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Hardy–Littlewood maximal operator on reflexive variable Lebesgue spaces over spaces of homogeneous type

by

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Abstract. We show that the Hardy–Littlewood maximal operator is bounded on a reflexive variable Lebesgue space $L^{p(\cdot)}$ over a space of homogeneous type (X,d,μ) if and only if it is bounded on its dual space $L^{p'(\cdot)}$, where $1/p(x)+1/p'(x)=1$ for $x\in X$. This result extends the corresponding result of Lars Diening from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces (X,d,μ) of homogeneous type.

1. Introduction. We begin with the definition of a space of homogeneous type (see, e.g., [C90a]). Given a set X and a function $d: X\times X\to[0,\infty)$, one says that (X,d) is a *quasi-metric space* if the following axioms hold:

(a) $d(x,y)=0$ if and only if $x=y$;
(b) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(c) for all $x,y,z\in X$ and some constant $\kappa\ge 1$,

$$d(x,y)\le\kappa(d(x,y)+d(y,z)). \tag{1.1}$$

For $x\in X$ and $r>0$, consider the ball $B(x,r)=\{y\in X: d(x,y)<r\}$. Given a quasi-metric space (X,d) and a positive measure μ that is defined on the σ-algebra generated by quasi-metric balls, one says that (X,d,μ) is a *space of homogeneous type* if there exists a constant $C_\mu\ge 1$ such that for any $x\in X$ and any $r>0$,

$$\mu(B(x,2r))\le C_\mu\mu(B(x,r)). \tag{1.2}$$

To avoid trivial measures, we will always assume that $0<\mu(B)<\infty$ for every ball B. Consequently, μ is a σ-finite measure.

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Given a complex-valued function $f \in L^1_{\rm loc}(X,d,\mu)$, we define its *Hardy–Littlewood maximal function* Mf by

$$(Mf)(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(x)|\,d\mu(x), \qquad x \in X,$$

where the supremum is taken over all balls $B \subset X$ containing $x \in X$. The *Hardy–Littlewood maximal operator* M is a sublinear operator acting by the rule $f \mapsto Mf$.

Let $L^0(X,d,\mu)$ denote the set of all complex-valued measurable functions on X and let $\mathcal{P}(X)$ denote the set of all measurable a.e. finite functions $p : X \to [1,\infty]$. For a measurable set $E \subset X$, put

$$p_-(E) := \operatorname*{ess\,inf}_{x\in E} p(x), \qquad p_+(E) := \operatorname*{ess\,sup}_{x\in E} p(x)$$

and

$$p_- := p_-(X), \qquad p_+ := p_+(X).$$

For $f \in L^0(X,d,\mu)$ and $p \in \mathcal{P}(X)$, consider the functional, which is called the *modular*, given by

$$\varrho_{p(\cdot)}(f) := \int_X |f(x)|^{p(x)}\,d\mu(x).$$

By definition, the variable Lebesgue space $L^{p(\cdot)}(X,d,\mu)$ consists of all functions $f \in L^0(X,d,\mu)$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$ depending on f. It is a Banach space with respect to the Luxemburg–Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\}.$$

If $p \in \mathcal{P}(X)$ is constant, then $L^{p(\cdot)}(X,d,\mu)$ is nothing but the standard Lebesgue space $L^p(X,d,\mu)$. Variable Lebesgue spaces are often called *Nakano spaces*. We refer to Maligranda's paper [M11] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces and to the monographs [CF13, DH$^+$11] for the basic properties of these spaces. We only mention that the space $L^{p(\cdot)}(X,d,\mu)$ is reflexive if and only if $1 < p_-, p_+ < \infty$. In this case, the dual space $[L^{p(\cdot)}(X,d,\mu)]^*$ is isomorphic to $L^{p'(\cdot)}(X,d,\mu)$, where $p' \in \mathcal{P}(X)$ is given by

$$1/p(x) + 1/p'(x) = 1, \quad x \in X$$

(see, e.g., [CF13, Proposition 2.79 and Corollary 2.81]).

One of the central problems of harmonic analysis on variable Lebesgue spaces is the problem of boundedness of the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(X,d,\mu)$. For a detailed history of this problem, we refer to the monographs [CF13, DH$^+$11, KM$^+$16]. We also mention that very recently Cruz-Uribe and Shukla [CS18, Theorem 1.1] proved a sufficient condition for the boundedness of the fractional maximal operator M_α, $0 \le \alpha < 1$, on reflexive variable Lebesgue spaces $L^{p(\cdot)}(X, d, \mu)$ over spaces of homogeneous type, which includes the case of the Hardy–Littlewood maximal operator as a special case when $\alpha = 0$.

In 2005, Diening [D05, Theorem 8.1] (see also [DH$^+$11, Theorem 5.7.2]) proved the following remarkable result: if $1 < p_-(\mathbb{R}^n)$, $p_+(\mathbb{R}^n) < 1$, then the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on its dual $L^{p'(\cdot)}(\mathbb{R}^n)$. Recently Lerner [L17, Theorem 1.1] generalized this result to the setting of weighted variable Lebesgue spaces $L^{p(\cdot)}_w(\mathbb{R}^n)$.

The aim of this paper is to present a self-contained proof of the following extension of Diening's theorem to the setting of spaces of homogeneous type.

THEOREM 1.1 (Main result). *Let (X, d, μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$ be such that $1 < p_-, p_+ < \infty$. The Hardy–Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ if and only if it is bounded on its dual space $L^{p'(\cdot)}(X, d, \mu)$.*

Our approach is based on the adaptation of Lerner's proof [L17], which is heavily based on the Calderón–Zygmund decomposition and dyadic maximal functions in the Euclidean setting of $\mathbb{R}^n$, to the setting of spaces of homogeneous type. This becomes possible thanks to the recently developed techniques of dyadic decomposition of spaces of homogeneous type due to Hytönen and Kairema [HK12] (see also previous works by Christ [C90a, C90b]). Note that these techniques were successfully applied in [AHT17, AW18, CS18, K19] to study various problems on spaces of homogeneous type (this list is far from being exhaustive).

The paper is organized as follows. In Section 2, we describe the construction by Hytönen and Kairema [HK12] of a system of adjacent dyadic grids on a space of homogeneous type. Elements of this system are called *dyadic cubes* and have many important properties of usual dyadic cubes in $\mathbb{R}^n$.

In Section 3, we recall the definition of Banach function spaces and the main result of [K19] (see also [L17, Theorem 3.1]) saying that if the Hardy–Littlewood maximal operator M is bounded on a Banach function space $\mathcal{E}(X, d, \mu)$, then its boundedness on the associate space $\mathcal{E}'(X, d, \mu)$ is equivalent to a certain condition $\mathcal{A}_\infty$. Since the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ is a Banach function space, in order to prove Theorem 1.1, it is sufficient to verify that $L^{p(\cdot)}(X, d, \mu)$ satisfies the condition $\mathcal{A}_\infty$.

In Section 4, we recall very useful relations between the norm and the modular in a variable Lebesgue space. This allows us to formulate a modular analogue of the condition $\mathcal{A}_\infty$ and show that this modular analogue implies the (norm) condition $\mathcal{A}_\infty$. The rest of the paper is devoted to the verification of the modular analogue of $\mathcal{A}_\infty$ (see Lemma 4.4).

In Section 5, we prepare for the proof of the main result, extending [L17, Lemmas 5.1–5.3 and 4.1] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type. Finally, in Section 6, we complete the proof of Theorem 1.1 following the scheme of the proof of [L17, Theorem 1.1].

2. Dyadic decomposition of spaces of homogeneous type

2.1. Construction of Hytönen and Kairema. Let (X, d, μ) be a space of homogeneous type. The doubling property of μ implies the following geometric doubling property of the quasi-metric d: any ball $B(x, r)$ can be covered by at most $N := N(C_\mu, \kappa)$ balls of radius $r/2$. It is not difficult to show that $N \le C_\mu^{6+3\log_2 \kappa}$.

An important tool for our proofs is the concept of an adjacent system of dyadic grids $\mathcal{D}^t$, $t \in \{1, \dots, K\}$, on a space of homogeneous type (X, d, μ). Christ [C90a, Theorem 11] (see also [C90b, Chap. VI, Theorem 14]) constructed a system of sets on (X, d, μ) which satisfy many of the properties of a system of dyadic cubes on the Euclidean space. His construction was further refined by Hytönen and Kairema [HK12, Theorem 2.2]. We will use the version from [AHT17, Theorem 4.1].

Theorem 2.1. *Let (X, d, μ) be a space of homogeneous type with the constant $\kappa \ge 1$ in inequality (1.1) and the geometric doubling constant N. Suppose the parameter $\delta \in (0, 1)$ satisfies $96\kappa^2\delta \le 1$. Then there exist an integer $K = K(\kappa, N, \delta)$, a countable set $\{z_\alpha^{k,t} : \alpha \in \mathcal{A}_k\}$ of points with $k \in \mathbb{Z}$ and $t \in \{1, \dots, K\}$, and a finite number of dyadic grids*

$$\mathcal{D}^t := \{Q_\alpha^{k,t} : k \in \mathbb{Z},\ \alpha \in \mathcal{A}_k\},$$

such that the following properties are fulfilled:

(a) *for every $t \in \{1, \dots, K\}$ and $k \in \mathbb{Z}$ one has*

  (i) $X = \bigcup_{\alpha \in \mathcal{A}_k} Q_\alpha^{k,t}$ *(disjoint union);*
  (ii) *if $Q, P \in \mathcal{D}^t$, then $Q \cap P \in \{\emptyset, Q, P\}$;*
  (iii) *if $Q_\alpha^{k,t} \in \mathcal{D}^t$, then*

$$B(z_\alpha^{k,t}, c_1\delta^k) \subset Q_\alpha^{k,t} \subset B(z_\alpha^{k,t}, C_1\delta^k), \tag{2.1}$$

  where $c_1 = (12\kappa^4)^{-1}$ and $C_1 := 4\kappa^2$;

(b) *for every $t \in \{1, \dots, K\}$ and every $k \in \mathbb{Z}$, if $Q_\alpha^{k,t} \in \mathcal{D}^t$, then there exists at least one $Q_\beta^{k+1,t} \in \mathcal{D}^t$, called a* child *of $Q_\alpha^{k,t}$, such that $Q_\beta^{k+1,t} \subset Q_\alpha^{k,t}$, and there exists exactly one $Q_\gamma^{k-1,t} \in \mathcal{D}^t$, the* parent *of $Q_\alpha^{k,t}$, such that $Q_\alpha^{k,t} \subset Q_\gamma^{k-1,t}$;*

(c) *for every ball $B = B(x, r)$, there exists*

$$Q_B \in \bigcup_{t=1}^{K} \mathcal{D}^t$$

such that $B \subset Q_B$ and $Q_B = Q_\alpha^{k-1,t}$ for some indices $\alpha \in \mathcal{A}_k$ and $t \in \{1, \dots, K\}$, where k is the unique integer such that

$$\delta^{k+1} < r \le \delta^k.$$

The collections $\mathcal{D}^t$, $t \in \{1, \dots, K\}$, are called *dyadic grids* on X. The sets $Q_\alpha^{k,t} \in \mathcal{D}^t$ are referred to as *dyadic cubes* with center $z_\alpha^{k,t}$ and sidelength δ^k; see (2.1). The sidelength of a cube $Q \in \mathcal{D}^t$ will be denoted by $\ell(Q)$. We emphasize that these sets are not cubes in the standard sense even if the underlying space is $\mathbb{R}^n$. Parts (a) and (b) of the above theorem describe dyadic grids $\mathcal{D}^t$, with $t \in \{1, \dots, K\}$, individually. In particular, (2.1) permits a comparison between a dyadic cube and quasi-metric balls. Part (c) guarantees the existence of a finite family of dyadic grids such that an arbitrary quasi-metric ball is contained in a dyadic cube in one of these grids. Such a finite family of dyadic grids is referred to as an *adjacent system of dyadic grids.*

2.2. Dyadic maximal function. Let $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$ be a fixed dyadic grid. One can define the *dyadic maximal function* $M^{\mathcal{D}} f$ of $f \in L^1_{\mathrm{loc}}(X, d, \mu)$ by

$$(M^{\mathcal{D}} f)(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x), \qquad x \in X,$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{D}$ containing x.

The following important theorem is proved by Hytönen and Kairema [HK12, Proposition 7.9].

THEOREM 2.2. *Let (X, d, μ) be a space of homogeneous type and let $\bigcup_{t=1}^{K} \mathcal{D}^t$ be the adjacent system of dyadic grids given by Theorem* 2.1. *There exists a constant $C_{HK}(X) \ge 1$ depending only on (X, d, μ) such that for every $f \in L^1_{\mathrm{loc}}(X, d, \mu)$ and a.e. $x \in X$, one has*

$$(M^{\mathcal{D}^t} f)(x) \le C_{HK}(X)(Mf)(x), \qquad t \in \{1, \dots, K\},$$

$$(Mf)(x) \le C_{HK}(X) \sum_{t=1}^{K} (M^{\mathcal{D}^t} f)(x).$$

2.3. Calderón–Zygmund decomposition of a cube. The following result is a consequence of Theorem 2.1.

LEMMA 2.3. *Suppose (X, d, μ) is a space of homogeneous type with the constants $\kappa \ge 1$ in inequality* (1.1) *and $C_\mu \ge 1$ in inequality* (1.2). *Let (X, d, μ)*

be equipped with an adjacent system of dyadic grids $\{\mathcal{D}^t, t=1,\dots,K\}$ *and let* $\delta\in(0,1)$ *be chosen as in Theorem* 2.1. *Then there is an* $\varepsilon=\varepsilon(\kappa,C_\mu,\delta)\in(0,1)$ *such that for every* $t\in\{1,\dots,K\}$ *and all* $Q,P\in\mathcal{D}^t$, *if* Q *is a child of* P, *then*

$$\mu(Q)\ge\varepsilon\mu(P).$$

Proof. See [AW18, Corollary 2.9] or [K19, Lemma 8]. ∎

Let (X,d,μ) be a space of homogeneous type and $\mathcal{D}=\mathcal{D}^{t_0}\in\bigcup_{t=1}^K\mathcal{D}^t$ be a dyadic grid. Fix $Q_0\in\mathcal{D}$. Then there exist $k_0\in\mathbb{Z}$ and $\alpha_0\in\mathcal{A}_{k_0}$ such that $Q_0=Q_{\alpha_0}^{k_0,t_0}$. Consider

$$\mathcal{D}(Q_0):=\{Q_\alpha^{k,t_0}:k\in\mathbb{Z},\,k\ge k_0,\,\alpha\in\mathcal{A}_k\}=\{Q'\in\mathcal{D}^{t_0}:Q'\subset Q\},\tag{2.2}$$

that is, the set of all dyadic cubes with respect to Q_0. The set $\mathcal{D}(Q_0)$ is formed by all dyadic descendants of the cube Q_0. For a measurable function f such that

$$\int_{Q_0}|f(x)|\,d\mu(x)<\infty,\tag{2.3}$$

consider the *local dyadic maximal function* of f defined by

$$(M^{\mathcal{D}(Q_0)}f)(x):=\sup_{Q\ni x,\,Q\in\mathcal{D}(Q_0)}\frac{1}{\mu(Q)}\int_Q|f(x)|\,d\mu(x),\qquad x\in Q_0.$$

Given a dyadic grid $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$, a *sparse family* $S\subset\mathcal{D}$ is a collection of dyadic cubes $Q\in\mathcal{D}$ for which there exists a collection of sets $\{E(Q)\}_{Q\in S}$ such that the sets $E(Q)$ are pairwise disjoint, $E(Q)\subset Q$, and

$$\mu(Q)\le2\mu(E(Q)).$$

We will need the following variation of the Calderón–Zygmund decomposition of the cube Q_0 (cf. [L17, Lemma 2.4]).

THEOREM 2.4. *Let* (X,d,μ) *be a space of homogeneous type,* $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$ *be a dyadic grid,* $Q_0\in\mathcal{D}$, *and* $\mathcal{D}(Q_0)$ *be defined by* (2.2). *Suppose* $\varepsilon\in(0,1)$ *is as in Lemma* 2.3. *For a nonzero measurable function* f *on* Q_0 *satisfying* (2.3) *and* $k\in\mathbb{N}$, *set*

$$\Omega_k(Q_0):=\left\{x\in Q_0:(M^{\mathcal{D}(Q_0)}f)(x)>\left(\frac{2}{\varepsilon}\right)^k\frac{1}{\mu(Q_0)}\int_{Q_0}|f(x)|\,d\mu(x)\right\}.\tag{2.4}$$

If $\Omega_k(Q_0)\ne\emptyset$, *then there exists a collection* $\{Q_j^k(Q_0)\}_{j\in J_k}\subset\mathcal{D}(Q_0)$ *that is pairwise disjoint, maximal with respect to inclusion, and such that*

$$\Omega_k(Q_0)=\bigcup_{j\in J_k}Q_j^k(Q_0).\tag{2.5}$$

The collection of cubes

$$S := \{Q_j^k(Q_0) : \Omega_k(Q_0) \neq \emptyset,\ j \in J_k\}$$

is sparse, and for all j and k, the sets

$$E(Q_j^k(Q_0)) := Q_j^k(Q_0) \setminus \Omega_{k+1}(Q_0)$$

satisfy

$$\mu(Q_j^k(Q_0)) \leq 2\mu\big(E(Q_j^k(Q_0))\big). \tag{2.6}$$

Proof. For each $k \in \mathbb{N}$ satisfying $\Omega_k(Q_0) \neq \emptyset$, the existence of a pairwise disjoint and inclusion-maximal collection $\{Q_j^k(Q_0)\}_{j\in J_k}$, such that (2.5) is fulfilled, follows from [K19, Theorem 9(a)]. Moreover, in view of the same theorem, for every $k \in \mathbb{N}$ satisfying $\Omega_k(Q_0) \neq \emptyset$ and $j \in J_k$, one has

$$\begin{aligned}\left(\frac{2}{\varepsilon}\right)^k \frac{1}{\mu(Q_0)} \int_{Q_0} |f(x)|\,d\mu(x) &< \frac{1}{\mu(Q_j^k(Q_0))} \int_{Q_j^k(Q_0)} |f(x)|\,d\mu(x) \\ &\leq \left(\frac{2}{\varepsilon}\right)^k \frac{1}{\varepsilon\mu(Q_0)} \int_{Q_0} |f(x)|\,d\mu(x).\end{aligned} \tag{2.7}$$

It remains to prove (2.6). Since $\Omega_{k+1}(Q_0) \subset \Omega_k(Q_0)$ and, for each fixed k, the cubes $Q_j^k(Q_0)$ are pairwise disjoint, it is clear that the sets $E(Q_j^k(Q_0))$ are pairwise disjoint for all j and k. If $Q_j^k(Q_0) \cap Q_i^{k+1}(Q_0) \neq \emptyset$, then by the maximality of the cubes in $\{Q_j^k(Q_0)\}_{j\in J_k}$ and the fact that $2/\varepsilon > 1$, we have $Q_i^{k+1}(Q_0) \subsetneq Q_j^k(Q_0)$. In view of (2.5) and (2.7), we see that

$$\begin{aligned}\mu\big(Q_j^k(Q_0) \cap \Omega_{k+1}(Q_0)\big) &= \sum_{\{i\,:\,Q_i^{k+1}(Q_0)\subsetneq Q_j^k(Q_0)\}} \mu(Q_i^{k+1}(Q_0)) \\ &\leq \sum_{\{i\,:\,Q_i^{k+1}(Q_0)\subsetneq Q_j^k(Q_0)\}} \frac{(\varepsilon/2)^{k+1} \int_{Q_i^{k+1}(Q_0)} |f(x)|\,d\mu(x)}{\frac{1}{\mu(Q_0)} \int_{Q_0} |f(x)|\,d\mu(x)} \\ &\leq \frac{(\varepsilon/2)^{k+1} \int_{Q_j^k(Q_0)} |f(x)|\,d\mu(x)}{\frac{1}{\mu(Q_0)} \int_{Q_0} |f(x)|\,d\mu(x)} \\ &\leq \left(\frac{\varepsilon}{2}\right)^{k+1} \left(\frac{2}{\varepsilon}\right)^k \frac{\mu(Q_j^k(Q_0))}{\varepsilon} = \frac{\mu(Q_j^k(Q_0))}{2}.\end{aligned}$$

Then

$$\begin{aligned}\mu\big(E(Q_j^k(Q_0))\big) &= \mu\big(Q_j^k(Q_0) \setminus \Omega_{k+1}(Q_0)\big) \\ &= \mu(Q_j^k(Q_0)) - \mu\big(Q_j^k(Q_0) \cap \Omega_{k+1}(Q_0)\big) \\ &\geq (1 - 1/2)\mu(Q_j^k(Q_0)),\end{aligned}$$

whence $\mu(Q_j^k(Q_0)) \le 2\mu(E(Q_j^k(Q_0)))$ for all j and k, which completes the proof of (2.6). ∎

3. Hardy–Littlewood maximal operator on the associate space of a Banach function space

3.1. Banach function spaces. Let us recall the definition of a Banach function space (see, e.g., [BS88, Chap. 1, Definition 1.1]). Let $L_+^0(X,d,\mu)$ be the set of all nonnegative measurable functions on X. The characteristic function of a set $E \subset X$ is denoted by χ_E. A mapping $\rho : L_+^0(X,d,\mu) \to [0,\infty]$ is called a *Banach function norm* if, for all functions f, g, f_n $(n \in \mathbb{N})$ in the set $L_+^0(X,d,\mu)$, for all constants $a \ge 0$, and for all measurable subsets E of X, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f+g) \le \rho(f) + \rho(g)$,

(A2) $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the lattice property),

(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\mu(E) < \infty \;\Rightarrow\; \rho(\chi_E) < \infty$,

(A5) $\int_E f(x)\,d\mu(x) \le C_E \rho(f)$

with a constant $C_E \in (0,\infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set $\mathcal{E}(X,d,\mu)$ of all functions $f \in L^0(X,d,\mu)$ for which $\rho(|f|) < \infty$ is called a *Banach function space.* For each $f \in \mathcal{E}(X,d,\mu)$, the norm of f is defined by

$$\|f\|_{\mathcal{E}} := \rho(|f|).$$

The set $\mathcal{E}(X,d,\mu)$ under the natural linear space operations and under this norm becomes a Banach space (see [BS88, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its *associate norm* ρ' is defined on $L_+^0(X,d,\mu)$ by

$$\rho'(g) := \sup\Big\{ \int_X f(x)g(x)\,d\mu(x) : f \in L_+^0(X,d,\mu),\ \rho(f) \le 1 \Big\}.$$

It is a Banach function norm itself [BS88, Chap. 1, Theorem 2.2]. The Banach function space $\mathcal{E}'(X,d,\mu)$ determined by the Banach function norm ρ' is called the *associate space* (or *Köthe dual*) of $\mathcal{E}(X,d,\mu)$.

3.2. The condition $\mathcal{A}_\infty$. Following [L17] and [K19, Definition 1], we say that a Banach function space $\mathcal{E}(X,d,\mu)$ over a space (X,d,μ) of homogeneous type satisfies the condition $\mathcal{A}_\infty$ if there exist constants $\Phi, \theta > 0$ such that for every dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$, every finite sparse family $S \subset \mathcal{D}$, every collection $\{\alpha_Q\}_{Q\in S}$ of nonnegative numbers, and every collection $\{G_Q\}_{Q\in S}$ of pairwise disjoint measurable sets such that $G_Q \subset Q$, one has

$$\Big\| \sum_{Q\in S} \alpha_Q \chi_{G_Q} \Big\|_{\mathcal{E}} \le \Phi\bigg(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)} \bigg)^{\theta} \Big\| \sum_{Q\in S} \alpha_Q \chi_Q \Big\|_{\mathcal{E}}. \tag{3.1}$$

The following result is a generalization of [L17, Theorem 3.1] from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

THEOREM 3.1 ([K19, Theorem 2]). *Let $\mathcal{E}(X,d,\mu)$ be a Banach function space over a space of homogeneous type (X,d,μ) and let $\mathcal{E}'(X,d,\mu)$ be its associate space.*

(a) *If the Hardy–Littlewood maximal operator M is bounded on $\mathcal{E}'(X,d,\mu)$, then $\mathcal{E}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$.*
(b) *If the Hardy–Littlewood maximal operator M is bounded on $\mathcal{E}(X,d,\mu)$, and $\mathcal{E}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$, then the M is bounded on $\mathcal{E}'(X,d,\mu)$.*

Since the variable Lebesgue space $L^{p(\cdot)}(X,d,\mu)$ is a Banach function space and, under the condition $1 < p_-$, $p_+ < \infty$, its associate space $[L^{p(\cdot)}(X,d,\mu)]'$ is isomorphic to the variable Lebesgue space $L^{p'(\cdot)}(X,d,\mu)$ (see, e.g., [CF13, Section 2.10.3]), Theorem 3.1(b) immediately implies the following.

COROLLARY 3.2. *Let (X,d,μ) be a space of homogeneous type and let $p \in \mathcal{P}(X)$ be such that $1 < p_-$, $p_+ < \infty$. If the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(X,d,\mu)$, and $L^{p(\cdot)}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$, then M is bounded on the dual space $L^{p'(\cdot)}(X,d,\mu)$.*

It follows from Corollary 3.2 that in order to prove Theorem 1.1, it is sufficient to verify that $L^{p(\cdot)}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$.

4. Norm inequalities and modular inequalities

4.1. Norm-modular unit ball property. In this subsection we formulate two very useful properties that relate norms and modulars in variable Lebesgue spaces.

LEMMA 4.1 (see, e.g., [DH⁺11, Lemma 3.2.4]). *Let (X,d,μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$. Then for every $f \in L^{p(\cdot)}(X,d,\mu)$ the inequalities $\|f\|_{L^{p(\cdot)}} \le 1$ and $\varrho_{p(\cdot)}(f) \le 1$ are equivalent.*

LEMMA 4.2 (see, e.g., [DH⁺11, Lemma 3.2.5]). *Let (X,d,μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$ be such that $1 < p_-$, $p_+ < \infty$. Then for every $f \in L^{p(\cdot)}(X,d,\mu)$,*

$$\min\{\varrho_{p(\cdot)}(f)^{1/p_-}, \varrho_{p(\cdot)}(f)^{1/p_+}\} \le \|f\|_{L^{p(\cdot)}} \le \max\{\varrho_{p(\cdot)}(f)^{1/p_-}, \varrho_{p(\cdot)}(f)^{1/p_+}\}.$$

4.2. Auxiliary lemma. The following auxiliary lemma illustrates the possibility of substitution of norm inequalities by modular inequalities.

LEMMA 4.3. *Let (X,d,μ) be a space of homogeneous type and let $p \in \mathcal{P}(X)$ satisfy $1 < p_-, p_+ < \infty$. Suppose $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$ is a dyadic grid. If $S \in \mathcal{D}$ is a finite family and $\{\alpha_Q\}_{Q\in S}$ is a family of nonnegative numbers such that*

$$\Big\|\sum_{Q\in S} \alpha_Q \chi_Q\Big\|_{L^{p(\cdot)}} \le 1,$$

then

$$\sum_{Q\in S}\int_Q \alpha_Q^{p(x)}\,\mu(x) \le 1.$$

Proof. It is clear that

$$\sum_{Q\in S}\int_Q \alpha_Q^{p(x)}\,d\mu(x) = \int_X \Big(\sum_{Q\in S}(\alpha_Q\chi_Q(x))^{p(x)}\Big)\,d\mu(x). \tag{4.1}$$

Since $1 < p_- \le p(x) \le p_+ < \infty$ for a.e. $x \in X$, one has

$$\sum_{Q\in S}(\alpha_Q\chi_Q(x))^{p(x)} \le \Big(\sum_{Q\in S}\alpha_Q\chi_Q(x)\Big)^{p(x)}. \tag{4.2}$$

Taking into account (4.1) and (4.2), it follows from Lemma 4.1 that

$$\sum_{Q\in S}\int_Q \alpha_Q^{p(x)}\,d\mu(x) \le \int_X \Big(\sum_{Q\in S}\alpha_Q\chi_Q(x)\Big)^{p(x)}\,d\mu(x) \le 1,$$

which completes the proof. ∎

4.3. Modular version of the condition $\mathcal{A}_\infty$. In this subsection we formulate a modular analogue of the condition $\mathcal{A}_\infty$ and show that it implies the (norm) condition $\mathcal{A}_\infty$.

LEMMA 4.4. *Let (X,d,μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$ satisfy $1 < p_-, p_+ < \infty$. If there exist constants $\Psi, \xi > 1$ such that for every dyadic grid $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$, every finite sparse family $S \subset \mathcal{D}$, every collection $\{G_Q\}_{Q\in S}$ of pairwise disjoint measurable sets such that $G_Q \subset Q$ and every collection $\{\alpha_Q\}_{Q\in S}$ of nonnegative numbers such that*

$$\Big\|\sum_{Q\in S} \alpha_Q \chi_Q\Big\|_{L^{p(\cdot)}} = 1, \tag{4.3}$$

one has

$$\sum_{Q\in S}\int_{G_Q} \alpha_Q^{p(x)}\,d\mu(x) \le \Psi\left(\max_{Q\in S}\frac{\mu(G_Q)}{\mu(Q)}\right)^{\xi}, \tag{4.4}$$

then the variable Lebesgue space $L^{p(\cdot)}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$.

Proof. Fix a dyadic grid $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$, a finite sparse family $S \subset \mathcal{D}$, and a collection $\{G_Q\}_{Q\in S}$ of pairwise disjoint measurable sets such that $G_Q \subset Q$. Let $\{\beta_Q\}_{Q\in S}$ be an arbitrary collection of nonnegative numbers. Put

$$\alpha_Q := \frac{\beta_Q}{\|\sum_{Q\in S} \beta_Q \chi_Q\|_{L^{p(\cdot)}}}. \tag{4.5}$$

Then (4.3) is fulfilled. Since the sets $\{G_Q\}_{Q\in S}$ are pairwise disjoint, we have

$$\Big(\sum_{Q\in S} \alpha_Q \chi_{G_Q}(x)\Big)^{p(x)} = \sum_{Q\in S} \alpha_Q^{p(x)} \chi_{G_Q}(x), \qquad x \in X.$$

Hence

$$\sigma := \sum_{Q\in S} \int_{G_Q} \alpha_Q^{p(x)}\, d\mu(x) = \int_X \Big(\sum_{Q\in S} \alpha_Q \chi_{G_Q}(x)\Big)^{p(x)} d\mu(x). \tag{4.6}$$

By Lemma 4.2, (4.4) and (4.6), we have

$$\begin{aligned}\Big\|\sum_{Q\in S} \alpha_Q \chi_{G_Q}\Big\|_{L^{p(\cdot)}} &\le \max\{\sigma^{1/p_-}, \sigma^{1/p_+}\} \\ &\le \max\bigg\{\Psi^{1/p_-}\Big(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\Big)^{\xi/p_-}, \Psi^{1/p_+}\Big(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\Big)^{\xi/p_+}\bigg\} \\ &\le \Psi^{1/p_-}\Big(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\Big)^{\xi/p_+}\end{aligned} \tag{4.7}$$

because $\Psi > 1$, $\mu(G_Q) \le \mu(Q)$ for all $Q \in S$ and $p_- \le p_+$. It follows from (4.5) and (4.7) that

$$\Big\|\sum_{Q\in S} \beta_Q \chi_{G_Q}\Big\|_{L^{p(\cdot)}} \le \Psi^{1/p_-}\Big(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\Big)^{\xi/p_+} \Big\|\sum_{Q\in S} \beta_Q \chi_Q\Big\|_{L^{p(\cdot)}},$$

that is, the space $L^{p(\cdot)}(X, d, \mu)$ satisfies the condition $\mathcal{A}_\infty$ with $\Phi = \Psi^{1/p_-}$ and $\theta = \xi/p_+$. ■

5. Preparations for the verification of the condition $\mathcal{A}_\infty$

5.1. First lemma. Let $\|M\|_{\mathcal{B}(L^{p(\cdot)})}$ denote the norm of the Hardy–Littlewood maximal operator on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$. As usual, for an exponent $r \in (1, \infty)$, let $r' = r/(r-1) \in (1, \infty)$ denote the conjugate exponent.

The preparation for the verification of the condition $\mathcal{A}_\infty$ in the proof of Theorem 1.1 consists of four steps. The first step is the proof of the following extension of [L17, Lemma 5.1] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

Lemma 5.1. *Let (X,d,μ) be a space of homogeneous type and $p\in\mathcal{P}(X)$ satisfy $1<p_-,p_+<\infty$. Suppose the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(X,d,\mu)$. There exist constants $A,\lambda>1$ such that for every dyadic grid $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$, every family $S_d\subset\mathcal{D}$ of pairwise disjoint cubes, every family $\{\alpha_Q\}_{Q\in S_d}$ of nonnegative numbers, if*

$$\sum_{Q\in S_d}\int_Q \alpha_Q^{p(x)}\,d\mu(x)\le 1, \tag{5.1}$$

then

$$\sum_{Q\in S_d}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda}\le A. \tag{5.2}$$

Proof. Let $\varepsilon\in(0,1)$ be as in Lemma 2.3. Fix a dyadic grid $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$ and a family of pairwise disjoint cubes $S_d\subset\mathcal{D}$. For $k\in\mathbb{N}$ and $Q\in S_d$, put

$$\Omega_k(Q):=\left\{x\in Q:(M^{\mathcal{D}(Q)}\alpha_Q^{p(\cdot)})(x)>\left(\frac{2}{\varepsilon}\right)^k\frac{1}{\mu(Q)}\int_Q\alpha_Q^{p(x)}\,d\mu(x)\right\}. \tag{5.3}$$

By Theorem 2.4, if these sets are nonempty, then they can be written as

$$\Omega_k(Q)=\bigcup_j Q_j^k(Q), \tag{5.4}$$

where $Q_j^k(Q)\in\mathcal{D}(Q)$ are pairwise disjoint cubes for all j and k, and

$$\mu(Q_j^k(Q)\setminus\Omega_{k+1}(Q))\ge\frac{1}{2}\mu(Q_j^k(Q)). \tag{5.5}$$

Fix $k\in\mathbb{N}$ and $Q\in S_d$. If $x\in\Omega_k(Q)$, then in view of (5.4) there exists j_0 such that $x\in Q_{j_0}^k(Q)$. It follows from (5.5) that

$$\begin{aligned}\chi_{\Omega_k(Q)}(x)&\le\frac{2\mu\big(Q_{j_0}^k(Q)\setminus\Omega_{k+1}(Q)\big)}{\mu(Q_{j_0}^k(Q))}\\&=\frac{2\mu\big(Q_{j_0}^k(Q)\cap(\Omega_k(Q)\setminus\Omega_{k+1}(Q))\big)}{\mu(Q_{j_0}^k(Q))}\\&=\frac{2}{\mu(Q_{j_0}^k(Q))}\int_{Q_{j_0}^k(Q)}\chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}(y)\,d\mu(y)\\&\le 2\sup_{Q'\ni x,\,Q'\in\mathcal{D}}\frac{1}{\mu(Q')}\int_{Q'}\chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}(y)\,d\mu(y),\end{aligned}$$

which implies that

$$\chi_{\Omega_k(Q)}(x)\le 2(M^{\mathcal{D}}\chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)})(x),\qquad x\in X.$$

Therefore, for all $k \in \mathbb{N}$ and $Q \in S_d$,

$$\alpha_Q \chi_{\Omega_k(Q)}(x) \le 2\big(M^{\mathcal{D}}(\alpha_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)})\big)(x) \le 2\Big(M^{\mathcal{D}}\Big(\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big)\Big)(x), \quad x \in X.$$

Since the cubes in S_d are pairwise disjoint, the sets in $\{\Omega_k(Q)\}_{Q\in S_d}$ are also pairwise disjoint for every fixed $k \in \mathbb{N}$. Hence, the above inequality implies that for $k \in \mathbb{N}$,

$$\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)}(x) \le 2\Big(M^{\mathcal{D}}\Big(\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big)\Big)(x), \quad x \in X. \tag{5.6}$$

It follows from the boundedness of M on $L^{p(\cdot)}(X, d, \mu)$, Theorem 2.2, and (5.6) that

$$\Big\|\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}} \le 2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}\Big\|\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}}. \tag{5.7}$$

Set

$$\widetilde{\alpha}_Q := \alpha_Q \Big(\Big\|\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}}\Big)^{-1}. \tag{5.8}$$

Then (5.7) can be rewritten as

$$\frac{1}{2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}} \le \Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}}. \tag{5.9}$$

It follows from (5.8) that

$$\Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \le \Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}} = 1. \tag{5.10}$$

Inequality (5.10) and Lemma 4.2 imply that

$$\Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \le \Big(\int_X \Big(\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}(x)\Big)^{p(x)} d\mu(x)\Big)^{1/p_+}. \tag{5.11}$$

Since the cubes in S_d are pairwise disjoint, so are the sets in the collection $\{\Omega_k(Q) \setminus \Omega_{k+1}(Q)\}_{Q\in S_d}$. Therefore, we deduce from (5.9) and (5.11) that

$$\left(\frac{1}{2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}}\right)^{p_+} \tag{5.12}$$
$$\leq \int_X \Big(\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)}(x)\Big)^{p(x)}\, d\mu(x)$$
$$= \sum_{Q\in S_d} \int_{\Omega_k(Q)\setminus\Omega_{k+1}(Q)} (\widetilde{\alpha}_Q)^{p(x)}\, d\mu(x)$$
$$= \sum_{Q\in S_d} \int_{\Omega_k(Q)} (\widetilde{\alpha}_Q)^{p(x)}\, d\mu(x) - \sum_{Q\in S_d} \int_{\Omega_{k+1}(Q)} (\widetilde{\alpha}_Q)^{p(x)}\, d\mu(x).$$

Again, taking into account that the cubes in S_d are pairwise disjoint, we deduce from (5.10) and Lemma 4.1 that

$$\sum_{Q\in S_d} \int_{\Omega_k(Q)} (\widetilde{\alpha}_Q)^{p(x)}\, d\mu(x) = \int_X \Big(\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)}(x)\Big)^{p(x)}\, d\mu(x) \leq \Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}} = 1. \tag{5.13}$$

Since $\Omega_{k+1}(Q) \subset \Omega_k(Q)$, it follows from (5.10) that

$$\Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \leq \Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}} = 1.$$

Hence, in view of Lemma 4.2, we have

$$\Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \leq \Big(\int_X \Big(\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_{k+1}(Q)}(x)\Big)^{p(x)}\, d\mu(x)\Big)^{1/p_+} = \Big(\sum_{Q\in S_d} \int_{\Omega_{k+1}(Q)} (\widetilde{\alpha}_Q)^{p(x)}\, d\mu(x)\Big)^{1/p_+}. \tag{5.14}$$

It follows from (5.12)–(5.14) that

$$\Big\|\sum_{Q\in S_d} \widetilde{\alpha}_Q \chi_{\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \leq \left\{1 - \left(\frac{1}{2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}}\right)^{p_+}\right\}^{1/p_+} =: \beta.$$

The above inequality and (5.8) imply that for $k \in \mathbb{N}$,

$$\Big\|\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_{k+1}(Q)}\Big\|_{L^{p(\cdot)}} \leq \beta \Big\|\sum_{Q\in S_d} \alpha_Q \chi_{\Omega_k(Q)}\Big\|_{L^{p(\cdot)}}. \tag{5.15}$$

It follows from (5.1) and Lemma 4.1 that

$$\Big\|\sum_{Q\in S_d} \alpha_Q \chi_Q\Big\|_{L^{p(\cdot)}} \leq 1. \tag{5.16}$$

Since $\Omega_1(Q) \subset Q$, applying (5.16) and then applying (5.15) $k-1$ times, we get

$$1 \geq \Big\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_1(Q)} \Big\|_{L^{p(\cdot)}} \geq \frac{1}{\beta} \Big\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_2(Q)} \Big\|_{L^{p(\cdot)}} \geq \dots$$
$$\geq \frac{1}{\beta^{k-1}} \Big\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \Big\|_{L^{p(\cdot)}}.$$

Thus

$$\Big\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \Big\|_{L^{p(\cdot)}} \leq \beta^{k-1}, \qquad k \in \mathbb{N}.$$

In view of Lemma 4.2, this inequality implies that

$$\sum_{Q \in S_d} \int_{\Omega_k(Q)} \alpha_Q^{p(x)} \, d\mu(x) \leq \beta^{p_-(k-1)}, \qquad k \in \mathbb{N}. \tag{5.17}$$

Fix $Q \in S_d$. Put $\Omega_0(Q) := Q$. Then it follows from (5.3) that for $k \in \mathbb{Z}_+$ and $x \in \Omega_k(Q) \setminus \Omega_{k+1}(Q)$, one has

$$\alpha_Q^{p(x)} \leq (M^{\mathcal{D}(Q)} \alpha_Q^{p(\cdot)})(x) \leq \left(\frac{2}{\varepsilon}\right)^{k+1} \frac{1}{\mu(Q)} \int_Q \alpha_Q^{p(y)} \, d\mu(y).$$

From this inequality we obtain, for every $\phi > 0$,

$$\begin{aligned} \int_Q \alpha_Q^{p(x)(1+\phi)} \, d\mu(x) &= \sum_{k=0}^{\infty} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \alpha_Q^{p(x)(1+\phi)} \, d\mu(x) \\ &\leq \left(\frac{1}{\mu(Q)} \int_Q \alpha_Q^{p(y)} \, d\mu(y)\right)^{\phi} \sum_{k=0}^{\infty} \left(\frac{2}{\varepsilon}\right)^{\phi(k+1)} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \alpha_Q^{p(x)} \, d\mu(x) \\ &\leq \left(\frac{1}{\mu(Q)} \int_Q \alpha_Q^{p(y)} \, d\mu(y)\right)^{\phi} \sum_{k=0}^{\infty} \left(\frac{2}{\varepsilon}\right)^{\phi(k+1)} \int_{\Omega_k(Q)} \alpha_Q^{p(x)} \, d\mu(x). \end{aligned} \tag{5.18}$$

It is easy to see that one can choose $\phi > 0$ such that

$$0 < (2/\varepsilon)^{\phi} \beta^{p_-} < 1.$$

Then

$$\sum_{k=0}^{\infty} [(2/\varepsilon)^{\phi} \beta^{p_-}]^{k+1} < \infty. \tag{5.19}$$

Take $\lambda := 1 + \phi$. By (5.18), we have

$$(5.20)\qquad \sum_{Q\in S_d}\mu(Q)\Big(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{\lambda p(x)}\,d\mu(x)\Big)^{1/\lambda}$$
$$\le \sum_{Q\in S_d}\mu(Q)\Big(\frac{1}{\mu(Q)}\Big(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{p(y)}\,d\mu(y)\Big)^{\phi}\Big)^{1/\lambda}$$
$$\times\Big(\sum_{k=0}^{\infty}(2/\varepsilon)^{\phi(k+1)}\int_{\Omega_k(Q)}\alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda}.$$

Since $\frac{1}{\lambda}=\frac{1}{1+\phi}$ and $\frac{1}{\lambda'}=\frac{\phi}{1+\phi}$, we have

$$(5.21)\qquad \mu(Q)\Big(\frac{1}{\mu(Q)}\Big(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{p(y)}\,d\mu(y)\Big)^{\phi}\Big)^{1/\lambda}$$
$$=\mu(Q)\Big(\frac{1}{(\mu(Q))^{1+\phi}}\Big)^{1/\lambda}\Big(\int_Q \alpha_Q^{p(x)}\,d\mu(x)\Big)^{\phi/\lambda}=\Big(\int_Q \alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda'}.$$

Combining (5.20) with (5.21), applying Hölder's inequality, and taking (5.1) into account, we obtain

$$(5.22)\qquad \sum_{Q\in S_d}\mu(Q)\Big(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{\lambda p(x)}\,d\mu(x)\Big)^{1/\lambda}$$
$$\le \sum_{Q\in S_d}\Big(\int_Q \alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda'}\Big(\sum_{k=0}^{\infty}\Big(\frac{2}{\varepsilon}\Big)^{\phi(k+1)}\int_{\Omega_k(Q)}\alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda}$$
$$\le \Big(\sum_{Q\in S_d}\int_Q \alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda'}\Big(\sum_{Q\in S_d}\sum_{k=0}^{\infty}\Big(\frac{2}{\varepsilon}\Big)^{\phi(k+1)}\int_{\Omega_k(Q)}\alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda}$$
$$=\Big(\sum_{k=0}^{\infty}\Big(\frac{2}{\varepsilon}\Big)^{\phi(k+1)}\sum_{Q\in S_d}\int_{\Omega_k(Q)}\alpha_Q^{p(x)}\,d\mu(x)\Big)^{1/\lambda}.$$

It follows from (5.1), (5.17) and (5.22) that

$$\sum_{Q\in S_d}\mu(Q)\Big(\frac{1}{\mu(Q)}\int_Q \alpha_Q^{\lambda p(x)}\,d\mu(x)\Big)^{1/\lambda}$$
$$\le\Big\{\Big(\frac{2}{\varepsilon}\Big)^{\phi}\sum_{Q\in S_d}\int_Q \alpha_Q^{p(x)}\,d\mu(x)+\sum_{k=1}^{\infty}\Big(\frac{2}{\varepsilon}\Big)^{\phi(k+1)}\beta^{p_-(k-1)}\Big\}^{1/\lambda}$$
$$\le\Big\{\Big(\frac{2}{\varepsilon}\Big)^{\phi}+\sum_{k=1}^{\infty}\Big(\frac{2}{\varepsilon}\Big)^{\phi(k+1)}\beta^{p_-(k-1)}\Big\}^{1/\lambda}=:A.$$

Combining $2/\varepsilon > 1$ and (5.19), we see that $A \in (1,\infty)$, which completes the proof of (5.2). ∎

5.2. Second lemma. The next lemma generalizes [L17, Lemma 5.2] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

LEMMA 5.2. *Let (X,d,μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$ satisfy $1 < p_-, p_+ < \infty$. Suppose the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(X,d,\mu)$. There exist constants $B,\lambda > 1$ and a measure ν on X such that for every dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ and every finite family $S_d \subset \mathcal{D}$ of pairwise disjoint cubes, the following properties hold:*

(i) *if $Q \in \mathcal{D}$ and $t \ge 0$ satisfy*

$$\int_Q t^{p(x)}\,d\mu(x) \le 1, \tag{5.23}$$

then

$$\mu(Q)\bigg(\frac{1}{\mu(Q)}\int_Q t^{\lambda p(x)}\,d\mu(x)\bigg)^{1/\lambda} \le B\int_Q t^{p(x)}\,d\mu(x) + \nu(Q); \tag{5.24}$$

(ii) $\sum_{Q\in S_d} \nu(Q) \le 2B$.

Proof. (i) Let $A,\lambda > 1$ be the constants from Lemma 5.1. Set

$$B := 2^{p_+/p_- + 1}A. \tag{5.25}$$

Fix a dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$. Given a cube $Q \in \mathcal{D}$, consider the functions

$$F_1(t) := \int_Q t^{p(x)}\,d\mu(x), \qquad F_2(t) := \mu(Q)\bigg(\frac{1}{\mu(Q)}\int_Q t^{\lambda p(x)}\,d\mu(x)\bigg)^{1/\lambda}, \qquad t \ge 0,$$

and the set

$$A(Q) := \{t > 0 : F_1(t) \le 1,\ F_2(t) > BF_1(t)\}.$$

Set

$$t_Q := \begin{cases} 0 & \text{if } A(Q) = \emptyset, \\ \sup A(Q) & \text{if } A(Q) \ne \emptyset. \end{cases}$$

We claim that

$$F_1(t_Q) < 1. \tag{5.26}$$

Indeed, if $F_1(t_Q) = 1$, then by the continuity of F_1 and F_2, we would have $F_2(t_Q) \ge B > A$, and this would contradict Lemma 5.1.

Further,

$$F_2(t_Q) = BF_1(t_Q). \tag{5.27}$$

Indeed, otherwise $F_2(t_Q) > BF_1(t_Q)$, which together with (5.26) and the continuity of F_1 and F_2 would imply that there exists $\varepsilon > 0$ such that

$$F_1(t_Q+\varepsilon) < 1, \qquad F_2(t_Q+\varepsilon) > BF_1(t_Q+\varepsilon),$$

and these inequalities would contradict the definition of t_Q.

Set

$$\nu(Q) := F_2(t_Q) \tag{5.28}$$

and suppose that (5.23) is fulfilled. Since F_2 is increasing, we see that

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda} \le \nu(Q), \qquad t \le t_Q. \tag{5.29}$$

On the other hand, if $t > t_Q$, then $t \notin A(Q)$, whence $F_2(t) \le BF_1(t)$, that is,

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda} \le B\int_Q t^{p(x)}\,d\mu(x), \qquad t > t_Q. \tag{5.30}$$

Combining (5.29) and (5.30), we immediately arrive at (5.24), as desired.

(ii) Consider an arbitrary finite family $S_d \subset \mathcal{D}$ of pairwise disjoint cubes. Among all subsets $\widetilde{S}_d \subset S_d$ such that

$$\sum_{Q\in\widetilde{S}_d}\int_Q t_Q^{p(x)}\,d\mu(x) \le 2, \tag{5.31}$$

we choose a maximal subset S'_d that is, a subset containing the largest number of cubes (it is not unique, in general).

We claim that $S'_d = S_d$. Indeed, assuming that $S'_d \subsetneq S_d$ and taking into account that F_1 is increasing, we have

$$\sum_{Q\in S'_d}\int_Q (t_Q/2^{1/p_-})^{p(x)}\,d\mu(x) \le \sum_{Q\in S'_d}\int_Q (t_Q^{p(x)}/2)\,d\mu(x) \le 1.$$

By Lemma 5.1, this implies that

$$\sum_{Q\in S'_d}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{t_Q}{2^{1/p_-}}\right)^{\lambda p(x)}d\mu(x)\right)^{1/\lambda} \le A.$$

Since F_2 is increasing, the above inequality yields

$$\sum_{Q\in S'_d}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{1}{2^{p_+/p_-}}\right)^{\lambda} t_Q^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda} \le A,$$

whence

$$\sum_{Q\in S'_d}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t_Q^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda} \le 2^{p_+/p_-}A.$$

This inequality and (5.25) and (5.27) imply that

$$\sum_{Q\in S'_d}\int_Q t_Q^{p(x)}\,d\mu(x) = \frac{1}{B}\sum_{Q\in S'_d}\mu(Q)\bigg(\frac{1}{\mu(Q)}\int_Q t_Q^{\lambda p(x)}\,d\mu(x)\bigg)^{1/\lambda} \le \frac{1}{2}.$$

Now let $P \in S_d \setminus S'_d$. Then, taking (5.26) into account, we get

$$\sum_{Q\in S'_d\cup\{P\}}\int_Q t_Q^{p(x)}\,d\mu(x) \le \frac{1}{2} + \int_P t_P^{p(x)}\,d\mu(x) < \frac{3}{2}.$$

This inequality, in view of (5.31), contradicts the maximality of S'_d. This proves that $S'_d = S_d$. It follows from (5.31), (5.27) and (5.28) that

$$\sum_{Q\in S_d}\nu(Q) = B\sum_{Q\in S_d}\int_Q t_Q^{p(x)}\,d\mu(x) \le 2B,$$

which completes the proof of (ii). ■

5.3. Third lemma. The next lemma is an extension of [L17, Lemma 5.3] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

LEMMA 5.3. *Let (X,d,μ) be a space of homogeneous type and $p \in \mathcal{P}(X)$ satisfy $1 < p_-, p_+ < \infty$. Suppose the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(X,d,\mu)$. There exist constants $D,\gamma > 1$ and $\zeta > 0$ such that for every dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ and every cube $Q \in \mathcal{D}$, if*

$$\min\bigg\{1, \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\bigg\} \le t \le \max\bigg\{1, \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\bigg\}, \tag{5.32}$$

then

$$\bigg(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\bigg)^{1/\gamma} \le \frac{D}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x). \tag{5.33}$$

Proof. Let $A > 0$ and $\lambda > 1$ be the constants of Lemma 5.1. Take any γ satisfying $1 < \gamma < \lambda$ and set

$$\zeta := \frac{\lambda-\gamma}{\gamma(1+\lambda)} > 0.$$

Fix a dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ and a cube $Q \in \mathcal{D}$. For any $\alpha > 0$, we have

$$\bigg(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\bigg)^{1/\gamma} = \bigg(\frac{1}{\mu(Q)}\int_Q t^{\gamma(p(x)-\alpha)}\,d\mu(x)\bigg)^{1/\gamma} t^{\alpha}. \tag{5.34}$$

It follows from (5.32) that either

$$1 \le t \le \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}, \tag{5.35}$$

or

$$\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}} \le t \le 1. \tag{5.36}$$

If (5.35) is fulfilled and $\gamma(p(x)-\alpha) \ge 0$, then

$$t^{\gamma(p(x)-\alpha)} \le \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma(p(x)-\alpha)} < 1 + \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma(p(x)-\alpha)}. \tag{5.37}$$

On the other hand, if (5.35) is fulfilled and $\gamma(p(x)-\alpha) < 0$, then

$$t^{\gamma(p(x)-\alpha)} \le 1 < 1 + \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma(p(x)-\alpha)}. \tag{5.38}$$

Analogously, if (5.36) is fulfilled and $\gamma(p(x)-\alpha) \ge 0$, then (5.38) holds. On the other hand, if (5.36) is fulfilled and $\gamma(p(x)-\alpha) < 0$, then (5.37) holds.

It follows from the above that if (5.32) holds, then for all $x \in X$ and all $\alpha > 0$,

$$t^{\gamma(p(x)-\alpha)} \le 1 + \|\chi_Q\|_{L^{p(\cdot)}}^{\gamma\alpha(1+\zeta)} \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)}.$$

Integrating this inequality over the cube Q yields

$$\int_Q t^{\gamma(p(x)-\alpha)}\, d\mu(x) \le \mu(Q) + \|\chi_Q\|_{L^{p(\cdot)}}^{\gamma\alpha(1+\zeta)} \int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)} d\mu(x).$$

Hence, since $(a^\gamma + b^\gamma)^{1/\gamma} \le a + b$ for $a, b \ge 0$ and $\gamma > 1$, we see that

$$\begin{aligned} &\left(\frac{1}{\mu(Q)} \int_Q t^{\gamma(p(x)-\alpha)}\, d\mu(x)\right)^{1/\gamma} \\ &\le \left(1 + \left[\|\chi_Q\|_{L^{p(\cdot)}}^{\alpha(1+\zeta)} \left(\frac{1}{\mu(Q)} \int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)} d\mu(x)\right)^{1/\gamma}\right]^\gamma\right)^{1/\gamma} \\ &\le 1 + \|\chi_Q\|_{L^{p(\cdot)}}^{\alpha(1+\zeta)} \left(\frac{1}{\mu(Q)} \int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)} d\mu(x)\right)^{1/\gamma}. \end{aligned} \tag{5.39}$$

Combining (5.34) and (5.39) we obtain, for $\alpha > 0$,

$$\begin{aligned} &\left(\frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)}\, d\mu(x)\right)^{1/\gamma} \\ &\qquad \le t^\alpha + \|\chi_Q\|_{L^{p(\cdot)}}^{\alpha(1+\zeta)} \left(\frac{1}{\mu(Q)} \int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)} d\mu(x)\right)^{1/\gamma} t^\alpha. \end{aligned} \tag{5.40}$$

Let $\alpha = m_p(Q)$ be a median value of p over Q, that is, a number satisfying

$$\max\left\{\frac{\mu\big(\{x\in Q : p(x) > m_p(Q)\}\big)}{\mu(Q)}, \frac{\mu\big(\{x\in Q : p(x) < m_p(Q)\}\big)}{\mu(Q)}\right\} \le \frac{1}{2}.$$

Set

$$E_1(Q) := \{x\in Q : p(x) \le m_p(Q)\}, \qquad E_2(Q) := \{x\in Q : p(x) \ge m_p(Q)\}.$$

It follows immediately from the definition of $m_p(Q)$ that

$$\mu(E_j(Q)) \ge \tfrac{1}{2}\mu(Q), \qquad j=1,2. \tag{5.41}$$

Then, for $t\in(0,\infty)$, we have

$$\begin{aligned} t^\alpha = t^{m_p(Q)} &\le \frac{2t^{m_p(Q)}\mu(E_j(Q))}{\mu(Q)} \\ &\le \begin{cases} \dfrac{2}{\mu(Q)} \displaystyle\int_{E_1(Q)} t^{p(x)}\,d\mu(x) & \text{if } t\in(0,1), \\ \dfrac{2}{\mu(Q)} \displaystyle\int_{E_2(Q)} t^{p(x)}\,d\mu(x) & \text{if } t\in[1,\infty) \end{cases} \\ &\le \frac{2}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x). \end{aligned} \tag{5.42}$$

We claim that

$$\chi_Q(x) \le 2(M^{\mathcal{D}}\chi_{E_j(Q)})(x), \qquad x\in X,\ j=1,2. \tag{5.43}$$

Indeed, if $x\notin Q$, then (5.43) is trivial. On the other hand, if $x\in Q$, then (5.41) implies that

$$\chi_Q(x) = \frac{\mu(Q)}{\mu(Q)} \le \frac{2\mu(E_j(Q))}{\mu(Q)} = \frac{2}{\mu(Q)}\int_Q \chi_{E_j(Q)}(y)\,d\mu(y) \le 2(M^{\mathcal{D}}\chi_{E_j(Q)})(x),$$

which completes the proof of (5.43).

It follows from (5.43), Theorem 2.2, and the boundedness of the Hardy–Littlewood maximal operator on $L^{p(\cdot)}(X,d,\mu)$ that

$$\|\chi_Q\|_{L^{p(\cdot)}} \le 2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}\|\chi_{E_j(Q)}\|_{L^{p(\cdot)}}, \qquad j=1,2. \tag{5.44}$$

In view of Lemma 4.2, we have

(5.45)

$$\begin{aligned} \|\chi_{E_j(Q)}\|_{L^{p(\cdot)}} &\le \max\big\{(\mu(E_j(Q)))^{1/p_-(E_j(Q))}, (\mu(E_j(Q)))^{1/p_+(E_j(Q))}\big\} \\ &\le \max\big\{(\mu(Q))^{1/p_-(E_j(Q))}, (\mu(Q))^{1/p_+(E_j(Q))}\big\}, \qquad j=1,2. \end{aligned}$$

Taking into account the definition of the sets $E_j(Q)$, we see that

$$p_-(E_1(Q)) \le p_+(E_1(Q)) \le m_p(Q) \le p_-(E_2(Q)) \le p_+(E_2(Q)).$$

Therefore, if $\mu(Q) \le 1$, then

$$\max\{(\mu(Q))^{1/p_-(E_1(Q))}, (\mu(Q))^{1/p_+(E_1(Q))}\} \le (\mu(Q))^{1/m_p(Q)}, \tag{5.46}$$

and if $\mu(Q) > 1$, then

$$\max\{(\mu(Q))^{1/p_-(E_2(Q))}, (\mu(Q))^{1/p_+(E_2(Q))}\} \le (\mu(Q))^{1/m_p(Q)}. \tag{5.47}$$

If (5.35) is fulfilled, then $\|\chi_Q\|_{L^{p(\cdot)}} \le 1$. Then, in view of Lemma 4.1, $\mu(Q) \le 1$. On the other hand if (5.36) is fulfilled, then $\|\chi_Q\|_{L^{p(\cdot)}} \ge 1$. Therefore, by Lemma 4.1, $\mu(Q) \ge 1$. Thus, if (5.32) is fulfilled, then (5.44)–(5.47) imply that

$$\|\chi_Q\|_{L^{p(\cdot)}} \le 2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})}(\mu(Q))^{1/m_p(Q)}. \tag{5.48}$$

Set

$$q := \frac{1+\lambda}{1+\gamma}, \qquad q' := \frac{q}{q-1}.$$

Then

$$\gamma q(1+\zeta) = \gamma\frac{1+\lambda}{1+\gamma}\left(1 + \frac{\lambda-\gamma}{\gamma(1+\lambda)}\right) = \lambda \tag{5.49}$$

and

$$\gamma q'\zeta = \gamma\frac{1+\lambda}{1+\gamma}\left(\frac{1+\lambda}{1+\gamma} - 1\right)^{-1}\frac{\lambda-\gamma}{\gamma(1+\lambda)} = 1. \tag{5.50}$$

Taking (5.49) and (5.50) into account, by Hölder's inequality with exponents $q, q' \in (1,\infty)$, we get

$$\begin{aligned} &\frac{1}{\mu(Q)}\int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)} d\mu(x) \\ &\qquad \le \left(\frac{1}{\mu(Q)}\int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{\gamma q(1+\zeta)p(x)} d\mu(x)\right)^{1/q} \\ &\qquad = \left(\frac{1}{\mu(Q)}\int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{\lambda p(x)} d\mu(x)\right)^{1/q}. \end{aligned} \tag{5.51}$$

Since

$$\int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{p(x)} d\mu(x) = \int_X \left(\frac{\chi_Q(x)}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{p(x)} d\mu(x) \le 1,$$

applying Lemma 5.1 with $S_d = \{Q\}$ and $\alpha_Q = 1/\|\chi_Q\|_{L^{p(\cdot)}}$ we obtain

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q \left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{\lambda p(x)} d\mu(x)\right)^{1/\lambda} \le A.$$

Hence

$$\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}\right)^{\lambda p(x)}d\mu(x)\right)^{1/q}\le\left(\frac{A}{\mu(Q)}\right)^{\lambda/q}. \tag{5.52}$$

Combining (5.51) and (5.52), we arrive at

$$\frac{1}{\mu(Q)}\int_Q\left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)}d\mu(x)\le\left(\frac{A}{\mu(Q)}\right)^{\lambda/q}.$$

It follows from the above estimate and from (5.48) that

$$\begin{aligned}\|\chi_Q\|_{L^{p(\cdot)}}^{\alpha(1+\zeta)}&\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}}\right)^{\gamma p(x)}d\mu(x)\right)^{1/\gamma}t^\alpha \\ &\le(2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})})^{\alpha(1+\zeta)}(\mu(Q))^{\frac{\alpha(1+\zeta)}{m_p(Q)}}\left(\frac{A}{\mu(Q)}\right)^{\lambda/(\gamma q)}t^\alpha \\ &=(2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})})^{m_p(Q)(1+\zeta)}A^{\lambda/(\gamma q)}(\mu(Q))^{1+\zeta-\lambda/(\gamma q)}t^\alpha.\end{aligned} \tag{5.53}$$

Taking into account the definitions of ζ and q, we see that

$$1+\zeta-\frac{\lambda}{\gamma q}=1+\frac{\lambda-\gamma}{\gamma(1+\lambda)}-\frac{1+\gamma}{1+\lambda}\cdot\frac{\lambda}{\gamma}=0. \tag{5.54}$$

Combining (5.40), (5.42), (5.53) and (5.54), we get

$$\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma}\le\frac{D}{2}t^{m_p(Q)}\le\frac{D}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x)$$

with

$$D:=1+(2C_{HK}(X)\|M\|_{\mathcal{B}(L^{p(\cdot)})})^{p_+(1+\zeta)}A^{\lambda/(\gamma q)},$$

which completes the proof of (5.33). ■

5.4. Fourth lemma. The next lemma is an extension of [L17, Lemma 4.1] with $w\equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

LEMMA 5.4. *Let (X,d,μ) be a space of homogeneous type and $p\in\mathcal{P}(X)$ satisfy $1<p_-,p_+<\infty$. If the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(X,d,\mu)$, then there exist constants $C,\gamma>1$ and $\eta>0$ and a measure ν on X such that for every dyadic grid $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$ and every finite family $S_d\subset\mathcal{D}$ of pairwise disjoint cubes the following properties hold:*

(i) *if $Q\in\mathcal{D}$ and $t>0$ satisfies $t\|\chi_Q\|_{L^{p(\cdot)}}\le 1$, then*

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma}\le C\int_Q t^{p(x)}\,d\mu(x)+2t^\eta\nu(Q)\chi_{(0,1)}(t); \tag{5.55}$$

(ii) $\sum_{Q\in S_d}\nu(Q)\le C$.

Proof. Let $B, D > 1$, and $1 < \gamma < \lambda$ be the constants given by Lemmas 5.2 and 5.3 and ν be the measure on X given by Lemma 5.2. Put

$$\zeta := \frac{\lambda - \gamma}{\gamma(1+\lambda)}$$

and take

$$C := \max\{(2B)^{1+\zeta}, D\}. \tag{5.56}$$

Then (ii) follows from Lemma 5.2(ii) because $C \ge 2B$.

Let us prove (i). If $t\|\chi_Q\|_{L^{p(\cdot)}} \le 1$ and $t \ge 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \le 1$ and therefore $\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta} \le \|\chi_Q\|_{L^{p(\cdot)}} \le 1$. It is easy to check that (5.32) is fulfilled. Then it follows from Lemma 5.3 and (5.56) that

$$\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma} \le \frac{C}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x),$$

which immediately implies (5.55) and completes the proof of part (i) for $t \ge 1$.

Assume that $t\|\chi_Q\|_{L^{p(\cdot)}} \le 1$ and $0 < t < 1$. If

$$\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma} \le \frac{C}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x), \tag{5.57}$$

then (5.55) is trivial.

Assume that (5.57) does not hold, that is,

$$\frac{1}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x) < \frac{1}{C}\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma}. \tag{5.58}$$

Set, as in the proof of Lemma 5.3,

$$q := \frac{1+\lambda}{1+\gamma}, \qquad q' := \frac{q}{q-1}.$$

By Hölder's inequality, (5.58) and (5.49), we have

$$\begin{aligned} \frac{1}{\mu(Q)}\int_Q t^{\frac{p(x)}{1+\zeta}}\,d\mu(x) &\le \left(\frac{1}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x)\right)^{\frac{1}{1+\zeta}} \\ &\le \left(\frac{1}{C}\right)^{\frac{1}{1+\zeta}}\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma q p(x)}\,d\mu(x)\right)^{\frac{1}{\gamma q(1+\zeta)}} \\ &= \left(\frac{1}{C}\right)^{\frac{1}{1+\zeta}}\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}. \end{aligned} \tag{5.59}$$

It follows from (5.56) and (5.58) that

$$\frac{D}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x)\le\frac{C}{\mu(Q)}\int_Q t^{p(x)}\,d\mu(x)<\left(\frac{1}{\mu(Q)}\int_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma},$$

that is, (5.33) does not hold. Therefore, by Lemma 5.3, condition (5.32) is not fulfilled. Since $0<t<1$, this means that

$$0<t<\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}^{1+\zeta}},$$

whence $\|t^{1/(1+\zeta)}\chi_Q\|_{L^{p(\cdot)}}\le 1$. Therefore, by Lemma 4.1,

$$\int_Q t^{\frac{p(x)}{1+\zeta}}\,d\mu(x)=\int_X (t^{\frac{1}{1+\zeta}}\chi_Q(x))^{p(x)}\,d\mu(x)\le 1,$$

that is, (5.23) is fulfilled with $t^{1/(1+\zeta)}$ in place of t. Then, by Lemma 5.2, (5.24) holds with $t^{1/(1+\zeta)}$ in place of t, that is,

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}\le B\int_Q t^{\frac{p(x)}{1+\zeta}}\,d\mu(x)+\nu(Q). \tag{5.60}$$

It follows from (5.59), (5.60) and (5.56) that

$$\begin{aligned}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}&\le B\left(\frac{1}{C}\right)^{\frac{1}{1+\zeta}}\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}+\nu(Q)\\&\le\frac{1}{2}\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}+\nu(Q).\end{aligned}$$

Thus

$$\left(\frac{1}{\mu(Q)}\int_Q t^{\frac{\lambda p(x)}{1+\zeta}}\,d\mu(x)\right)^{1/\lambda}\le 2\nu(Q). \tag{5.61}$$

Since $0<t<1$, we have

$$t^{\frac{p(x)}{1+\zeta}}=t^{p(x)}t^{-\frac{\zeta p(x)}{1+\zeta}}\le t^{p(x)}t^{-\frac{\zeta p_-}{1+\zeta}}. \tag{5.62}$$

Inequalities (5.61) and (5.62) imply that

$$\mu(Q)\left(\frac{1}{\mu(Q)}t^{-\frac{\lambda\zeta p_-}{1+\zeta}}\int_Q t^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda}\le 2\nu(Q),$$

whence

$$\mu(Q)\left(\frac{1}{Q}\int_Q t^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda}\le 2t^{\eta}\nu(Q) \tag{5.63}$$

with

$$\eta := \frac{\zeta p_-}{1+\zeta}.$$

Since $1 < \gamma < \lambda$, by Hölder's inequality we have

$$\left(\frac{1}{\mu(Q)}\int\limits_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma} \le \left(\frac{1}{\mu(Q)}\int\limits_Q t^{\lambda p(x)}\,d\mu(x)\right)^{1/\lambda}. \tag{5.64}$$

Combining (5.63) and (5.64), we arrive at

$$\mu(Q)\left(\frac{1}{\mu(Q)}\int\limits_Q t^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma} \le 2t^{\eta}\nu(Q),$$

which implies (5.55) and completes the proof of (i) for $0 < t < 1$. ■

6. Proof of Theorem 1.1. It is sufficient to show that if the Hardy–Littlewood maximal operator M is bounded on the space $L^{p(\cdot)}(X,d,\mu)$, then it is also bounded on $L^{p'(\cdot)}(X,d,\mu)$. In turn, in view of Corollary 3.2 it is enough to verify the condition $\mathcal{A}_\infty$. To do this, we will apply Lemma 4.4.

Let $\mathcal{D} \in \bigcup_{t=1}^{K}\mathcal{D}^t$ be a dyadic grid, $S \subset \mathcal{D}$ be a finite sparse family, and $\{G_Q\}_{Q\in S}$ be a collection of nonnegative numbers such that

$$\Big\|\sum_{Q\in S}\alpha_Q\chi_Q\Big\|_{L^{p(\cdot)}} = 1.$$

Then for every $Q \in S$,

$$\alpha_Q\|\chi_Q\|_{L^{p(\cdot)}} \le \Big\|\sum_{Q\in S}\alpha_Q\chi_Q\Big\|_{L^{p(\cdot)}} = 1.$$

Let $C, \gamma > 1$ and $\eta > 0$ be the constants and ν be the measure from Lemma 5.4. Suppose $Q \in S$ is such that $\alpha_Q \ge 1$. Applying Hölder's inequality and Lemma 5.4, we get

$$\begin{aligned}\int\limits_{G_Q}\alpha_Q^{p(x)}\,d\mu(x) &= \frac{\mu(Q)}{\mu(Q)}\int\limits_Q \alpha_Q^{p(x)}\chi_{G_Q}(x)\,d\mu(x)\\ &\le \left(\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}\mu(Q)\left(\frac{1}{\mu(Q)}\int\limits_Q \alpha_Q^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma}\\ &\le C\left(\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}\int\limits_Q \alpha_Q^{p(x)}\,d\mu(x).\end{aligned}$$

Combining this inequality with Lemma 4.3, we get

$$\sum_{Q\in S\,:\,\alpha_Q\geq 1}\int_{G_Q}\alpha_Q^{p(x)}\,d\mu(x)\leq C\sum_{Q\in S\,:\,\alpha_Q\geq 1}\left(\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}\int_Q\alpha_Q^{p(x)}\,d\mu(x)$$
$$\leq C\left(\max_{Q\in S}\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}\sum_{Q\in S}\int_Q\alpha_Q^{p(x)}\,d\mu(x)\leq C\left(\max_{Q\in S}\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}. \tag{6.1}$$

For $k\in\mathbb{N}$, put

$$S_k:=\{Q\in S:2^{-k}\leq\alpha_Q<2^{-k+1}\}. \tag{6.2}$$

If $S_k\neq\emptyset$, then there exist $i_k\in\mathbb{N}$ and cubes $Q_1^k,\dots,Q_{i_k}^k$ such that $\bigcup_{Q\in S_k}Q=\bigcup_{j=1}^{i_k}Q_j^k$; if $i,j\in\{1,\dots,i_k\}$ and $i\neq j$, then $Q_i^k\cap Q_j^k=\emptyset$; and for every $Q\in S_k$, there is $j\in\{1,\dots,i_k\}$ such that $Q\subset Q_j^k$.

For $k\in\mathbb{N}$ and $S_k\neq\emptyset$, put

$$\psi_{Q_j^k}(x)=\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\chi_{G_Q}(x),\qquad j\in\{1,\dots,i_k\}. \tag{6.3}$$

Then, taking (6.2) into account one has, for all $j\in\{1,\dots,i_k\}$,

$$\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\int_{G_Q}\alpha_Q^{p(x)}\,d\mu(x)=\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\int_{Q_j^k}\alpha_Q^{p(x)}\chi_{G_Q}(x)\,d\mu(x)$$
$$\leq\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\int_{Q_j^k}(2^{-k+1})^{p(x)}\chi_{G_Q}(x)\,d\mu(x)=\int_{Q_j^k}(2^{-k+1})^{p(x)}\psi_{Q_j^k}(x)\,d\mu(x)$$
$$\leq 2^{p_+}\int_{Q_j^k}2^{-kp(x)}\psi_{Q_j^k}(x)\,d\mu(x)\leq 2^{p_+}\int_{Q_j^k}\alpha_{Q_j^k}^{p(x)}\psi_{Q_j^k}(x)\,d\mu(x). \tag{6.4}$$

By Hölder's inequality, for $k\in\mathbb{N}$ with $S_k\neq\emptyset$ and $j\in\{1,\dots,i_k\}$, one has

$$\int_{Q_j^k}\alpha_{Q_j^k}^{p(x)}\psi_{Q_j^k}(x)\,d\mu(x)\leq\mu(Q_j^k)\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k}\alpha_{Q_j^k}^{\gamma p(x)}\,d\mu(x)\right)^{1/\gamma}$$
$$\times\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k}\psi_{Q_j^k}^{\gamma'}(x)\,d\mu(x)\right)^{1/\gamma'}. \tag{6.5}$$

It follows from (6.2) and the hypothesis that the sets $\{G_Q\}_{Q\in S}$ are pairwise disjoint that

$$\int_{Q_j^k}\psi_{Q_j^k}^{\gamma'}(x)\,d\mu(x)=\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\mu(G_Q)\leq\left(\max_{Q\in S}\frac{\mu(G_Q}{\mu(Q)}\right)\sum_{Q\in S_k\,:\,Q\subset Q_j^k}\mu(Q). \tag{6.6}$$

Since S is a sparse family, there exists a collection of pairwise disjoint sets $\{E(Q)\}_{Q\in S}$ such that $E(Q)\subset Q$ and $\mu(Q)\leq 2\mu(E(Q))$. Hence, for all $k\in\mathbb{N}$

such that $S_k \neq \emptyset$ and all $j \in \{1, \dots, i_k\}$,

$$\sum_{Q \in S_k \,:\, Q \subset Q_j^k} \mu(Q) \le 2 \sum_{Q \in S_k \,:\, Q \subset Q_j^k} \mu(E(Q)) = 2\mu\Big(\bigcup_{Q \in S_k \,:\, Q \subset Q_j^k} E(Q) \Big) \le 2\mu(Q_j^k). \tag{6.7}$$

On the other hand, taking into account that $\alpha_{Q_j^k} < 1$, we deduce from Lemma 5.4 that

$$\mu(Q_j^k) \bigg(\frac{1}{\mu(Q_j^k)} \int_{Q_j^k} \alpha_{Q_j^k}^{\gamma p(x)} \, d\mu(x) \bigg)^{1/\gamma} \le C \int_{Q_j^k} \alpha_{Q_j^k}^{p(x)} \, d\mu(x) + 2\alpha_{Q_j^k}^{\eta} \nu(Q_j^k). \tag{6.8}$$

Combining (6.4)–(6.8), we obtain for every $k \in \mathbb{N}$ such that $S_k \neq \emptyset$ and every $j \in \{1, \dots, i_k\}$,

$$\sum_{Q \in S_k \,:\, Q \subset Q_j^k} \int_{G_Q} \alpha_Q^{p(x)} \, d\mu(x) \le 2^{p_+ + \frac{1}{\gamma'}} \bigg(\frac{\mu(G_Q)}{\mu(Q)} \bigg)^{1/\gamma'} \bigg(C \int_{Q_j^k} \alpha_{Q_j^k}^{p(x)} \, d\mu(x) + 2\alpha_{Q_j^k}^{\eta} \nu(Q_j^k) \bigg).$$

Then

$$\begin{aligned} \sum_{Q \in S \,:\, \alpha_Q < 1} \int_{G_Q} \alpha_Q^{p(x)} \, d\mu(x) &= \sum_{k \in \mathbb{N} \,:\, S_k \neq \emptyset} \sum_{Q \in S_k} \int_{G_Q} \alpha_Q^{p(x)} \, d\mu(x) \\ &= \sum_{k \in \mathbb{N} \,:\, S_k \neq \emptyset} \sum_{j=1}^{i_k} \sum_{Q \in S_k \,:\, Q \subset Q_j^k} \int_{G_Q} \alpha_Q^{p(x)} \, d\mu(x) \\ &\le 2^{p_+ + 1/\gamma'} \bigg(\max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \bigg)^{1/\gamma'} \\ &\quad \times \sum_{k \in \mathbb{N} \,:\, S_k \neq \emptyset} \sum_{j=1}^{i_k} \bigg(C \int_{Q_j^k} \alpha_{Q_j^k}^{p(x)} \, d\mu(x) + 2\alpha_{Q_j^k}^{\eta} \nu(Q_j^k) \bigg). \end{aligned} \tag{6.9}$$

It follows from Lemma 4.3 that

$$\sum_{k \in \mathbb{N} \,:\, S_k \neq \emptyset} \sum_{j=1}^{i_k} \int_{Q_j^k} \alpha_{Q_j^k}^{p(x)} \, d\mu(x) \le \sum_{Q \in S} \int_Q \alpha_Q^{p(x)} \, d\mu(x) \le 1. \tag{6.10}$$

Since for every fixed k, the cubes $Q_1^k, \dots, Q_{i_k}^k$ are pairwise disjoint, it follows from Lemma 5.4(ii) and (6.2) that

$$\begin{aligned}(6.11)\qquad \sum_{k\in\mathbb{N}\,:\,S_k\neq\emptyset}\sum_{j=1}^{i_k}\alpha_{Q_j^k}^{\eta}\nu(Q_j^k) &\le \sum_{k\in\mathbb{N}\,:\,S_k\neq\emptyset}(2^{-k+1})^{\eta}\sum_{j=1}^{i_k}\nu(Q_j^k)\\ &\le C\sum_{k\in\mathbb{N}\,:\,S_k\neq\emptyset}2^{(-k+1)\eta}\le C2^{\eta}\sum_{k=1}^{\infty}\left(\frac{1}{2^{\eta}}\right)^{k}=:C_1.\end{aligned}$$

It follows from (6.9)–(6.11) that

$$(6.12)\qquad \sum_{Q\in S\,:\,\alpha_Q<1}\int_{G_Q}\alpha_Q^{p(x)}\,d\mu(x)\le 2^{p_+ +1/\gamma'}(C+C_1)\left(\max_{Q\in S}\frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}.$$

Combining (6.1) and (6.12), we see that

$$\sum_{Q\in S}\int_{G_Q}\alpha_Q^{p(x)}\,d\mu(x)\le \Psi\left(\max_{Q\in S}\frac{\mu(G_Q)}{\mu(Q)}\right)^{\xi}$$

with $\Psi := C + 2^{p_+ +1/\gamma'}(C+2C_1)$ and $\xi = 1/\gamma'$. Hence (4.3) implies (4.4). By Lemma 4.4, the space $L^{p(\cdot)}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$. Thus, the Hardy–Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p'(\cdot)}(X,d,\mu)$ in view of Corollary 3.2. □

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