in the steady state must be constant. Therefore, the distribution of agents along $[0, N)$ compatible with the steady state is uniform, with density $1/N$. It is possible to have other distributions of agents if agents have different consumption patterns within holding periods. To study the steady state, however, it is natural to have agents with the same consumption pattern and consequently a uniform distribution.²

Consider now the pattern of consumption of each agent. The first order conditions of the maximization problem of the agent with respect to consumption imply $c(t, n) = e^{-\rho t} / [P(t) \lambda(n) Q(T_j)]^\eta$, $t \in (T_j, T_{j+1})$, $j = 1, 2, \dots$, where $\lambda(n)$ is the

²A proof that the uniform distribution is the only distribution of agents compatible with a steady state in which agents have the same consumption pattern is in Grossman (1989, Appendix B). Grossman (1989, 1987) studies the transitional effects of changes in monetary policy in a model with fixed N .

Lagrange multiplier of (2). Write consumption within holding periods as $c(t, n) = c_0 e^{\eta(r-\pi-\rho)t} e^{-\eta r(t-T_j)}$, taking the largest j , such that $t \in [T_j(n), T_{j+1}(n))$, where c_0 , common for all agents, denotes consumption at the beginning of a holding period. As shown in the Appendix, aggregating $c(t, n)$ at time t implies $C(t) = c_0 e^{(r-\rho-\pi)\eta t} (1 - e^{-\eta r N}) / (\eta r N)$.

Therefore, aggregate consumption grows at the rate $(r - \rho - \pi)\eta$, for arbitrary r and π . If r increases, the growth rate of aggregate consumption increases, as in an economy with a representative agent. In equilibrium, $r = \rho + \pi$, as a constant Y requires aggregate consumption to be constant in equilibrium. The real interest rate, defined as $r - \pi$, is then constant and equal to ρ .

As $r = \rho + \pi$ in the steady state, individual consumption follows the pattern $c(t, n) = c_0 e^{-\eta r(t-T_j)}$, decreasing within holding periods. The agents start a holding period with consumption c_0 and consume $c_0 e^{-\eta r N}$ just before a new transfer. However, aggregate consumption is constant. A constant aggregate consumption coexists with decreasing individual consumption, although individual consumption decreases faster when the nominal interest rate increases. The same happens with money. Each agent has a decreasing money-income ratio equal to zero at the end of holding periods, but the aggregate money-income ratio is constant over time.³

Increasing r or η increases consumption in the beginning of holding periods. Eventually, it makes agents try to increase consumption by setting $c(t, n) < aY$ in the end of holding periods. As $M(t, n) = \int_t^{T_{j+1}} P(\tau)[c(\tau, n) - aY]d\tau$, $t \in (T_j, T_{j+1})$, this would imply $M(t, n) < 0$. The constraint $M(t, n) \geq 0$ rules out this possibility. However, instead of imposing this constraint, it is simpler to solve the model by assuming that $M(t, n) \geq 0$ holds, substituting the expression of $M^+(T_j(n), n)$ in (2), and then checking if $M(t, n) \geq 0$ holds. To guarantee that $M(t, n) \geq 0$, I check if $c(t, n) \geq aY$. An interesting property of the model is that $c(t, n) > aY$ always holds for the relevant range of η and r . That is, for η between 0 and 5 and r from 0 to 16 percent per year (I discuss the data in more detail later). This property facilitates the characterization of the demand for money and of the welfare cost.

The value of c_0 is obtained with the market clearing condition for goods. In the steady state, it implies $(1/N) \int_0^N c_0 e^{-\eta r x} dx + (1/N)\gamma Y = Y$. Therefore, $c_0(r, N) = (1 - \frac{\gamma}{N})(\frac{1 - e^{-\eta r N}}{\eta r N})^{-1} Y$. The consumption-income ratio, $\hat{c}(t, n) = c(t, n)/Y$, is then $\hat{c}(t, n) = \hat{c}_0(r, N) e^{-\eta r(t-T_j)}$, independent of Y , where $\hat{c}_0(r, N) = c_0(r, N)/Y$ is the consumption-income ratio at the beginning of a holding period.

We obtain N with the first order conditions for T_j and for consumption, as described in the Appendix. The optimal interval between transfers N is the positive root of

$$(5) \quad \hat{c}_0(r, N) rN \left[\frac{1 - e^{-r(\eta-1)N}}{r(\eta-1)N} - \frac{1 - e^{-[\rho+r(\eta-1)]N}}{[\rho+r(\eta-1)]N} \right] \\ = \rho\gamma + arN \left[\frac{e^{rN} - 1}{rN} - \frac{e^{(r-\rho)N} - 1}{(r-\rho)N} \right], \text{ for } \eta \neq 1, \text{ and}$$

³Caplin and Leahy (2010) discuss other models with (S, s) policies with differences between individual and aggregate behaviors.

$$(6) \quad \hat{c}_0(r, N) rN \left[1 - \frac{1 - e^{-\rho N}}{\rho N} \right] \\ = \rho\gamma + arN \left[\frac{e^{rN} - 1}{rN} - \frac{e^{(r-\rho)N} - 1}{(r - \rho)N} \right], \text{ for } \eta = 1,$$

$$\text{where } \hat{c}_0(r, N) = \left(1 - \frac{\gamma}{N} \right) \left(\frac{1 - e^{-\eta r N}}{\eta r N} \right)^{-1}.$$

To analyze these equations, write (5) as $A = B$, where $B \equiv \rho\gamma$, and A is defined with the remaining terms (the reasoning is the same with equation 6). A is the marginal loss of increasing N and B is the marginal benefit of increasing N , both in real terms. B is the marginal benefit because the agent postpones the payment of the transfer cost when N increases. This term appears in the first order conditions as $P(T_j)Q(T_j)[r(T_j) - \pi(T_j)]\gamma$. So, the benefit of postponing the payment depends on the difference between the nominal interest rate and inflation at T_j . In the steady state, $r(T_j) - \pi(T_j) = \rho$.

The term A increases when N increases, as the money transferred at T_j loses value for inflation from T_j to T_{j+1} . When N is large, the losses are large, as any remaining balance from the transfer at T_j values little at a time close to T_{j+1} . In this case, $A > B$. When N is small, the losses are small, which implies $A < B$. The optimal N equalizes marginal benefits with marginal losses, setting $A = B$.

An increase in η or in a makes N increase. The intuition for η is that agents can decrease the losses A by making the consumption profile steeper during holding periods. The agents then lose less for inflation because they leave little of the money transferred at T_j for the end of holding periods. In the same way, a decreases the losses for inflation as an increase in a implies that agents receive money that can be promptly used in the goods markets. Therefore, an increase in η or in a with constant N makes $A < B$. To re-establish the equilibrium, N increases.

Turn now to r and γ , the classical factors that influence the holding period. If r increases, the losses A during (T_j, T_{j+1}) increase. To re-establish the equality $A = B$, the agents decrease A by decreasing the holding period N . If γ increases, the benefit of postponing the payment of the transfer cost increases, making $A < B$. It is then optimal to increase N . Therefore, N decreases with the nominal interest rate and increases with the transfer cost, $\partial N / \partial r < 0$ and $\partial N / \partial \gamma > 0$. (The proofs of the effects of the parameters on N are in Silva 2011.)

We can use the expressions (5) and (6) if $\hat{c}(t, n) \geq a$. As consumption decreases within N , it is sufficient to check whether $\hat{c}_0 e^{-rnN} > a$. With $r = 16$ percent per year, $\eta = 5$, and $a = 0.6$, for example, we have $\hat{c}_0 e^{-rnN} = 0.71 > 0.6$, and so we can use the expressions above.

A second-order Taylor expansion around zero of the exponential terms in (6), with $a = 0$, implies $N \approx \sqrt{2\gamma/r}$, the square-root formula for the interval between transfers. The approximation does not hold with $a > 0$. With $a = 0.6$, $N \approx \sqrt{2\gamma/0.4r}$, 60 percent higher than the square root approximation.

The aggregate demand for money is $(1/N) \int M(t, n) dn$, and so the money-income ratio is $m(r) = 1/(P(t)Y) (1/N) \int M(t, n) dn$. In the steady state, as $m(r)$ is constant,

the rate of inflation is equal to the growth rate of the stock of money. Therefore, $m(r) = 1/(P_0Y) (1/N) \int M_0(n) dn$, using initial money holdings across agents. The values of $M_0(n)$ are obtained by finding the quantity of money necessary for agent n to consume at the steady-state rate during $[0, n]$. The calculations and the values of $M_0(n)$ are shown in the Appendix. The values of $M_0(n)$ imply that $M^-(n) = 0$ in the steady state.⁴

The money-income ratio is then

$$(7) \quad m(r) = \frac{\hat{c}_0(r, N) e^{-\eta r N(r)}}{\rho + r(\eta - 1)} \left[\frac{e^{\eta r N(r)} - 1}{\eta r N(r)} - \frac{e^{(r-\rho)N(r)} - 1}{(r - \rho)N(r)} \right] \\ - a \frac{1}{r - \rho} \left[\frac{e^{(r-\rho)N(r)} - 1}{(r - \rho)N(r)} - 1 \right],$$

where $N(r)$ is given by (5) and (6), and $\hat{c}_0(r, N) = (1 - \gamma/N)[(1 - e^{-\eta r N})/(\eta r N)]^{-1}$. The calculations to obtain $m(r)$ are in the Appendix. $m(r)$ does not depend on Y as $M(t)/P(t)$ is linear in Y (or, as the income-elasticity of $M(t)/P(t)$ is equal to one). The money-income ratio depends on the interest rate, preference parameters, and γ . I write $m(r)$ to emphasize the role of r .

Figures 1 and 2 show data on the money-income ratio along with $m(r)$ given by equation (7), with $a = 0$, for fixed and optimal N .⁵ The interest-elasticity of $m(r)$ is close to $-1/2$ for $0.2 \leq \eta \leq 10$, using (7) to calculate the elasticity numerically.⁶ With fixed N , $m(r)$ is close to a straight line for $\eta \leq 10$. The model better fits the data with optimal N .

A. Calibration, Data, and the Behavior of the Demand for Money

There are four parameters to calibrate: a , ρ , η , and γ . I use M1 for the monetary aggregate and the short-term commercial paper rate for the interest rate. The data-set is similar to the one used in Lucas (2000). M1 and the commercial paper rate were also used by Dotsey and Ireland (1996), Lagos and Wright (2005), Craig and Rocheteau (2008), among others. Data are annual from 1900 to 1997 (the last year in which the commercial paper rate data are available from the same source). There is a discussion about whether M1 is the best aggregate to study money demand (Teles and Zhou 2005).⁷ The main reason that I use M1 and the commercial paper

⁴The last step to characterize the equilibrium is to find the values of $B_0(n)$. See Silva (2011) for this last step. An agent with $(M_0(n), B_0(n))$ then chooses $T_1 = n$, $T_{j+1} - T_j = N$, and consumes at the steady-state rate. An economy with $(M_0(n), B_0(n))$ distributed to agents $n \in [0, N]$ is in the steady state equilibrium for all $t \geq 0$.

⁵As $m \equiv M/(PY)$, having M expressed in dollars and GDP expressed in dollars per year implies that the units of m are in years. The interpretation is that agents carry in money the equivalent of a fraction m of the production of one year. For example, $m = 0.26$, the average money-income ratio during the twentieth century for the United States, means that agents carry in money the equivalent of the production of about one quarter.

⁶The smallest interest-elasticity in modulus in this range of η and r is -0.49 for $\eta = 0.2$ and $r = 0.1$, and the largest is -0.52 for $\eta = 10$ and $r = 16$. In either case, close to $-1/2$. The modulus of the elasticity increases with r and η , but little. Only for large η and r the elasticity is significantly different from $-1/2$. For example, the elasticity is -0.70 for $r = 16$ and $\eta = 50$.

⁷Another aspect is that some components of M1 pay a small but positive interest rate, which could affect the welfare cost. See Lucas (2000, 270) and Cysne (2003).

rate is to facilitate comparison of the estimates obtained here with the estimates obtained in the literature.

I set $a = 0$ and $a = 0.6$. With $a = 0$, the economy behaves as a standard cash-in-advance model: the agents work, the proceeds from work are separated from the agents, and the agents use their income to buy goods only after reallocating bonds and money. If the length of the time period is one quarter, agents can consume from their sales only in one quarter. With $a > 0$, the agents can use some of their income to buy goods before reallocating bonds and money. For $a = 0.6$, I follow the calibration in Alvarez, Atkeson, and Edmond (2009) and Khan and Thomas (2010). Alvarez, Atkeson, and Edmond (2009) and Khan and Thomas (2010) interpret a as the fraction of income received as wages.

I set the intertemporal discount ρ to 3 percent per year. With this, going from 10 percent inflation to zero requires decreasing r from 13 percent to 3 percent. I vary the elasticity of intertemporal substitution η from 0.1 to values as high as 100 (when $a = 0$). The estimates of η are usually below 10 (Mehra and Prescott 1985, Bansal and Yaron 2004, for example; Hansen and Singleton 1982 and Attanasio and Weber 1989 have larger estimates of η). I use the values of η above 10 to show that setting fixed periods only matches the data when η is high. When N is optimal, η has little effect on the money-income ratio.

The parameter that changes the demand for money the most is the transfer cost γ . A higher γ shifts the money-income ratio upward in the $r \times m$ diagram. I set γ so that $m(r)$ matches the historical average on the money-income ratio and interest rates (obtained with their geometric means). Lucas (2000) follows the same method; and, similarly, Alvarez, Atkeson, and Edmond (2009) and Khan and Thomas (2010) use the historical average of M2 velocity. The mean interest rate during 1900–1997 is 3.6 percent per year, and the mean money-income ratio is 0.26. The data, therefore, say that agents hold, on average, about 90 days of income in money. These high historical values imply $\gamma = 1.79$ for $a = 0$ and $\eta = 1$.⁸

Figure 1 shows the data and $m(r)$ for $a = 0$ and $\eta = 0.1, 1, 10$, and 50. It also shows the interval between transfers N for each η and r . The curves overlap, as η has little effect on N and on the money-income ratio when N is optimal. I include $\eta = 50$ to compare the results with those with a fixed N . Although the model simplifies many features of an actual economy (for example, it abstracts from precautionary motives for holding money, and aggregates households and firms in one agent), the fit of the money-income ratio is surprisingly good.

The model is able to explain the high money-income ratio of the 1940s (the points in the upper left corner; in 1946, $r = 0.81$ and $m = 0.48$) and the low money-income ratio of the 1980s (the points in the bottom right corner; in 1981, $r = 14.8$ and $m = 0.14$). It explains the decrease in the money-income ratio from 1945 to 1981; the money-income ratio decreased because the interest rate increased. The model does not explain the high m in the beginning of the century, with interest rates between 3 and 6 percent (just above the curve), and the low m in the 1990s (just

⁸ $\gamma = 1.785, 1.791, 1.845, \text{ and } 2.205$ for $a = 0$, and $\eta = 0.1, 1, 10, \text{ and } 50$.

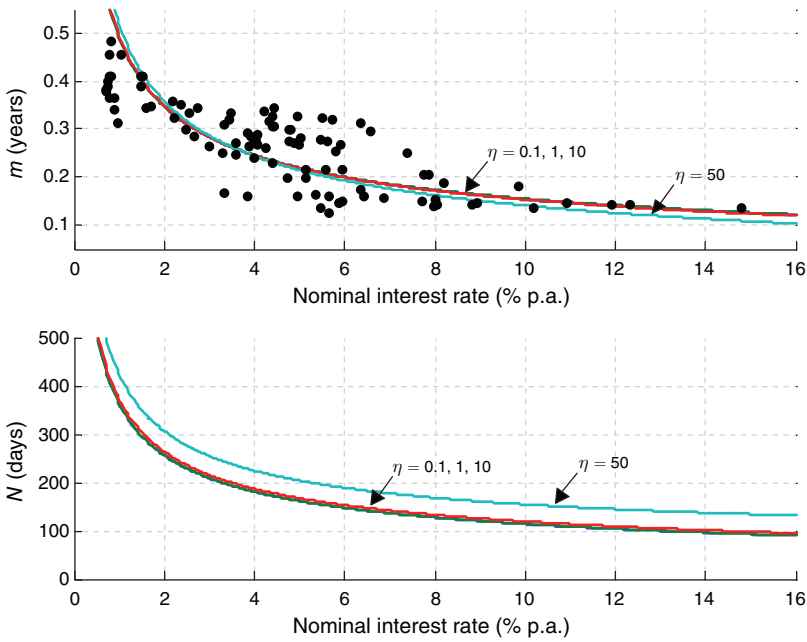


FIGURE 1. OPTIMAL N

Notes: Money-income ratio and interval between transfers when N is endogenous. US annual data, 1900–1997, M1 and commercial paper rate. $a = 0$. η : elasticity of intertemporal substitution.

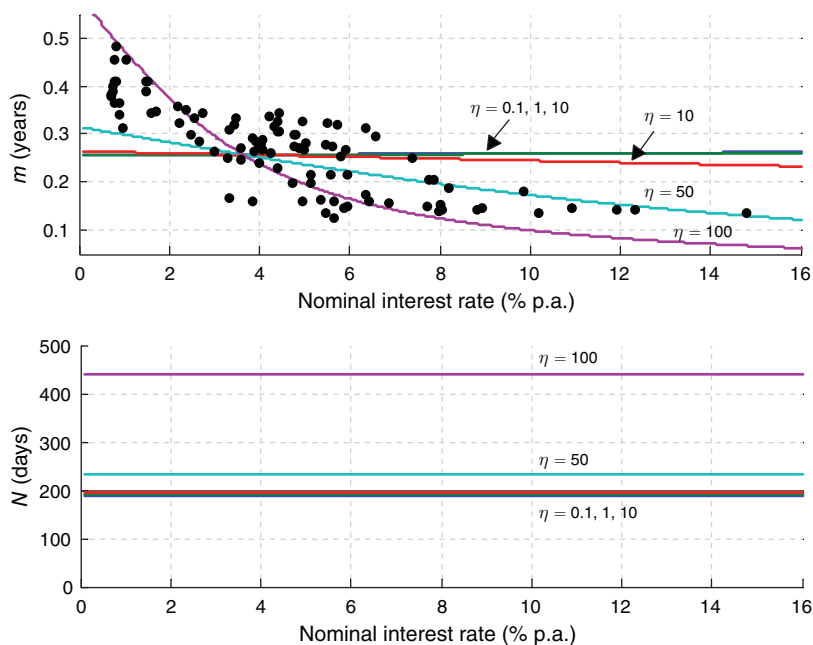
below the curve), with approximately the same interest rates. But the model does describe the general pattern of the data.

The curves in Figure 1 overlap with the Baumol-Tobin money-income ratio $\sqrt{\gamma/(2r)}$. Lucas (2000) argues that the Baumol-Tobin money demand has a good fit to the US data.

The money-income ratio decreases as the interest rate increases because the interval between transfers is endogenous. $m(r)$ is approximately constant in r if the interval between transfers is fixed (Romer 1986 finds similar results with logarithmic utility). What makes m decrease is the ability of agents to change N . That is, the ability to change the frequency of portfolio rebalancing. If N is fixed, m is constant in r .

With fixed N , m could decrease because agents vary consumption within N . This effect is relevant only for high elasticities of substitution. As shown in Figure 2, the money-income ratio with fixed N approximates the data only if η is greater than 50 (N is fixed at the optimal value of N when r is equal to the historical mean of the data, $m(r)$ with N fixed and N optimal coincide at the mean r). Letting agents change N is important to explain the empirical fact that the money-income ratio decreases when the interest rate increases. Some early evidence about this behavior is in Meltzer (1963) and Lucas (1988).

The values of N implied by the model are large. With $r = 4$ percent per year and $\eta = 1$, the interval between transfers is 181 days, which implies about two transfers per year. These transfers are from high-yielding assets to cash. They are not ATM

FIGURE 2. FIXED N

Notes: Money-income ratio and interval between transfers when N is fixed. US annual data, 1900–1997, M1 and commercial paper rate. $a = 0$. η : elasticity of intertemporal substitution.

withdrawals. Transfers convert bonds into money, whereas ATM withdrawals convert deposits into cash but do not change the quantity of money. There are two reasons for the values of γ and N . First, the average money-income ratio is high. Second, households and firms hold a large quantity of money and make infrequent transfers.

First, the average money-income ratio, about one fourth of a year, is high in the data. With per capita income of \$35,000 in 2000, $m = 0.26$ implies that each person in the United States holds about \$9,000. Therefore, $\gamma = 1.79$ and $N = 181$ days so that $m(r)$ matches the high historical money-income ratio. If we use more recent data, $m = 0.1$ in 2000, then γ decreases to 0.27 and N to 70 days. I divide r and ρ by 365 to obtain N in days. The unit of Y is then goods per day. As γY is in goods per transfer, the unit of γ is days per transfer. So $\gamma = 0.27$ means about one fourth of a day. The cost per transfer is then $0.27 \times 35,000/365 = \26 . (Alternatively, dividing r and ρ by 260 working days implies $\gamma = 0.19$ working days per transfer, $N = 52$ working days and, again, $0.19 \times 35,000/260 = \26 .) However, we cannot use only the more recent data to estimate the demand for money and the welfare cost of inflation. Moreover, to compare the results on the demand for money and the welfare cost, we need to use the same dataset as in previous studies.

Second, households and firms in fact trade infrequently and hold large quantities of money. This behavior requires a large γ . Vissing-Jorgensen (2002), for example, shows that a large fraction of households trade assets with higher yields less than once a year. Christiano, Eichenbaum, and Evans (1996) show that households take a long time to adjust their portfolios. Alvarez, Atkeson, and Edmond (2009) state

that only about half of the households that held stocks bought or sold stocks in 1998–2001 (I later calibrate the model with 60 percent of income directly deposited into the bank account, as in Alvarez, Atkeson, and Edmond). Bates, Kahle, and Stulz (2009) find that the cash-assets ratio has been increasing strongly among firms (as stated above, US firms in 2000 held 62 percent of M1). These pieces of evidence are puzzling given the financial innovations of the last decades. It is beyond the objectives of this paper to study the individual behavior of households or firms. For studies on household money holdings and firm cash holdings, see Alvarez and Lippi (2009) and Bates, Kahle, and Stulz (2009).

The large value of N is common in the literature (Edmond and Weill 2008). Alvarez, Atkeson, and Edmond (2009) set the transfer interval from 1.5 to 3 years (they calibrate N directly because N is fixed in their model). Guerron-Quintana (2009), in a model with portfolio adjustments, estimates that agents adjust their portfolios every 3 to 6 quarters. The calibration in Khan and Thomas (2010) implies average transfer intervals from 1.2 to 2.4 years. Here, the calibration implies N of about six months, a smaller N mainly because I use M1 instead of M2 and because so far I studied the case $a = 0$. I close this section by studying the case with $a = 0.6$, the same value used in Alvarez, Atkeson, and Edmond (2009), and Khan and Thomas (2010).

When $a = 0.6$, the agents receive 60 percent of their income in money. The remainder is deposited in the brokerage account. As the agents receive a fraction of their income in money without cost, they have less incentives to trade bonds for money frequently. This effect increases N and, keeping the same γ as with $a = 0$, decreases the financial cost. The money demand shifts downward. In order to reestablish the match between m and the data, the parameter γ has to increase. The increase in γ is such that γ/N is approximately constant with $a = 0$ or 0.6.

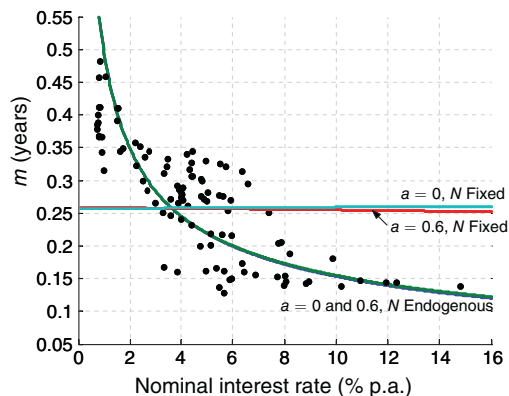
The money-income ratio changes little with a equal to zero or large, such as 60 percent, when N is endogenous. This can be seen in Figure 3. The figure shows $m(r)$ with $a = 0$ and 0.6, and with $\eta = 1$ in the left panel and $\eta = 5$ in the right panel. When N is optimal, the two curves for m overlap with $\eta = 1$. With $\eta = 5$, we can distinguish the two curves with different values of a , but the difference is small. When N is fixed, $a = 0.6$ makes m decreasing in r , but the difference between $a = 0$ and 0.6 is clear only with $\eta = 5$. When N is optimal, a has almost no effect.

The reason for the small impact of a with N endogenous is that agents best manage their money holdings by varying the transfer intervals. A higher a is compensated by a lower frequency of exchanges of bonds for money. That is, a higher N . This shifts m downward without changing its elasticity. Once γ is recalibrated, the money-income ratio returns to its previous position; γ increases from 1.79 to 4.66 for $\eta = 1$ when a increases to 0.6. With $r = 4$ percent, N increases from 181 days to 467 days. γ/N is approximately constant when $a = 0$ or 0.6, $\gamma/N = 1$ percent. In the next section, I relate γ/N to the welfare cost of inflation.

III. The Welfare Cost of Inflation

Agents use part of their resources for financial services when $r > 0$. The optimal monetary policy is to set $r = 0$. The Friedman rule applies (Friedman 1969).

Panel A. Money-income ratio, $a = 0$ and $a = 0.6$; $\eta = 1$ (log utility)



Panel B. Money-income ratio, $a = 0$ and $a = 0.6$; $\eta = 5$

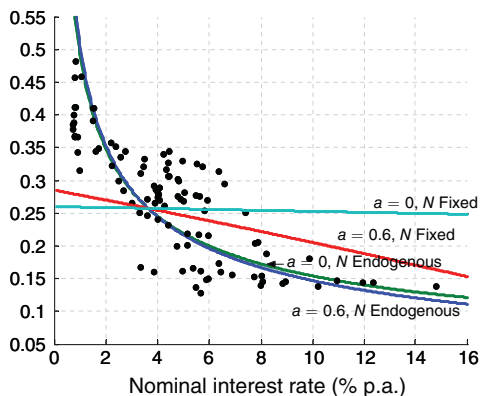


FIGURE 3. MONEY-INCOME RATIO FOR $a = 0, 0.6, \eta = 1, 5$, AND N FIXED OR ENDOGENOUS

Notes: Left: $\eta = 1$ (log utility). Right: $\eta = 5$. A fraction $a > 0$ in money affects the economy only with η high and N fixed. $a > 0$ as high as 0.6 changes little the money-income ratio when N is endogenous.

The income compensation $w(r)$, so that agents are indifferent between r and \bar{r} , is defined as $U^T[r, (1 + w(r)) Y] = U^T(\bar{r}, Y)$, where $U^T(r, Y)$ is the aggregate utility from all agents with equal weight, $U^T(r, Y) = 1/\rho \int_0^N \frac{[c_0(r; Y) e^{-\eta t}]^{1-1/\eta}}{1-1/\eta} dt$. Therefore, the income compensation is

$$(8) \quad 1 + w(r) = \frac{c_0(\bar{r})}{c_0(r)} \left[\left(\frac{1 - e^{-r(\eta-1)N}}{r(\eta-1)N} \right)^{-1} \frac{1 - e^{-\bar{r}(\eta-1)\bar{N}}}{\bar{r}(\eta-1)\bar{N}} \right]^{\frac{1}{1-1/\eta}}, \text{ for } \eta \neq 1, \text{ and}$$

$$1 + w(r) = \frac{c_0(\bar{r})}{c_0(r)} \exp\left(\frac{rN}{2} - \frac{\bar{r}\bar{N}}{2}\right), \text{ for } \eta = 1.$$

Table 2 shows the welfare cost of 10 percent instead of 0 inflation ($r = 13$ and $\bar{r} = 3$). It shows the role of the fraction of income in money a , and the role of fixed or endogenous N . As mentioned earlier, $a = 0$ or 0.6 has a small effect on the money-income ratio when N is endogenous. In the same way, increasing a from 0 to 0.6 slightly increases, but changes little, the welfare cost when N is endogenous.

A fraction a of 60 percent could greatly change the welfare cost because agents gain access to a large part of their income earlier. Surprisingly, it has no effect. What matters is the fraction of income devoted to financial transfers—the product between the cost of one transfer as a fraction of income and the frequency of transfers, $\gamma \times 1/N$.

With $r = 4$ percent per year and log utility, $\gamma/N = 1$ percent (for $a = 0$ or 0.6), which means that agents devote 1 percent of their time to manage money holdings. Workers in the United States worked about 1,900 hours per year, on average, from

TABLE 2—WELFARE COST OF 10 PERCENT INFLATION INSTEAD OF 0 INFLATION
(in percentage of income)

η	N Endogenous		N Fixed	
	$a = 0.6$	$a = 0$	$a = 0.6$	$a = 0$
0.2	0.97	0.95	0.02	0.00
0.5	0.97	0.95	0.06	0.01
1	0.97	0.95	0.12	0.02
2	0.98	0.95	0.25	0.04
5	1.03	0.95	0.66*	0.09
Constant consumption	0.96	0.95	0.00	0.00
Lucas-Bailey	0.92			

Notes: Welfare cost: $w(r)$ from $r = 13$ percent p.a. to $r = 3$ percent p.a. η : elasticity of intertemporal substitution. Constant consumption: $\eta = 0.001$. Lucas-Bailey: area under the Baumol-Tobin money demand.

*: for 9.5 percent inflation ($c > aY$ binds for 10 percent inflation, $\eta = 5$, $a = 0.6$, and N Fixed).

1950 to 1997 (OECD data). When $r = 4$ percent, therefore, the model estimates about 22 minutes per week devoted to financial services.

γ/N approximates the welfare cost of a positive interest rate. The cost of 10 percent inflation instead of 0 inflation can then be approximated by the difference between $\gamma/N(r)$ when $r = 13$ and 3 percent, $\gamma/N_{r=13} - \gamma/N_{r=3}$. With log utility, this difference equals 0.93 percent. The welfare cost of 10 percent inflation using (8) for $a = 0$ and 0.6 is 0.95 and 0.97, respectively.

γ/N and the welfare cost can also be related by the method used by Bailey (1956), calculating the area under the demand for money above equilibrium money holdings. As stated in Lucas (2000), $\int_0^r m(x) dx - rm(r)$. The model generates a demand for money close to the Baumol-Tobin money demand $\sqrt{\gamma/(2r)}$. In this case, the method of Bailey yields $\sqrt{r\gamma/2}$, which is approximately γ/N when $N \approx \sqrt{2\gamma/r}$. Lucas (2000) uses the method of Bailey with a Baumol-Tobin money demand as a benchmark for the welfare cost of inflation. The Lucas-Bailey measure $\sqrt{r\gamma/2}$ is smaller, but close to $w(r)$, as shown in Table 2.

$w(r)$ decreases substantially when N is fixed as it can be seen in Table 2. With $a = 0$, the welfare cost is approximately zero. One way of understanding this result is that the demand for money is approximately constant with N fixed. As the area under the demand for money above equilibrium money holdings approximates the welfare cost, the welfare cost with N fixed is close to zero.

The parameter η increases $w(r)$ when N is fixed because the demand for money is more elastic. The elasticity of the demand for money increases with η because a higher η allows agents to vary consumption more easily within holding periods. With N fixed, agents can decrease m only by increasing the variation of consumption, which is easier to do when η increases (the effect on $w(r)$ is small when N is endogenous because the demand for money is already elastic in this case). The effect of a is stronger than the effect of η because $a = 0.6$ implies a demand for money more negatively sloped, but this effect is important only if η is high; a increases $w(r)$ from 0 to 0.1 percent with log utility and to 0.7 percent with $\eta = 5$. Even with $\eta = 5$, it is still lower than the welfare cost with N endogenous. The

relation of $w(r)$ with the demand for money is used to explain the effects of the parameters. With the exception of the Lucas-Bailey case, all values in Table 2 were obtained from (8).⁹

Why does an endogenous N make the welfare cost increase? In principle, an endogenous N had to imply a smaller welfare cost, as agents choose N to maximize utility. What happens in the model is similar to the effects of an increase in a distortionary tax, such as the income tax. With endogenous labor supply, it is optimal for each agent to decrease the labor supply if there is an increase in the income tax rate. In the new equilibrium, the decrease in labor reduces aggregate output (an effect not considered by the agents when solving their maximization problems) and reduces welfare by more than the gains from reducing labor supply.

Here, it is optimal for each agent to decrease N when r increases. The agents take into account that they have to pay the adjustment cost more often and willingly do so. However, they do not take into account that the equilibrium changes when r increases. The model also takes into account the higher variation of consumption within holding periods, but the more relevant effect is the increase in financial services. In particular, total resources diverted from consumption are equal to $(1/N)\gamma Y$. With endogenous N , the welfare cost increases because the model takes into account the increase in the use of financial services.

Another way of understanding this result is that each agent is better off by reducing N when r increases, taking the real values of money and bonds as given. But the new equilibrium implies a different price level P_0 and different real values of money and bonds. These equilibrium changes are not taken into account by the agents when they solve their individual maximization problems. Figure 4 shows this mechanism. The figure shows the utility of a particular agent, agent $n = 0$, after solving the maximization problem of the agent. The real values of $M_0(n)$ and $B_0(n)$ have to be such that $1/N(r)$ agents make transfers at each time. Therefore, each steady state with a nominal interest rate r implies a different price level P_0 and different real values of M_0 and B_0 for each agent.

In the first diagram of Figure 4, the real values of M_0 and B_0 are kept constant at their values compatible with the historical mean of the interest rate, $r = 3.6$ percent per year (it could be any value of r). As the interest rate increases, the utility decreases whether or not the agent is able to change N . Given the real values of M_0 and B_0 , utility decreases less with N endogenous, as the agent adapts to the higher inflation by decreasing N .¹⁰

However, the real values of M_0 and B_0 compatible with a steady state with $r = 3.6$ percent are not compatible with a steady-state equilibrium with $r > 3.6$ percent. With a higher r , the price level changes. So, the real values of M_0 and B_0 change. Their new real values satisfy the increase in the frequency of transfers of all

⁹We have to check whether $c(t) \geq aY$ to use the expressions in (8). The constraint binds only when N is fixed and $\eta = 5$ (the reason is that N , fixed at the value that matches the historical data, cannot decrease under 10 percent inflation; the table reports $w(r)$ for 9.5 percent inflation in this case). $c(t) \geq aY$ never binds in the cases considered for N endogenous.

¹⁰To obtain the figures, I also increased inflation, using $\pi = r - \rho$, in order to imply the same $c_0(T_j)$ at the beginning of each holding period.

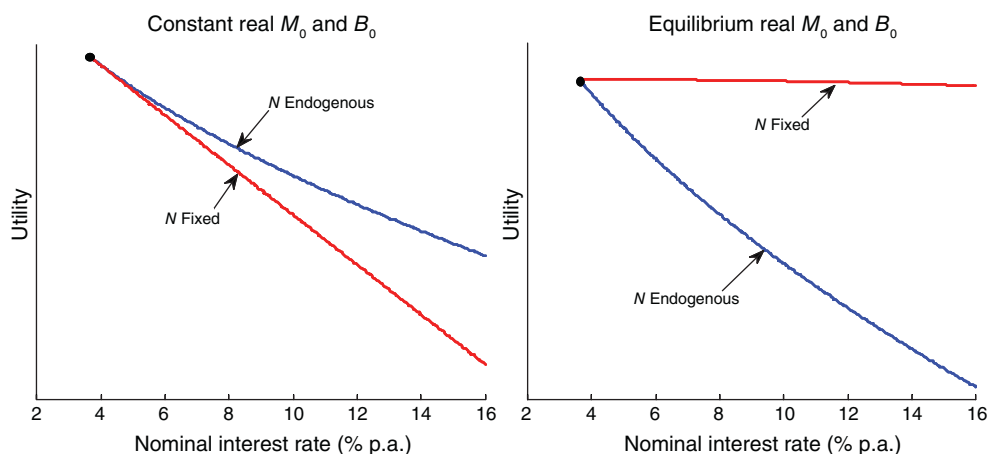


FIGURE 4. UTILITY OF A SINGLE AGENT

Notes: Left: real M_0 and B_0 constant at their equilibrium values under the mean nominal interest rate, $r = 3.6$ percent. Right: real M_0 and B_0 compatible with the steady-state equilibrium for each interest rate. The welfare cost with N endogenous is higher when we consider the general equilibrium effects.

agents and the new market clearing conditions. When we recalculate the utility with the equilibrium real values of M_0 and B_0 , utility decreases faster with N endogenous.

For both N fixed and N endogenous, in addition to the transfer cost, the variation of $c(t, n)$ within holding periods decreases welfare. The effect of the variation of consumption is small for both fixed and endogenous N for the levels of inflation considered here. The effect of the increase in financial services is more important. However, when r is high, the variation of consumption with N fixed can be so high that the welfare cost can be higher with N fixed (the variation of consumption is smaller with N endogenous because agents can adjust N to decrease the variation of consumption). This only happens with very high interest rates. The welfare cost with N fixed is equal to the welfare cost with N endogenous when $r = 261$ percent per year, and it is higher with N fixed for r beyond this point. With this nominal interest rate, $w(r) = 7.86$ percent of income.

I focus on moderate inflation in Table 2. I do not concentrate on the welfare cost of small positive interest rates instead of the Friedman rule or of very high inflation. Mulligan and Sala-i-Martin (2000) argue that the demand for money changes for low inflation because having or not having a bank account, that is the extensive margin, becomes more important than the rebalancing frequency. Ireland (2009) argues that a semi-log money demand better matches the data for low inflation. For high inflation, in addition to changes in the money demand, we would have to consider other factors, such as the loss of information caused by inflation (Harberger 1998). Studying moderate inflation emphasizes the effects of the frequency of financial transactions. For moderate inflation, $w(r)$ changes little with log-log, as we have here, or semi-log money demands (Lucas 2000). Moreover, rates of inflation between 0 and 10 percent per year are the most common rates of inflation in OECD countries.

TABLE 3—WELFARE COST OF 10 PERCENT INFLATION INSTEAD OF 0 INFLATION
(in percentage of income). THE EFFECT OF INTRODUCING CAPITAL AND LABOR

	N Endogenous	N Fixed	Difference
Standard labor decision	1.33	0.36	0.97
Indivisible labor decision	1.32	0.50	0.83
No capital or labor (first model)	0.95	0.02	0.93

Notes: Welfare cost: $w(r)$ from $r = 13$ percent p.a. to $r = 3$ percent p.a. N Fixed: optimal choice of N under $r = 3.64$ percent p.a., the geometric average of r over the period.

IV. The Model with Capital and Labor

I now introduce the decisions on capital and labor. This extension allows us to compare a cash-in-advance model with endogenous periods, as we have here, with a standard cash-in-advance model. Both with capital and endogenous labor supply. I show that the calculations of the welfare costs are robust. The welfare cost increases in parallel for fixed and endogenous periods. The welfare cost of 10 percent instead of 0 inflation is still 1 percentage point higher with endogenous periods.

The expressions for the interval between transfers and for the money-income ratio remain unchanged with capital and labor. As a result, the area under the demand for money does not change. As the welfare cost increases, this extension puts in evidence that the area under the demand for money is an approximation for the welfare cost, but it does not take into account the full welfare cost of inflation (a result also emphasized, for example, in Lagos and Wright 2005). With capital and labor, the welfare cost of inflation increases about 0.3 percentage points for N fixed and N endogenous. With N endogenous, it increases to 1.33 percent of income. Table 3 summarizes the effects of introducing capital and labor on the welfare cost of inflation.

I use a standard cash-in-advance model with capital and labor, the difference is the decision on the size of the holding periods. The model is similar to the models in Cooley and Hansen (1989) and Cooley (1995).

Production is given by $Y(t) = Y_0 K(t)^\theta H(t)^{1-\theta}$, where $K(t)$ and $H(t)$ are aggregate capital and hours of work at time t and $0 < \theta < 1$. Capital depreciates at the rate δ . Individual capital and hours of work are given by $k(t, s)$ and $h(t, s)$, where now $s = (M_0, B_0, k_0)$. Profit maximization implies that real wages $w(t)$ and the real interest rate on capital $r^k(t)$ are given by $w(t) = (1 - \theta) Y_0 [K(t)/H(t)]^\theta$ and $r^k(t) = \theta Y_0 [K(t)/H(t)]^{-(1-\theta)}$.

The government offers bonds that pay a nominal interest rate $r(t)$. To avoid the opportunity of arbitrage between government bonds and capital, we must have $r(t) - \pi(t) = r^k(t) - \delta$. If this condition is violated, agents would arbitrarily increase the quantity of bonds or capital in their portfolios.

Preferences now take into account consumption and hours of work. I consider the logarithmic preferences $u(c, h) = \log c + \alpha \log(1 - h)$, $\alpha > 0$, and the preferences for indivisible labor $u(c, h) = \log c - Bh$, $B > 0$. I focus the exposition on the logarithmic preferences, with standard labor decision. Both preferences are derived from $u(c, h) = [c(1 - h)^\alpha]^{1-1/\eta} / (1 - 1/\eta)$ with $\eta = 1$ and are compatible with a balanced growth path (King, Plosser, and Rebelo 1988). The preferences for

indivisible labor were considered by Cooley and Hansen (1989). They are obtained with $\eta = 1$ and the additional assumption that agents can only work zero or a certain positive number of hours (Hansen 1985).

The separability of consumption and hours worked obtained with $\eta = 1$ implies that, in the steady state, hours worked within holding periods are constant. Having constant hours worked simplifies the analysis and facilitates the comparison of the welfare cost of inflation with fixed or endogenous trading frequency. Having $\eta = 1$ also facilitates the comparison with other estimates for the welfare cost of inflation obtained in the literature.¹¹

Agents make their decisions at $t = 0$, given prices and their initial holdings of money M_0 , bonds B_0 , and capital k_0 . The maximization problem is

$$(9) \quad \max_{c, h, T_j, M} \sum_{j=0}^{\infty} \int_{T_j(s)}^{T_{j+1}(s)} e^{-\rho t} [\log c(t, s) + \alpha \log(1 - h(t, s))] dt,$$

subject to

$$(10) \quad \sum_{j=1}^{\infty} Q(T_j(s)) [M^+(T_j(s), s) + P(T_j) \gamma Y(t)] \leq \sum_{j=1}^{\infty} Q(T_j(s)) M^-(T_j) + W_0(s),$$

where $M^+(T_j(s), s) = \int_{T_j(s)}^{T_{j+1}(s)} P(t) c(t, s) dt$, $\dot{M}(t, s) = -P(t) c(t, s)$, and

$$(11) \quad W_0(s) = B_0 + P_0 k_0 + \int_0^{\infty} Q(t) P(t) w(t) h(t, s) dt.$$

With indivisible labor, the utility function changes to $\log c(t, s) - Bh(t, s)$. The problem is written for the case with $a = 0$, which simplifies the characterization of the equilibrium. The purchase of capital and the income from capital appear in the term $P_0 k_0$ in the present value budget constraint. At $T_j(s)$, agent s sells bonds and capital to start the holding period with $M^+(T_j(s), s)$ in money to be used in the goods market. During a holding period $[T_j, T_{j+1})$, real holdings of bonds and capital grow at the rates $r(t) - \pi(t)$ and $r^k(t) - \delta$. As $r(t) - \pi(t) = r^k(t) - \delta$, the agents are indifferent to the evolution of the two assets.¹²

¹¹ It is possible to obtain analytical formulas with $\eta \neq 1$. However, the variation of hours worked requires a more extensive analysis and deviates the focus from the comparison of the welfare cost of inflation with fixed or endogenous trading frequency. From the results of the previous sections, the welfare cost calculations with $\eta \neq 1$ change little. Especially when we take into account that the empirical evidence on η frequently points to values smaller than two.

¹² To obtain the present value budget constraint (10) use the constraint $M^+(T_j) + B^+(T_j) + P(T_j) k^+(T_j) + P(T_j) \Gamma = M^-(T_j) + B^-(T_j) + P(T_j) k^-(T_j)$, $j = 1, 2, \dots$, where B^+ , B^- , k^+ , and k^- are the quantities of bonds and capital just after and just before the transfer, and the fact that bonds and capital follow $\dot{B}(t) = r(t) B(t) + P(t) w(t) \times h(t)$ and $\dot{k}(t) = (r^k(t) - \delta) k$ during holding periods (as $r(t) - \pi(t) = r^k(t) - \delta$, it does not matter if labor income is invested in bonds, as the equation for \dot{B} implies). Substituting recursively and using the conditions $\lim_{j \rightarrow +\infty} Q(T_j) B^+(T_j) = 0$ and $\lim_{j \rightarrow +\infty} Q(T_j) P(T_j) k^+(T_j) = 0$ imply (10). In a different context, Ljungqvist and Sargent (2004, p. 482) also obtain a present value budget constraint with capital and labor.

The transfer cost Γ is given by $\gamma Y(t)$. If agents took $Y(t)$ in $\Gamma = \gamma Y(t)$ as their own income, they would decrease labor and capital discontinuously at time T_j to pay a smaller transfer cost. To rule out this possibility, each agent takes aggregate output $Y(t)$ as given, as the individual participation in aggregate output is small. As a result, the transfer cost $\Gamma = \gamma Y(t)$ is also taken as given.

The market clearing conditions for money and bonds are the same as before. The market clearing condition for goods now includes aggregate investment, $\dot{K}(t) + \delta K(t)$. Given a distribution of agents F , the market clearing condition for capital and hours of work are $K(t) = \int k(t,s) dF(s)$ and $H(t) = \int h(t,s) dF(s)$.

A. Solving the Model with Capital and Labor

As in the case without capital and labor, focus on the equilibrium in the steady state. In particular, the nominal interest rate is constant and the inflation rate is constant. Now, moreover, the aggregate quantities of capital and labor are constant.

The first order conditions for consumption imply $c(t,n) = c_0 e^{(r-\rho-\pi)T_j(n)} \times e^{-(\rho+\pi)(t-T_j(n))}$, $t \in (t_j, T_{j+1})$, $n \in [0, N)$. This expression implies that aggregate consumption is $C(t) = c_0 e^{(r-\rho-\pi)t} (1 - e^{-rN}) / (rN)$. Therefore, the same relation holds between the nominal interest rate and inflation to imply constant aggregate consumption, $r = \rho + \pi$. The non-arbitrage condition then implies $r^k = \rho + \delta$.

With $r^k = \rho + \delta$ and the expressions of w and r^k from the maximization problem of the firms, the capital-hours ratio and the capital-output ratio are $K/H = [\theta Y_0 / (\rho + \delta)]^{1/(1-\theta)}$ and $K/Y = \theta / (\rho + \delta)$. As $\dot{K} = 0$ in the steady state, the investment-output ratio in the steady state is given by $\delta K/Y = \delta \theta / (\rho + \delta)$.

The market clearing condition for goods implies $C + \delta K + (1/N) \gamma Y = Y$. Therefore,

$$(12) \quad c_0(r) = \left(1 - \frac{1}{N} \gamma - \delta \frac{\theta}{\rho + \delta}\right) \left(\frac{1 - e^{-rN}}{rN}\right)^{-1} Y_0 \left(\frac{K}{H}\right)^\theta H(r).$$

Dividing by $Y = Y_0 K^\theta H^{1-\theta}$, we obtain the expression for the consumption-income ratio just after a transfer, $\hat{c}_0(r) = (1 - \frac{1}{N} \gamma - \delta \frac{\theta}{\rho + \delta}) (\frac{1 - e^{-rN}}{rN})^{-1}$, which is, apart from the term $\delta \theta / (\rho + \delta)$, the same expression as in the case without capital and labor.

As the expressions for N and for the money-income ratio in (5)–(6), and (7) depend only on \hat{c}_0 , the value of N and the demand for money have the same expressions in this economy with capital and labor.

Therefore, the fact that agents can now react to the change in inflation with changes in capital and labor does not change the demand for money. In particular, this extension shows that the area under the demand for money works as an approximation for the welfare cost of inflation, but it does not account for the whole cost. Because the demand for money is the same, the area under it from $r = 3$ to $r = 13$ percent is also the same. However, the welfare cost of inflation increases for both N fixed and N endogenous.

The first order conditions for $h(t,n)$ imply $1 - h(t,n) = \alpha / (\lambda P_0 w)$, with $\lambda = 1 / (P_0 c_0)$. As w is constant, $h(t,n) = h$ is constant over time. With the expression of w , we obtain the equilibrium value of the hours of work,

$$(13) \quad h(r) = \frac{(1 - \theta)}{(1 - \theta) + \alpha \hat{c}_0(r)},$$

where $\hat{c}_0(r)$ is the consumption-income ratio. (The preferences for indivisible labor yield $h(r) = (1 - \theta)/(B\hat{c}_0(r))$ following similar steps. The expressions for $\hat{c}_0(r)$, $N(r)$, $m(r)$, and $w(r)$ remain unchanged with indivisible labor.) Given that h is constant across time and across agents, aggregate hours of work are given by $H = 1/N \int_0^N h(t, n) dn = h$. With this, we have obtained all equilibrium variables to calculate the welfare cost of inflation in this new economy.

The welfare cost of inflation is defined in the same way as in Section III: as the income compensation $w(r)$ to leave agents indifferent between an economy with $r > \bar{r}$ and an economy with \bar{r} , $U^T[c(r, (1 + w(r))Y(r)), h(r)] = U^T[c(\bar{r}, Y(\bar{r})), h(\bar{r})]$. Income now varies with the nominal interest rate. The preferences $u(c, h) = \log c + \alpha \log(1 - h)$ imply

$$(14) \quad 1 + w(r) = \frac{c_0(\bar{r})}{c_0(r)} \left(\frac{1 - h(\bar{r})}{1 - h(r)} \right)^\alpha \exp\left(\frac{rN}{2} - \frac{\bar{r}N}{2}\right),$$

where $c_0(r)$, $h(r)$, and N are given by (12), (13), and (6). With indivisible labor, $1 + w(r) = (c_0(\bar{r})/c_0(r)) \exp[B(h(r) - h(\bar{r}))] \exp((rN/2) - (\bar{r}N/2))$ following similar steps.

B. Calibration

In addition to ρ and γ , the extension for capital and labor requires the calibration of θ , δ , α , and B (α for the standard labor decision or B for indivisible labor). All parameters are in Table 4.

I use the same value for ρ and the same method to obtain γ . That is, $\rho = 3$ percent per year, and γ is obtained to imply that the demand for money passes through the geometric mean of the historical data on r and $m(r)$. With capital and labor, the value of c_0 decreases as depreciation requires constant investment δK in the steady state. As a result, γ increases to match the money-income ratio at its historical mean, $m(r) = 0.257$, when $r = 3.64$ percent per year.

I set α and B so that the equilibrium value of hours of work $h(r)$ is equal to 0.3 when the nominal interest rate is equal to its historical mean, which implies $\alpha = 2.065$ and $B = 2.95$. For θ and δ , I use $\theta = 0.36$ and $\delta = 0.10$ percent per year, the same values used by Cooley and Hansen (1989).

N is endogenous in the main model, and so it is not a parameter to be calibrated. But it is a parameter in the model with fixed periods. With fixed periods, as before, I fix the value of N to the optimal N when the interest rate is equal to its historical mean (N is fixed at 190 days in the first model and 264 days with capital and labor). In this way, the models with endogenous and fixed N yield the same equilibrium values when $r = 3.64$ percent per year. When the interest rate changes, their equilibrium values move apart. In particular, when the interest rate increases, the money-income ratio is approximately constant with N fixed, while it decreases with N endogenous.

TABLE 4—PARAMETERS

	ρ	γ	θ	δ	α	B
First model	3 percent p.a.	1.79	—	—	—	—
With capital and labor						
Standard labor decision	3 percent p.a.	2.49	0.36	10 percent p.a.	2.065	—
Indivisible labor	3 percent p.a.	2.49	0.36	10 percent p.a.	—	2.950

Notes: ρ : calibrated so that $r = 3$ percent p.a. implies zero inflation. γ : calibrated so that $m(r)$ matches the US historical average when r is the historical nominal interest rate, that is, $m(3.64 \text{ percent}) = 0.257$. α and B : calibrated so that $h(3.64 \text{ percent}) = 0.3$. θ and δ are taken from Cooley and Hansen (1989).

C. Results with Capital and Labor

The welfare cost of 10 percent instead of 0 inflation increases about 0.3 percentage points for N fixed and for N endogenous, as previously stated. The difference between the estimations of the welfare cost of inflation is still about 1 percentage point. The estimation of the welfare cost is a little larger, with N fixed and indivisible labor decision, 0.5 percent of income. In this case, the welfare cost is 0.8 percentage points higher with N endogenous. In any case, the money-income ratio is still approximately constant with N fixed, with standard or indivisible labor decisions. Output, capital, hours of work, and consumption have similar behavior with standard or indivisible labor. The results are in Table 5. Table 5 includes the results for the first model, without the decisions on capital and labor.

When inflation increases, the variation in consumption within holding periods increases, $c_0^+(T_j)/c_0^-(T_j)$ increases. With fixed periods, this variation is larger as agents cannot adapt their frequency of transfers to the higher inflation. As inflation increases, the variation of consumption can be so high that eventually the welfare cost of inflation is higher with N fixed. As stated in Section III, this happens with very high inflation. Without the decision on the supply of labor, the welfare costs with N fixed and N endogenous are equal for $r = 261$ percent per year.

As the welfare cost increases with the labor decision, the value of r , for which $w(r)$ with fixed and endogenous N are equal, decreases, but it is still high. It is $r = 73$ percent with the standard labor decision, which implies $w(r) = 5.12$ percent of income; and it is $r = 55$ percent with indivisible labor, which implies $w(r) = 4.11$ percent of income. With inflation of 100 percent instead of 0, the welfare cost is higher with N fixed. It is 8.86 percent with standard labor decision and 11.26 percent with indivisible labor. The welfare cost is 6.26 percent and 6.25 percent, respectively, with N endogenous. On the other hand, as the money-income ratio is approximately constant in r with N fixed, but it is slightly increasing, the money-income ratio increases 10 percent with N fixed under 100 percent inflation, while it decreases 80 percent with N endogenous. The welfare cost is higher with N fixed when inflation is 100 percent per year, but the model with N fixed implies higher real balances even though inflation is high. As the model with endogenous N implies a smaller money-income ratio, the welfare cost of 6.25 percent is more plausible. However, the same comments about the welfare cost for very low or very high inflation rates made in Section III apply here. The model is intended to study the effects of the choice of the trading frequency for moderate inflation, between zero and 10 percent per year.

TABLE 5—EQUILIBRIUM VALUES AND THE WELFARE COST OF INFLATION

	Inflation (percent p.a.)			
	0		10	
	<i>N</i> endogenous	<i>N</i> fixed	<i>N</i> endogenous	<i>N</i> fixed
Standard labor decision				
Output*	1.00	1.00	1.00	0.98
Capital*	1.00	1.00	1.00	0.98
Consumption*	1.00	1.00	0.99	0.98
Hours of work	0.30	0.30	0.30	0.29
$c^+(T_j)/c^-(T_j)$	1.02	1.02	1.05	1.10
Holding period (days)	291	264	139	264
Money-income ratio	0.28	0.26	0.14	0.26
Relative to 0 percent inflation	1.0	1.0	0.5	1.0
Welfare cost (percent)	0.00	0.00	1.33	0.36
Indivisible labor decision				
Output*	1.00	1.00	1.00	0.96
Capital*	1.00	1.00	1.00	0.96
Consumption*	1.00	1.00	0.99	0.96
Hours of work	0.30	0.30	0.30	0.29
$c^+(T_j)/c^-(T_j)$	1.02	1.02	1.05	1.10
Holding period (days)	291	264	139	264
Money-income ratio	0.28	0.26	0.14	0.26
Relative to 0 percent inflation	1.0	1.0	0.5	1.0
Welfare cost (percent)	0.00	0.00	1.32	0.50
First model				
Output*	1.00	1.00	1.00	1.00
Consumption*	1.00	1.00	0.99	1.00
$c^+(T_j)/c^-(T_j)$	1.02	1.02	1.04	1.07
Holding period (days)	209	190	100	190
Money-income ratio	0.28	0.26	0.14	0.26
Relative to 0 percent inflation	1.0	1.0	0.5	1.0
Welfare cost (percent)	0.00	0.00	0.95	0.02

* Relative to zero inflation. Welfare Cost: percentage increase in income to make agents indifferent between 10 percent and zero inflation, $w(r)$ from $r = 13$ percent p.a. to $r = 3$ percent p.a. *N* Fixed: optimal choice of *N* under $r = 3.64$ percent p.a., the geometric mean of r in the period.

Surprisingly, the results on hours worked and output change whether there are fixed or endogenous periods. A well-known result in cash-in-advance models is that hours worked and output decrease when inflation increases. The reason is that the value of work decreases, as agents can use their labor income only after they have worked and after the increase in prices from one period to the other. Agents, therefore, decrease their consumption of goods and, by decreasing labor, increase their consumption of leisure. These results are reproduced in Table 5 with *N* fixed. Hours of work, capital, and output decrease as inflation increases when *N* is fixed.

When *N* is endogenous, the increase in the frequency of transfers requires an increase in the use of resources in financial services. The term $(1/N)\gamma Y$ increases in the market clearing condition. To compensate the decrease in resources available for consumption, the agents maintain their hours of work and maintain the level of output.¹³ Although output and hours worked are different with fixed or endogenous

¹³The hours of work and output increase very little. From their values relative to zero inflation, they increase from 1 to 1.0002 when inflation increases from 0 to 10 percent.

periods, the conclusions about the welfare cost of inflation do not change. Welfare always decreases with inflation.

As output does not change with inflation, we have a situation as obtained empirically by McCandless and Weber (1995), in which inflation and output have no correlation in the long run. On the other hand, Cooley and Hansen (1989) show evidence of a negative correlation between inflation and output. McCandless and Weber (1995) point out that there is uncertainty about the empirical relation between inflation and output. With endogenous periods and logarithmic utility, the model predicts no correlation between output and inflation.

Inflation still implies high welfare costs with endogenous periods, although output does not change with inflation. The reason is that agents are working more to increase the financial sector, which does not increase their utility. Aggregate consumption decreases with inflation. As a result, the welfare cost is still high with endogenous periods.

Although there is uncertainty on the relation between inflation and output, the evidence on the relation between the money-income ratio and the nominal interest rate is strong. The money-income ratio decreases with the nominal interest rate (Meltzer 1963, Lucas 1988, 2000, among others). The model with endogenous periods matches these predictions as can be seen in Table 5. With 10 percent inflation, the money-income ratio decreases to 50 percent of its level under 0 inflation with endogenous N , while it is constant with N fixed. This behavior allows the money-income ratio to match the data more closely with endogenous N , as shown in Figures 1 and 2.

The conclusion is that the introduction of the decisions of capital and labor imply new results, but it does not affect the conclusions on the welfare cost of inflation. The welfare cost increases for both N fixed and N endogenous. The money-income ratio is still decreasing with the interest rate with N endogenous, and it is approximately constant with N fixed. The conclusions about the effect of the decision on the transfer frequency on the welfare cost of inflation are robust.

V. Related Literature

The model uses a transfer cost to imply a decision on the time to exchange bonds for money. Alternatively, agents in Khan and Thomas (2010) draw a transfer cost from a random distribution in a model based on Alvarez, Atkeson, and Kehoe (2002). Because I study the welfare cost of inflation in the steady state, it simplifies the calculations to have the same transfer cost for all agents and to focus on the transfer times. With this, I can determine analytically the transfer times and the distribution of money holdings for each inflation rate. Moreover, by studying the transfer times, I can directly compare the welfare cost in economies with fixed and optimal transfer times.

The model requires heterogeneity in money holdings because the transfer cost rules out an equilibrium with a representative agent. Agents are heterogeneous only in their money holdings, they are affected by inflation in the same way. It is possible to extend the model to study the distributional effects of inflation, but this is beyond the objectives of the paper. Erosa and Ventura (2002) and Boel and Camera (2009),

for example, have agents heterogeneous in wealth and in other characteristics to analyze the distributional effects of inflation.

The transfer cost is constant across agents and through time. Hasn't financial innovation made γ decrease through time? Following the reasoning in Reynard (2004), we can obtain a stable money demand by having decreasing γ combined with increasing market participation. However, it simplifies the analysis to have a constant γ . As the comparison of $m(r)$ and the data shows, a constant γ is powerful in explaining the aggregate demand for money over the century.

Cooley and Hansen (1989) have an important contribution on the effects of inflation. It is interesting to compare their results about the welfare cost, with their economy with unitary velocity, with the results here, with an elastic demand for money. Their results imply a welfare cost of 0.28 percent in terms of income of 10 percent inflation instead of 0 inflation (they report the welfare cost with respect to the Friedman rule; $0.28 = 0.387 - 0.107$ is the difference between the costs of 10 percent and 0 inflation). Cooley and Hansen (1989) have a cash-in-advance economy in which agents are constrained to trade bonds for money every quarter (they also have a calibration with monthly periods, with smaller estimates).

The corresponding value here is for $\eta = 1$ and $a = 0$, as Cooley and Hansen (1989) use logarithmic utility and agents receive their income only at the end of each quarter. From Table 2, the welfare cost is close to 0 (0.02 for $a = 0$ or 0.12 for $a = 0.6$; N with fixed periods is about 2 quarters, and here agents can smooth consumption within holding periods. The point is that the welfare cost is close to zero with fixed periods). The difference from 0.28 and approximately 0 comes from the decision on capital and labor, as Cooley and Hansen (1989) include capital and labor. Here, the welfare cost also increases with the introduction of capital and labor, as seen in Table 3, but the major portion of the welfare cost comes from the increase in financial transactions.

Cooley and Hansen (1991) include the decision on cash and credit goods. With credit goods, they obtain an interest-elastic demand for money. However, credit goods affect little the welfare cost of inflation. It decreases the value above from 0.28 to 0.27 (the paper also includes distortionary taxation, 0.27 does not consider distortionary taxation). As in Cooley and Hansen (1989), most of the effects of inflation come from the decrease in the labor supply. As found here, including the effects of the increase in financial transactions substantially increases the welfare cost estimations. Here, with capital and labor and the effect of financial transactions, the welfare cost of 10 percent instead of 0 inflation increases to 1.3 percent.

Another way of introducing money is through a shopping-time technology, as in McCallum and Goodfriend (1987). It is used, for example, in Guidotti and Vegh (1993), Correia and Teles (1996), Lucas (2000), and da Costa and Werning (2008).¹⁴ However, a shopping-time technology still requires a functional form and setting fixed periods between trades. To specifically address the role of optimal or fixed periods, I compare two models in which the only difference is the choice of the trading period.

¹⁴Gillman (1993), and Aiyagari, Braun, and Eckstein (1998) have different transactions technologies to relate time and credit.

Other factors have been considered to study the welfare cost of inflation. Distortionary taxation is considered in Cooley and Hansen (1991), Guidotti and Vegh (1993), Correia and Teles (1996), and da Costa and Werning (2008). Khan, King, and Wolman (2003) consider sticky prices and other factors. In the tradition of Kiyotaki and Wright (1989), other models justify the use of money from micro foundations (Lagos and Wright 2005, Rocheteau and Wright 2005, and Craig and Rocheteau 2008). Gomme (1993) and Dotsey and Ireland (1996) consider endogenous growth. Considering additional factors increases the welfare cost of inflation. I concentrate on the effects of letting agents choose their holding periods. The conclusion here about the impact of the frequency of trades does not conflict with the findings in these papers. One percent in terms of income obtained here is the effect of the increase in the use of financial services.

VI. Conclusions

I find that letting agents reallocate bonds and money in fixed periods substantially underestimates the welfare cost of inflation. I calculate the welfare cost of inflation in two cash-in-advance economies. In one economy, the agents reallocate their money and bond holdings in fixed periods. In the other, the agents choose the reallocation periods.

This change implies large differences for the welfare cost of inflation. Taking into account the increase in the frequency of financial trades increases the estimates of the welfare cost.

The welfare cost of 1 percent of income is only the welfare cost caused by the increase in financial transactions. Lagos and Wright (2005), Dotsey and Ireland (1996), and Cooley and Hansen (1989, 1991) are examples of studies that estimate the welfare cost of inflation from other causes (among others, from the increase in search costs, the decrease in output growth, and the decrease in labor supply). One percent of income is already \$100 billion in 2000 dollars. The total welfare cost would combine all these estimations. Even for moderate inflation, the estimations of the total cost of inflation point to a substantial figure.

APPENDIX

A. Optimal Interval Between Transfers N (equations 5 and 6)

The Lagrangian of the problem (1)–(3) is $L = \sum_{j=0}^{\infty} \int_{T_j(s)}^{T_{j+1}(s)} e^{-\rho t} u(c(t,s)) dt + \lambda(s) \{ \sum_{j=1}^{\infty} Q(T_j(s)) M^-(T_j(s), s) + W_0 - \sum_{j=1}^{\infty} Q(T_j(s)) [M^+(T_j(s), s) - P(T_j(s)) \Gamma] \} + \mu(s) \{ \int_0^{T_1(s)} aP(t) Y(t) dt + M_0 - \int_0^{T_1(s)} P(t) c(t,s) dt - M^-(T_1(s), s) \}$, where $M^+(T_j(s), s) = \int_{T_j(s)}^{T_{j+1}(s)} [P(t) c(t,s) - aP(t) Y] dt$, as $M^-(T_{j+1}(s), s) = 0$, $j = 1, 2, \dots$. The constraint relative to μ is obtained by integrating equation (3) from $t = 0$ to $t = T_1$.

The first order conditions for $c(t,s)$ imply $e^{-\rho t} u'(c(t,s)) = \lambda(s) Q(T_j) P(t)$ for $t \in (T_j, T_{j+1})$, $j = 1, 2, \dots$. At $t = T_j$ and $t = T_{j+1}$, we have $e^{-\rho T_j} u'(c^+(T_j, s)) = \lambda(s) Q(T_j) \times P(T_j)$, and $e^{-\rho T_{j+1}} u'(c^-(T_{j+1}, s)) = \lambda(s) Q(T_j) P(T_{j+1})$.

The first order conditions for $T_j(s)$, $j = 2, 3, \dots$, imply $e^{-\rho T_j} u(c^-(T_j)) - e^{-\rho T_j} \times u(c^+(T_j)) = \lambda \dot{Q}(T_j) \int_{T_j}^{T_{j+1}} P(t)c(t) dt - \lambda Q(T_j) P(T_j) c^+(T_j) + \lambda Q(T_{j-1}) P(T_j) \times c^-(T_j) - \lambda \dot{Q}(T_j) \int_{T_j}^{T_{j+1}} aP(t) Y dt + \lambda Q(T_j) aP(T_j) Y - \lambda Q(T_{j-1}) aP(T_j) Y + \lambda \gamma Y \{ \dot{Q}(T_j) P(T_j) + Q(T_j) \dot{P}(T_j) \}$, removing the index for the agent s to simplify.

With $u(c) = \frac{c^{1-1/\eta}}{1-1/\eta}$, dividing by $Q(T_j) P(T_j) Y$, and rearranging, yields

$$(A1) \quad \gamma[r(T_j) - \pi(T_j)] + \frac{e^{-\rho T_j} c^+(T_j)^{-1/\eta}}{\lambda P(T_j) Q(T_j)} \frac{1}{1 - 1/\eta} \left[\frac{c^+(T_j)^{1/\eta}}{c^-(T_j)^{1/\eta}} \hat{c}^-(T_j) - \hat{c}^+(T_j) \right]$$

$$= -r(T_j) \int_{T_j}^{T_{j+1}} \frac{P(t) \hat{c}(t)}{P(T_j)} dt + \frac{Q(T_{j-1})}{Q(T_j)} \hat{c}^-(T_j)$$

$$- \hat{c}^+(T_j) + r(T_j) \int_{T_j}^{T_{j+1}} \frac{aP(t)}{P(T_j)} dt - a \left[\frac{Q(T_{j-1})}{Q(T_j)} - 1 \right].$$

Using the first order conditions for consumption, $e^{-\rho T_j} c^+(T_j)^{-1/\eta} = \lambda P(T_j) Q(T_j)$ and $c^+(T_j)^{1/\eta}/c^-(T_j)^{1/\eta} = Q(T_{j-1})/Q(T_j)$. Therefore, the equation simplifies to

$$(A2) \quad \gamma[r(T_j) - \pi(T_j)] + \frac{1}{\eta - 1} \left[\frac{Q(T_{j-1})}{Q(T_j)} \hat{c}^-(T_j) - \hat{c}^+(T_j) \right]$$

$$= -r(T_j) \int_{T_j}^{T_{j+1}} \frac{P(t) \hat{c}(t)}{P(T_j)} dt + r(T_j) \int_{T_j}^{T_{j+1}} \frac{aP(t)}{P(T_j)} dt - a \left[\frac{Q(T_{j-1})}{Q(T_j)} - 1 \right].$$

In the steady state, $r(t) = r$, $\pi(t) = \pi$, $Q(T_{j-1})/Q(T_j) = e^{rN}$. Write $P(t) = P_0 e^{\pi t}$. So,

$$(A3) \quad (r - \pi)\gamma + \frac{[e^{rN} \hat{c}^-(T_j) - \hat{c}^+(T_j)]}{\eta - 1}$$

$$= -r \int_{T_j}^{T_{j+1}} e^{\pi(t-T_j)} \hat{c}(t) dt - a(e^{rN} - 1) + r \int_{T_j}^{T_{j+1}} a e^{\pi(t-T_j)} dt.$$

Moreover, $\hat{c}^+(T_j) = \hat{c}_0$, $\hat{c}^-(T_j) = e^{-rN} \hat{c}_0$, and $\hat{c}(t) = \hat{c}_0 e^{-\eta r(t-T_j)}$, which implies

$$(A4) \quad (r - \pi)\gamma = \hat{c}_0 \left[\frac{1 - e^{-r(\eta-1)N}}{\eta - 1} - r \frac{1 - e^{-(\eta r - \pi)N}}{\eta r - \pi} \right]$$

$$- a \left[(e^{rN} - 1) - r \frac{e^{\pi N} - 1}{\pi} \right].$$

To obtain the expression in the text, use the fact that $r - \pi = \rho$ and $(\eta r - \pi) = \rho + r(\eta - 1)$. The steps for $\eta = 1$ are analogous.

B. Individual and Aggregate Consumption

At an arbitrary time t , the agents $n \in [0, N)$ are in different positions in their holding periods. With constant interest rate and inflation, we can use the first order conditions to identify the holding period of each agent and obtain their levels of consumption.

The first conditions for consumption imply $c^+(T_j, n) = e^{\eta(r-\pi-\rho)T_j} / [P_0\lambda(n)]^\eta$. Let consumption be given by $c^+(T_j, n) = c_0 e^{\eta(r-\pi-\rho)T_j}$ for an agent that makes a transfer at T_j . Using the fact that $\dot{c}(t, n)/c(t, n) = -\eta(\pi + \rho)$ within holding periods, we can write consumption of agent n as $c(t, n) = c_0 e^{\eta(r-\pi-\rho)t} e^{-\eta r(t-T_j(n))}$, taking the highest integer j such that $T_j(n) \leq t < T_{j+1}(n)$.

Let $T_j(n) = n + (j - 1)N$, for $j \geq 1$, and $T_j(n) = 0$, for $j = 0$, denote the time in which the holding period $j + 1$ starts. The first holding period starts at $t = 0$ and ends at T_1 . For an arbitrary $t > jN$, the agents will be in their holding periods $j + 1$ or $j + 2$. At $t = T_1(n)$, $j = 1$, agent n starts the 2nd holding period.

Given $t \geq N$, let j be the highest integer such that $t - jN \geq 0$ (the argument is similar for $0 \leq t < N$). If $n \in [0, t - jN)$, then n is in the holding period $j + 2$. If $n \in [t - jN, N)$, then n is in the holding period $j + 1$. For example, for $t = N + (N/2)$, we have $j = 1 \Rightarrow t - jN = N/2$. Therefore, agents $n \in [0, N/2)$ are in their third holding period, and agents $n \in [N/2, N)$ are in their second holding period. With $c(t, n) = c_0 e^{\eta(r-\pi-\rho)t} e^{-\eta r(t-T_j)}$, aggregate consumption is then given by

$$(B1) \quad C(t) = \frac{1}{N} \int_0^{t-jN} c_0 e^{\eta(r-\pi-\rho)t} e^{-\eta r(t-T_{j+1})} dn + \frac{1}{N} \int_{t-jN}^N c_0 e^{\eta(r-\pi-\rho)t} e^{-\eta r(t-T_j)} dn,$$

taking into account the different holding periods.

With a change of variables $s \equiv T_{j+1} = n + jN$ on the first integral and $s \equiv T_j = n + (j - 1)N$ on the second integral, we obtain $C(t) = c_0 e^{\eta(r-\pi-\rho)t} (1/N) \times \int_{t-N}^t e^{-\eta r(t-s)} ds$. With $x \equiv t - s$, we obtain

$$(B2) \quad C(t) = c_0 e^{\eta(r-\pi-\rho)t} \frac{1}{N} \int_0^N e^{-\eta r x} dx.$$

So, aggregate consumption is given by $C(t) = c_0 e^{\eta(r-\pi-\rho)t} (1 - e^{-\eta r N}) / (\eta r N)$. Therefore, in equilibrium, to imply constant aggregate consumption, we must have $r = \rho + \pi$.

C. Money-Income Ratio (equation 7)

To obtain $M_0(n)$, the initial money holdings so that agent n can consume at the steady state until the next transfer, solve $M_0(n) = \int_0^n P(t)c(t, n) dt - \int_0^n aP(t) Y dt$. For a complete holding period, all agents start with c_0 and end with $c_0 e^{-\eta r N}$. For the holding period $[0, n)$, only agent $n = 0$ makes a transfer at $t = 0$ and starts with

$c(0,0) = c_0$. The other agents makes the first transfer at $t = n$ and start with $c(0,n) = c_0 e^{-\eta r N} e^{\eta r n}$. As consumption follows $\dot{c}(t,n)/c(t,n) = -\eta r$, we have $c(t,n) = c_0 e^{-\eta r N} e^{\eta r n} e^{-\eta r t}$, $0 \leq t < n$. Substituting in the integral above, with $P(t) = P_0 e^{\pi t}$ and $r = \rho + \pi$, implies $M_0(n) = P_0 Y n \left[\hat{c}_0 e^{-N \eta r} e^{\eta r n} \frac{1 - e^{-[\rho + r(\eta - 1)]n}}{[\rho + r(\eta - 1)]n} - a \frac{e^{(r - \rho)n} - 1}{(r - \rho)n} \right]$.

The aggregate demand for money at $t = 0$ is then given by $M(0) = (1/N) \times \int_0^N M_0(n) dn$. The money-income ratio is given by $m(r) = M(0)/(P_0 Y)$. Substituting the expression of $M_0(n)$ and solving the integral yields equation (7).

REFERENCES

- Aiyagari, S. Rao, R. Anton Braun, and Zvi Eckstein. 1998. "Transaction Services, Inflation, and Welfare." *Journal of Political Economy* 106 (6): 1274–1301.
- Alvarez, Fernando, Andrew Atkeson, and Chris Edmond. 2009. "Sluggish Responses of Prices and Inflation to Monetary Shocks in an Inventory Model of Money Demand." *Quarterly Journal of Economics* 124 (3): 911–67.
- Alvarez, Fernando, Andrew Atkeson, and Patrick J. Kehoe. 2002. "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets." *Journal of Political Economy* 110 (1): 73–112.
- Alvarez, Fernando, and Francesco Lippi. 2009. "Financial Innovation and the Transactions Demand for Cash." *Econometrica* 77 (2): 363–402.
- Attanasio, Orazio P., and Guglielmo Weber. 1989. "Intertemporal Substitution, Risk Aversion and the Euler Equation for Consumption." *Economic Journal* 99 (395): 59–73.
- Bailey, Martin J. 1956. "The Welfare Cost of Inflationary Finance." *Journal of Political Economy* 64 (2): 93–110.
- Bansal, Ravi, and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance* 59 (4): 1481–1509.
- Bates, Thomas W., Kathleen M. Kahle, and Rene M. Stulz. 2009. "Why Do U.S. Firms Hold So Much More Cash Than They Used To?" *Journal of Finance* 64 (5): 1985–2021.
- Baumol, William J. 1952. "The Transactions Demand for Cash: An Inventory Theoretic Approach." *Quarterly Journal of Economics* 66 (4): 545–56.
- Boel, Paola, and Gabriele Camera. 2009. "Financial Sophistication and the Distribution of the Welfare Cost of Inflation." *Journal of Monetary Economics* 56 (7): 968–78.
- Bover, Olympia, and Nadine Watson. 2005. "Are There Economies of Scale in the Demand for Money by Firms? Some Panel Data Estimates." *Journal of Monetary Economics* 52 (8): 1569–89.
- Caplin, Andrew, and John Leahy. 2010. "Economic Theory and the World of Practice: A Celebration of the (S, s) Model." *Journal of Economic Perspectives* 24 (1): 183–202.
- Chiu, Jonathan. 2007. "Endogenously Segmented Asset Market in an Inventory Theoretic Model of Money Demand." Bank of Canada Working Paper 2007-46.
- Christiano, Lawrence J., Martin Eichenbaum, and Charles Evans. 1996. "The Effects of Monetary Policy Shocks: Evidence from the Flow of Funds." *Review of Economics and Statistics* 78 (1): 16–34.
- Cooley, Thomas F., ed. 1995. *Frontiers of Business Cycle Research*. Princeton: Princeton University Press.
- Cooley, Thomas F., and Gary D. Hansen. 1989. "The Inflation Tax in a Real Business Cycle Model." *American Economic Review* 79 (4): 733–48.
- Cooley, Thomas F., and Gary D. Hansen. 1991. "The Welfare Costs of Moderate Inflation." *Journal of Money, Credit, and Banking* 23 (3): 483–503.
- Correia, Isabel, and Pedro Teles. 1996. "Is the Friedman Rule Optimal When Money Is an Intermediate Good?" *Journal of Monetary Economics* 38 (2): 223–44.
- Craig, Ben, and Guillaume Rocheteau. 2008. "Inflation and Welfare: A Search Approach." *Journal of Money, Credit, and Banking* 40 (1): 89–119.
- Cysne, Rubens P. 2003. "Divisia Index, Inflation, and Welfare." *Journal of Money, Credit, and Banking* 35 (2): 221–38.
- da Costa, Carlos E., and Ivan Werning. 2008. "On the Optimality of the Friedman Rule with Heterogeneous Agents and Nonlinear Income Taxation." *Journal of Political Economy* 116 (1): 82–112.
- Dotsey, Michael, and Peter N. Ireland. 1996. "The Welfare Cost of Inflation in General Equilibrium." *Journal of Monetary Economics* 37 (1): 29–47.

- Edmond, Chris, and Pierre-Olivier Weill.** 2008. "Liquidity Effects, Models of." In *The New Palgrave Dictionary of Economics*, 2nd ed., edited by Steven N. Durlauf and Lawrence E. Blume. New York: Palgrave Macmillan.
- Erosa, Andres, and Gustavo Ventura.** 2002. "On Inflation as a Regressive Consumption Tax." *Journal of Monetary Economics* 49 (4): 761–95.
- Friedman, Milton.** 1969. "The Optimum Quantity of Money." In *The Optimum Quantity of Money and Other Essays*, 1–50. Chicago: Aldine.
- Gillman, Max.** 1993. "The Welfare Cost of Inflation in a Cash-in-Advance Economy with Costly Credit." *Journal of Monetary Economics* 31 (1): 97–115.
- Gomme, Paul.** 1993. "Money and Growth Revisited: Measuring the Costs of Inflation in an Endogenous Growth Model." *Journal of Monetary Economics* 32 (1): 51–77.
- Grossman, Sanford J.** 1989. "Monetary Dynamics with Proportional Transaction Costs and Fixed Payment Periods." National Bureau of Economic Research Working Paper 1663.
- Grossman, Sanford J.** 1987. "Monetary Dynamics with Proportional Transaction Costs and Fixed Payment Periods." In *New Approaches to Monetary Economics: Proceedings of the Second International Symposium in Economic Theory and Econometrics*, edited by William A. Barnett and Kenneth J. Singleton, 3–41. Cambridge, UK: Cambridge University Press.
- Grossman, Sanford, and Laurence Weiss.** 1983. "A Transactions-Based Model of the Monetary Transmission Mechanism." *American Economic Review* 73 (5): 871–80.
- Guerron-Quintana, Pablo A.** 2009. "Money Demand Heterogeneity and the Great Moderation." *Journal of Monetary Economics* 56 (2): 255–66.
- Guidotti, Pablo E., and Carlos A. Vegh.** 1993. "The Optimal Inflation Tax When Money Reduces Transactions Costs: A Reconsideration." *Journal of Monetary Economics* 31 (2): 189–205.
- Hansen, Gary D.** 1985. "Indivisible Labor and the Business Cycle." *Journal of Monetary Economics* 16 (3): 309–27.
- Hansen, Lars Peter, and Kenneth J. Singleton.** 1982. "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models." *Econometrica* 50 (5): 1269–86.
- Harberger, Arnold C.** 1998. "A Vision of the Growth Process." *American Economic Review* 88 (1): 1–32.
- Heathcote, Jonathan.** 1998. "Interest Rates in a General Equilibrium Baumol-Tobin Model." http://jonathanheathcote.com/bau_tob.pdf.
- Hodrick, Robert J., Narayana R. Kocherlakota, and Deborah Lucas.** 1991. "The Variability of Velocity in Cash-in-Advance Models." *Journal of Political Economy* 99 (2): 358–84.
- Ireland, Peter N.** 2009. "On the Welfare Cost of Inflation and the Recent Behavior of Money Demand." *American Economic Review* 99 (3): 1040–52.
- Jovanovic, Boyan.** 1982. "Inflation and Welfare in the Steady State." *Journal of Political Economy* 90 (3): 561–77.
- Khan, Aubhik, Robert G. King, and Alexander L. Wolman.** 2003. "Optimal Monetary Policy." *Review of Economic Studies*, 70(4): 825–60.
- Khan, Aubhik, and Julia K. Thomas.** 2010. "Inflation and Interest Rates with Endogenous Market Segmentation." <http://www.aubhik-khan.net/SegMktsApril2010.pdf>.
- King, Robert G., Charles I. Plosser, and Sergio T. Rebelo.** 1988. "Production, Growth and Business Cycles: I. The Basic Neoclassical Model." *Journal of Monetary Economics* 21 (2/3): 195–232.
- Kiyotaki, Nobuhiro, and Randall Wright.** 1989. "On Money as a Medium of Exchange." *Journal of Political Economy* 97 (4): 927–54.
- Lagos, Ricardo, and Randall Wright.** 2005. "A Unified Framework for Monetary Theory and Policy Analysis." *Journal of Political Economy* 113 (3): 463–84.
- Ljungqvist, Lars, and Thomas J. Sargent.** 2004. *Recursive Macroeconomic Theory*, 2nd ed. Cambridge, MA: MIT Press.
- Lucas, Robert E., Jr.** 1988. "Money Demand in the United States: a Quantitative Review." *Carnegie-Rochester Conference Series on Public Policy* 29: 137–67.
- Lucas, Robert E., Jr.** 1990. "Liquidity and Interest Rates." *Journal of Economic Theory* 50 (2): 237–64.
- Lucas, Robert E., Jr.** 2000. "Inflation and Welfare." *Econometrica* 68 (2): 247–74.
- McCallum, Bennett T., and Marvin S. Goodfriend.** 1987. "Demand for Money: Theoretical Studies." In *The New Palgrave: A Dictionary of Economics*, edited by John Eatwell, Murray Milgate, and Peter Newman, 775–81. London: Macmillan Press.
- McCandless, George T., Jr., and Warren E. Weber.** 1995. "Some Monetary Facts." *Federal Reserve Bank of Minneapolis Quarterly Review* 19 (3): 2–11.
- Mehra, Rajnish, and Edward C. Prescott.** 1985. "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15 (2): 145–61.

- Meltzer, Allan H.** 1963. "The Demand for Money: The Evidence from the Time Series." *Journal of Political Economy* 71 (3): 219–46.
- Mulligan, Casey B., and Xavier Sala-i-Martin.** 2000. "Extensive Margins and the Demand for Money at Low Interest Rates." *Journal of Political Economy* 108 (5): 961–91.
- Reynard, Samuel.** 2004. "Financial Market Participation and the Apparent Instability of Money Demand." *Journal of Monetary Economics* 51 (6): 1297–1317.
- Rocheteau, Guillaume, and Randall Wright.** 2005. "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium." *Econometrica* 73 (1): 175–202.
- Rodriguez Mendizabal, Hugo.** 2006. "The Behavior of Money Velocity in High and Low Inflation Countries." *Journal of Money, Credit, and Banking* 38 (1): 209–28.
- Romer, David.** 1986. "A Simple General Equilibrium Version of the Baumol-Tobin Model." *Quarterly Journal of Economics* 101 (4): 663–85.
- Rotemberg, Julio J.** 1984. "A Monetary Equilibrium Model with Transactions Costs." *Journal of Political Economy* 92 (1): 40–58.
- Silva, André C.** 2011. "Individual and Aggregate Money Demands." Nova School of Business and Economics Working Paper 557.
- Silva, André C.** 2012. "Rebalancing Frequency and the Welfare Cost of Inflation: Dataset." *American Economic Journal: Macroeconomics*. <http://dx.doi.org/10.1257/mac.4.2.153>.
- Teles, Pedro, and Ruilin Zhou.** 2005. "A Stable Money Demand: Looking for the Right Monetary Aggregate." *Federal Reserve Bank of Chicago Economic Perspectives* 29 (1): 50–63.
- Tobin, James.** 1956. "The Interest-Elasticity of Transactions Demand for Cash." *Review of Economics and Statistics* 38 (3): 241–47.
- Vissing-Jorgensen, Annette.** 2002. "Towards an Explanation of Household Portfolio Choice Heterogeneity: Nonfinancial Income and Participation Cost Structures." National Bureau of Economic Research Working Paper 8884.