Reciprocal matrices: properties and approximation by a transitive matrix

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Abstract

Reciprocal matrices and, in particular, transitive matrices, appear in several applied areas. Among other applications, they have an important role in decision theory in the context of the Analytical Hierarchical Process, introduced by Saaty. In this paper we study the possible ranks of a reciprocal matrix and give a procedure to construct a reciprocal matrix with the rank and the off-diagonal entries of an arbitrary row (column) prescribed. We apply some techniques from graph theory to the study of transitive matrices, namely to determine the maximum number of equal entries, and distinct from ± 1, in a transitive matrix. We then focus on the n-by-n reciprocal matrix, denoted by $C(n,x)$, with all entries above the main diagonal equal to $x>0$. We show that there is a Toeplitz transitive matrix and a transitive matrix preserving the maximum possible number of entries of $C(n,x)$ whose distance to $C(n,x)$, measured in the Frobenius norm, is smaller than the one of the transitive matrix suggested by Saaty,

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constructed from the right Perron eigenvector of $C(n,x)$. We illustrate our results with some numerical examples.

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1 Introduction

Pairwise comparisons are commonly used in science, as well as in current practice, when comparing entities. We may construct a pairwise comparison matrix (PC matrix) to easily process our relative comparisons data (such as reliability) and then conceive mathematical methods and techniques for decision making. The method of pairwise comparisons, one of the earliest scientific methods in social sciences, was already used by Condorcet [5] in its primitive form win/loss. In [23, 24], Saaty proposed a multicriteria decision making method, called the Analytical Hierarchical Process (AHP), which solves decision problems by prioritizing alternatives. The idea is that a vector that gives the priorities for the different alternatives is defined and from it a transitive pairwise comparison matrix is constructed. Briefly, given $n \geq 3$ alternate decisions D_i, then the i,j entry of the matrix indicates the strengh with which alternative i dominates alternative j relatively to a certain criterion.

In practice, due to human feelings, preferences and other emotions, when making decisions people may not be consistent. Therefore, pairwise comparison matrices arising in the real world are reciprocal matrices with all entries positive which, in general, reflect inconsistencies. Saaty [26] proposed that the vector of priorities of the alternatives is the Perron-Frobenius right eigenvector of the PC matrix. With this vector, a pairwise comparison matrix reflecting no inconsistencies can be constructed.

An n-by-n matrix $A = [a_{ij}]$ with real nonzero entries is said to be *reciprocal* if $a_{ij} = \frac{1}{a_{ji}}$ for all $i,j = 1,\ldots,n$. In particular $a_{ii} = 1$ for all $i = 1,\ldots,n$. The matrix A is said to be *transitive* or *consistent* if $a_{ij}a_{jk} = a_{ik}$ for all $i,j,k = 1,\ldots,n$. A transitive matrix is reciprocal. The converse is not true in general, though it holds for $n = 2$. When a reciprocal (resp. transitive) matrix A is entrywise positive, we will say that A is positive reciprocal (resp. transitive).

Since Saaty's first work, many researchers have studied reciprocal matrices and have proposed alternative methods of estimating the vector of alternatives priorities from a positive reciprocal matrix and discussed the implications of the different methods (see, for instance, [1, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 28, 32, 33] and the references therein). Besides decision theory, reciprocal matrices appear in many other contexts as, for instance, in economics and engineering [2, 29, 30, 31]. In many applications, reciprocal matrices have additional structures as, for example, a Toeplitz structure, which may be important to preserve when approximating the matrix by a transitive matrix. Due to their wide applications in practice, as for example in the study of differential equations and time series analysis [27, 21, 22], Toeplitz matrices have been the subject of a very productive research area. Recall that a matrix $A = [a_{ij}]$ is *Toeplitz* if $a_{i_1 j_1} = a_{i_2 j_2}$ when $i_1 - j_1 = i_2 - j_2$.

In this paper we investigate some properties of reciprocal matrices and, in particular, of transitive matrices, and study the approximation of a certain simple positive reciprocal Toeplitz matrix by a transitive matrix. We measure the quality of our approximations in the least squares sense, that is, the best transitive approximation $B = [b_{ij}]$ of an n-by-n positive reciprocal matrix $A = [a_{ij}]$ is the one that minimizes the Frobenius matrix norm $||A - B||_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} (a_{ij} - b_{ij})^2\right)^{\frac{1}{2}}$. Though not free from criticism, this approach was proposed by several authors (see, for example, [12, 25]).

The paper is organized as follows. In Section 2 we present some useful known properties of reciprocal and, in particular, of transitive matrices. Due to the importance of the rank of a matrix in several applications, namely in solving linear systems, we start by studying in Section 3 the possible ranks of a reciprocal matrix. It is shown that any rank between 1 and the size n of the matrix, except 2, can be attained. Our proof is constructive in the sense that it gives a procedure to solve the following inverse problem: obtain a reciprocal matrix with the rank and the off-diagonal entries of a line of the matrix prescribed. An example of this procedure is also provided. In Section 4 we apply techniques from graph theory (see [7] for details) to the study of transitive matrices. We show that there is a unique transitive matrix with certain sets of prescribed entries, as long as the positions of these entries satisfy a condition stated in terms of graphs. We also determine the maximum possible number of equal entries, distinct from ± 1, in a transitive matrix. In Section 5 we focus on the n-by-n reciprocal Toeplitz matrix, denoted by $C(n, x)$, with all entries above the main diagonal equal to $x \in \mathbb{R}\backslash\{0\}$. We start by describing the eigenvalues and eigenvectors of these matrices. We observe that in [19] an explicit expression for the Perron eigenvalue and the corresponding eigenvector is given, when $x > 0$, but no reference is made to the remaining eigenvalues. When $x > 0$, we then construct the transitive matrices corresponding to the left and right eigenvectors of $C(n, x)$ associated with the Perron eigenvalue and note that they coincide and have a Toeplitz structure. A transitive matrix (not Toeplitz) associated with $C(n, x)$ preserving the maximum number of entries of $C(n, x)$ is constructed and it is shown that its distance to $C(n, x)$, measured in the Frobenius matrix norm, is smaller than the one of the transitive matrix obtained from the Perron eigenvectors. We also show that there is a transitive Toeplitz matrix closer to $C(n, x)$ than the transitive matrix constructed from the Perron eigenvector. In Section 6 we present numerical examples that illustrate the results in Section 5. Finally, we draw some concluding remarks in Section 7.

2 Preliminaries

In this section we introduce some notation and concepts concerning positive reciprocal and transitive matrices that will be useful throughout the paper. For a summary of well-known properties of these matrices, see, for example, [11, 28].

Let $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T \in \mathbb{C}^{n\times 1}$, with $w_i \neq 0$ for $i = 1, \ldots, n$. We denote by $w^{(-1)}$ the row vector $\begin{bmatrix} \frac{1}{w_1} & \cdots & \frac{1}{w_n} \end{bmatrix}$.

It is well known that $A = [a_{ij}]$ is an n-by-n transitive matrix if and only if there is an n-by-1 positive vector w such that $A = ww^{(-1)}$. It follows that a positive reciprocal matrix A is transitive if and only if $rank(A) = 1$. The eigenvalues of a transitive matrix are 0 with geometric multiplicity $n-1$ and $n = tr(A)$ with algebraic (and geometric) multiplicity 1.

Given an n-by-n complex matrix A and an n-by-1 complex vector w, we say that w is a *(right) eigenvector* (resp. *left eigenvector*) of A if $w \neq 0$ and there is $\lambda \in \mathbb{C}$ such that $Aw = \lambda w$ (resp. $w^T A = \lambda w^T$). When A is positive the *Perron eigenvalue* of A is its the largest eigenvalue of A which, according to the Perron-Frobenius Theorem, is simple, positive and greater than the magnitude of any other eigenvalue [17]. We call the eigenvectors associated with the Perron eigenvalue the *Perron eigenvectors.* There is a positive Perron eigenvector and all the others are proportional. The one whose first entry is 1 will be called the *principal eigenvector* of A.

Given a positive reciprocal matrix A, Saaty proposed the transitive matrix $ww^{(-1)}$, where w is a right eigenvector of A associated with its Perron eigenvalue, to approximate A by a transitive matrix. Note that w is also a right eigenvector of $ww^{(-1)}$ associated with its Perron eigenvalue. In a similar way, as suggested for example in [18], the transitive matrix $\left(w'^{(-1)}\right)^T (w')^T$, where w' is a left eigenvector of A associated with the Perron eigenvalue, could be considered. Here ^{T} denotes matrix transposition. Again, note that w' is a left eigenvector of $\left(w'^{(-1)}\right)^T (w')^T$ associated with its Perron eigenvalue. Since the Perron eigenvalue is simple, these rank 1 matrices do not depend on the particular choice of the Perron eigenvector.

In this paper we will call $ww^{(-1)}$ the *(right) Perron transitive matrix* associated with A and $\left(w'^{(-1)}\right)^T (w')^T$ the *left Perron transitive matrix* associated with A.

3 Rank of a reciprocal matrix

In this section we show that an n-by-n reciprocal matrix (not necessarily positive) can have any rank from 1 to n, except 2. We then give a constructive proof of the existence of a reciprocal matrix with prescribed rank r, $1 \leq r \leq n$, $r \neq 2$, and off-diagonal entries of a certain row (column). By applying a permutation similarity and/or a transposition, we may assume, without loss of generality, that the prescribed entries are in the first row.

Let $A = [a_{ij}]$ be an n-by-n reciprocal matrix. For $1 < k < j \leq n$, let

$$s_{kj} = a_{1k}a_{kj} - a_{1j}.$$

Note that a_{kj} just occurs in s_{kj} and, if the first row of A and the s_{kj}'s, $1 < k < j \leq n$, are given, then each a_{kj} is determined from s_{kj}.

Denote row i of A by r_i. Let $A_1 = [b_{ij}]$ be obtained from A by replacing row i with

$$(a_{1i} \cdots a_{i-1,i})r_i - (a_{2i} \cdots a_{i-1,i})r_1,$$

$i = 2, \ldots, n$. Then, the entry i, j of A_1, with $i > 1$, is given by

$$b_{ij} = \begin{cases} (a_{1i} \cdots a_{i-1,i})a_{ij} - (a_{2i} \cdots a_{i-1,i})a_{1j} & \text{if } j \neq 1 \text{ and } j \neq i \\ 0 & \text{if } j = 1 \text{ or } i = j. \end{cases}$$

that is,

$$b_{ij} = \begin{cases} (a_{2i} \cdots a_{i-1,i})s_{ij} & \text{if } i < j \\ -\frac{1}{a_{ji}}(a_{2i} \cdots a_{i-1,i})s_{ji} & \text{if } i > j > 1 \\ 0 & \text{if } j = 1 \text{ or } i = j. \end{cases}$$

Note that A and A_1 have the same rank. Denote by $\mathcal{T}(A)$ the submatrix of A_1 obtained by deleting the first row and column.

Example 1 *Consider the case $n = 5$. Then*

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ \frac{1}{a_{12}} & 1 & a_{23} & a_{24} & a_{25} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & a_{34} & a_{35} \\ \frac{1}{a_{14}} & \frac{1}{a_{24}} & \frac{1}{a_{34}} & 1 & a_{45} \\ \frac{1}{a_{15}} & \frac{1}{a_{25}} & \frac{1}{a_{35}} & \frac{1}{a_{45}} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 1 & a_{12} & a_{12}a_{23} & a_{12}a_{24} & a_{12}a_{25} \\ a_{23} & a_{13} & a_{13}a_{23} & a_{13}a_{23}a_{34} & a_{13}a_{23}a_{35} \\ a_{24}a_{34} & a_{14}a_{34} & a_{14}a_{24} & a_{14}a_{24}a_{34} & a_{14}a_{24}a_{34}a_{45} \\ a_{25}a_{35}a_{45} & a_{15}a_{35}a_{45} & a_{15}a_{25}a_{45} & a_{15}a_{25}a_{35} & a_{15}a_{25}a_{35}a_{45} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & s_{23} & s_{24} & s_{25} \\ 0 & -s_{23} & 0 & a_{23}s_{34} & a_{23}s_{35} \\ 0 & -a_{34}s_{24} & -a_{24}s_{34} & 0 & a_{24}a_{34}s_{45} \\ 0 & -a_{35}a_{45}s_{25} & -a_{25}a_{45}s_{35} & -a_{25}a_{35}s_{45} & 0 \end{bmatrix} = A_1.$$

Then

$$\mathcal{T}(A) = \begin{bmatrix} 0 & s_{23} & s_{24} & s_{25} \\ -s_{23} & 0 & a_{23}s_{34} & a_{23}s_{35} \\ -a_{34}s_{24} & -a_{24}s_{34} & 0 & a_{24}a_{34}s_{45} \\ -a_{35}a_{45}s_{25} & -a_{25}a_{45}s_{35} & -a_{25}a_{35}s_{45} & 0 \end{bmatrix}. \qquad (1)$$

Remark 2 *Clearly, $rank(A) = 1 + rank(\mathcal{T}(A))$. Moreover, if there are i, j with $2 \leq i < j \leq n$ such that $s_{ij} \neq 0$, then $rank(\mathcal{T}(A)) \geq 2$ implying $rank(A) \geq 3$, otherwise $rank(\mathcal{T}(A)) = 0$ and $rank(A) = 1$.*

We next study the possible ranks of $\mathcal{T}(A)$. Given an n-by-n matrix C and $1 \leq k \leq n$, denote by $C[k]$ the principal submatrix of C indexed by rows and columns $1, \ldots, k$. The following observation is obvious.

Remark 3 *Given an n-by-n reciprocal matrix A, we have*

$$\mathcal{T}(A[n-1]) = (\mathcal{T}(A))\,[n-2].$$

Lemma 4 *Let $A = [a_{ij}]$ be an n-by-n reciprocal matrix with $n > 2$. Then $\mathcal{T}(A)$ (and, thus, A) is nonsingular if one of the following conditions holds:*

- *n is odd, and $s_{ij} \neq 0$ for all i, j with $2 \leq i < j \leq n$ and $i + j = n + 2$, and either $s_{ij} = 0$ for all i, j with $2 \leq i < j \leq n$ and $i + j > n + 2$, or $s_{ij} = 0$ for all i, j with $2 \leq i < j \leq n$ and $i + j < n + 2$;*
- *n is even, $s_{23}s_{24}s_{34}(a_{24} - a_{23}a_{34}) \neq 0$, and $s_{2i+1,2i+2} \neq 0$ and $s_{l_i,2i+2} = 0$ for all i, l_i with $2 \leq i \leq \frac{n}{2} - 1$ and $2 \leq l_i \leq 2i$.*

Moreover, if the first row of A is positive and $s_{ij} \geq 0$ for $2 \leq i < j \leq n$, then A is positive.

Proof. If n is odd and the claimed conditions hold, then $\mathcal{T}(A)$ has nonzero anti-diagonal entries and zeros above or below the anti-diagonal.

If n is even and the claimed conditions hold, then it can be easily seen that

$$\det(\mathcal{T}(A)) = k \prod_{i=2}^{\frac{n-2}{2}} s_{2i+1,2i+2}^2 \det \begin{bmatrix} 0 & s_{23} & s_{24} \\ -s_{23} & 0 & a_{23}s_{34} \\ -a_{34}s_{24} & -a_{24}s_{34} & 0 \end{bmatrix} \neq 0,$$

where k is a product of a_{ij}'s.

The second claim follows easily from the definition of the s_{ij}'s. ■

Remark 5 *From Lemma 4 and Remark 2, we conclude that for any $n > 2$ there is a nonsingular positive n-by-n reciprocal matrix where the off-diagonal entries of the first row are any positive real prescribed numbers.*

Lemma 6 *Let $r \in \{0, \ldots, n-1\} \backslash \{1\}$, with $n \geq 2$, and $b_{1,2}, \ldots, b_{1,n}$ be positive real numbers. Then there exists an n-by-n positive reciprocal matrix $A = [a_{ij}]$ such that $rank(\mathcal{T}(A)) = r$ and $a_{1p} = b_{1p}$ for $p = 2, \ldots, n$.*

Proof. The proof is by induction on n. If $n \leq 3$, the result follows easily. Now suppose that $n \geq 4$ and the result holds for $n - 1$.

If $r = 0$ let $a_{1p} = b_{1p}$ for $p = 2, \ldots, n$ and $a_{kj} = \frac{a_{1j}}{a_{1k}}$ for $1 < k < j \leq n$, so that $s_{kj} = 0$ for $1 < k < j$.

If $r = n - 1$ the result follows from Lemma 4 and Remark 5.

Suppose that $2 \leq r < n - 1$. By the induction hypothesis, there exists an $(n-1) \times (n-1)$ positive reciprocal matrix $C = [c_{ij}]$ such that $rank(\mathcal{T}(C)) = r$ and $c_{1p} = b_{1p}$ for $p = 2, \ldots, n-1$. Now let $a_{1n} = b_{1n}$ and $a_{kn} = \frac{a_{1n}}{c_{1k}}$ for

$1 < k < n$. Let $A = [a_{ij}]$, where $A[n-1] = C$. Since $s_{kn} = 0$ for $1 < k < n$, we have

$$\mathcal{T}(A) = \begin{bmatrix} \mathcal{T}(C) & 0 \\ 0 & 0 \end{bmatrix},$$

implying that $rank(\mathcal{T}(A)) = r$. ∎

From Lemma 6 and Remark 2, we obtain the following description of the possible ranks of a reciprocal matrix.

Theorem 7 *Let $n \geq 2$ and $b_{1,2}, \ldots, b_{1,n}$ be positive real numbers. If A is an n-by-n reciprocal matrix, then $rank(A) \neq 2$. Conversely, if $r \in \{0, \ldots, n\} \backslash \{2\}$, there exists an n-by-n positive reciprocal matrix $A = [a_{ij}]$ such that $rank(A) = r$ and $a_{1p} = b_{1p}$ for $p = 2, \ldots, n$.*

Lemma 6 gives us a procedure to construct an n-by-n reciprocal matrix A of rank r, with $1 \leq r \leq n$ and $r \neq 2$, and prescribed off-diagonal entries of the first row. In the next two examples we illustrate this procedure. In the first one we start by constructing a nonsingular matrix of odd size and from it we obtain a singular matrix. In the second one we construct a nonsingular matrix of even size.

Example 8 *We will give a positive 5-by-5 reciprocal matrix $A = [a_{ij}]$ with first row $\begin{bmatrix} 1 & 2 & 4 & 4 & 8 \end{bmatrix}$ and such that $rank(A) = 3$. We first construct a nonsingular positive 3-by-3 reciprocal matrix $A_1 = A[3]$, using Lemma 4. Let $a_{12} = 2$, $a_{13} = 4$ and $s_{23} = 2$. Then*

$$2 = s_{23} = a_{12}a_{23} - a_{13} \Leftrightarrow a_{23} = 3.$$

Thus,

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{4} & \frac{1}{3} & 1 \end{bmatrix}.$$

Using the procedure described in Lemma 6, we now construct a positive 4-by-4 reciprocal matrix A_2 such that $A[4] = A_2$ and $rank(A_2) = 3$. Let $a_{14} = 4$ and $s_{24} = s_{34} = 0$. Then

$$0 = s_{24} = a_{12}a_{24} - a_{14} \Leftrightarrow a_{24} = 2$$
$$0 = s_{34} = a_{13}a_{34} - a_{14} \Leftrightarrow a_{34} = 1,$$

implying that

$$A_2 = \begin{bmatrix} 1 & 2 & 4 & 4 \\ \frac{1}{2} & 1 & 3 & 2 \\ \frac{1}{4} & \frac{1}{3} & 1 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 & 1 \end{bmatrix}.$$

Now we construct A by adding one row and one column to A_2. Let $a_{15} = 8$ and $s_{25} = s_{35} = s_{45} = 0$. Then,

$$\begin{aligned} 0 = s_{25} = a_{12}a_{25} - a_{15} &\Leftrightarrow a_{25} = 4 \\ 0 = s_{35} = a_{13}a_{35} - a_{15} &\Leftrightarrow a_{35} = 2 \\ 0 = s_{45} = a_{14}a_{45} - a_{15} &\Leftrightarrow a_{45} = 2, \end{aligned}$$

implying that

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 & 8 \\ \frac{1}{2} & 1 & 3 & 2 & 4 \\ \frac{1}{4} & \frac{1}{3} & 1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 & 1 & 2 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

Example 9 *Using Lemma 4, we give a nonsingular positive 6-by-6 reciprocal matrix $A = [a_{ij}]$ with all entries in the first row equal to 1. Let $a_{1i} = 1$, $i = 1, \ldots, 6$. Now we choose the remaining entries so that $s_{23}s_{24}s_{34}s_{56} \neq 0$, $a_{24} \neq a_{23}a_{34}$ and $s_{26} = s_{36} = s_{46} = 0$. Letting $s_{23} = s_{24} = s_{34} = s_{56} = 1$, we obtain*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & a_{25} & 1 \\ 1 & \frac{1}{2} & 1 & 2 & a_{35} & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & 1 & a_{45} & 1 \\ 1 & \frac{1}{a_{25}} & \frac{1}{a_{35}} & \frac{1}{a_{45}} & 1 & 2 \\ 1 & 1 & 1 & 1 & \frac{1}{2} & 1 \end{bmatrix},$$

which is nonsingular for any a_{25}, a_{35}, a_{45}.

We next identify explicitly a class of positive reciprocal matrices with given rank $r \neq 2$. Denote by J_n the n-by-n matrix with all entries equal to 1 and by e_m the m-by-1 column vector with all entries equal to 1.

The matrix $A = J_n$ has *rank* 1. Suppose that $n > 2$ and let $a > 0$ be a real number. Then the matrix

$$W_{n,a} = J_n + \begin{bmatrix} 0 & aI_{\frac{n}{2}} \\ -\frac{a}{1+a} I_{\frac{n}{2}} & 0 \end{bmatrix},$$

if n is even, or

$$W_{n,a} = J_n + \begin{bmatrix} 0 & 0 & aI_{\frac{n-1}{2}} \\ 0 & 0 & 0 \\ -\frac{a}{1+a} I_{\frac{n-1}{2}} & 0 & 0 \end{bmatrix},$$

if n is odd, is nonsingular. To see this, note that, if n is odd, subtracting row $\frac{n+1}{2}$ from any other row of $W_{n,a}$, we obtain the matrix

$$\begin{bmatrix} 0 & 0 & aI_{\frac{n-1}{2}} \\ e^T_{\frac{n-1}{2}} & 1 & e^T_{\frac{n-1}{2}} \\ -\frac{a}{1+a} I_{\frac{n-1}{2}} & 0 & 0 \end{bmatrix},$$

which can be easily seen to be nonsingular. If n is even, we have

$$W_{n,a}\left[\begin{array}{cc} I_{\frac{n}{2}} & 0 \\ -I_{\frac{n}{2}} & I_{\frac{n}{2}} \end{array}\right]\left[\begin{array}{cc} I_{\frac{n}{2}} & \frac{1}{a}(J_{\frac{n}{2}}+aI_{\frac{n}{2}}) \\ 0 & I_{\frac{n}{2}} \end{array}\right]=\left[\begin{array}{cc} -aI_{\frac{n}{2}} & 0 \\ -\frac{a}{1+a}I & \frac{a}{1+a}(J_{\frac{n}{2}}-I_{\frac{n}{2}}) \end{array}\right],$$

which is nonsingular since $J-I_{\frac{n}{2}}$ is nonsingular.

If $2<r\leq n-1$, the following reciprocal matrix has rank r:

$$A=J_n+\left[\begin{array}{cc|c} \multicolumn{2}{c|}{W_{r,a}} & \begin{array}{c}(1+a)e_{n-r}^T \\ J_{r-1,n-r}\end{array} \\ \hline \frac{1}{1+a}e_{n-r} & J_{n-r,r-1} & J_{n-r,n-r} \end{array}\right],$$

as the last $n-r$ columns of A coincide with the rth column and $W_{r,a}$ is nonsingular.

We conclude this section with an easy consequence of Theorem 7 that was already stated in [28] for positive reciprocal matrices.

Corollary 10 *Let A be an n-by-n reciprocal matrix and $P(\lambda)$ be the characteristic polynomial of A. Then the coefficient of λ^{n-2} is 0.*

Proof. It is well-known [17] that the coefficient of λ^{n-2} is the sum of the 2-by-2 principal minors of A. Since any 2-by-2 principal submatrix of A is a reciprocal matrix, the claim follows from Theorem 7. ■

4 Transitive matrices and graphs

The next theorem shows that a unique transitive matrix can be constructed from the knowledge of the entries (which can be any nonzero real numbers) in certain $n-1$ positions.

Theorem 11 *Let $S=\{\{i_1,j_1\},\ldots,\{i_{n-1},j_{n-1}\}\}$, with $i_s,j_s\in\{1,\ldots,n\}$ for $s=1,\ldots,n-1$. Let G be the undirected graph with vertex set $\{1,\ldots,n\}$ and such that there is an edge between vertices i and j if and only if $\{i,j\}\in S$. Let $b_{i_1,j_1},\ldots,b_{i_{n-1},j_{n-1}}$ be $n-1$ nonzero real numbers.*

If G is a tree, then there is a unique transitive matrix $A=[a_{ij}]$ with $a_{i_sj_s}=b_{i_sj_s}$ for $s\in\{1,\ldots,n-1\}$. Moreover, A is positive if $b_{i_l,j_l}>0$, $l=1,\ldots,n-1$.

Proof. Suppose that G is a tree. Let $A=[a_{ij}]$ with $a_{i_sj_s}=b_{i_sj_s}$ and $a_{j_si_s}=\frac{1}{b_{i_sj_s}}$ for $s\in\{1,\ldots,n-1\}$. Let $a_{ii}=1$, for $i=1,\ldots,n$.

Let $x,y\in\{1,\ldots,n\}$, $x\neq y$. Since G is a tree, there is a unique path in G between vertices x and y where all vertices are distinct, say $x,p_2,\cdots,p_{l-1},y$. Because $\{p_i,p_{i+1}\}$ is an edge in G, $i=1,\ldots,l-1$, where p_1 is vertex x and p_l is vertex y, we have that $\{p_i,p_{i+1}\}\in S$. So, we can define

$$a_{xy}=a_{xp_2}a_{p_2p_3}\cdots a_{p_{l-1}y}.$$

Now we show that the obtained matrix A is transitive. Let $i, j, l \in \{1, \ldots, n\}$ be distinct. Let $t_1, t_2, \cdots t_{r-1}, t_r$, where t_1 is vertex i and t_r is vertex l, be the unique path in G, between i and l, with all vertices distinct. Let $s_1, s_2, \cdots s_{z-1}, s_z$, where s_1 is vertex l and s_z is vertex j, be the unique path in G, between l and j, with all vertices distinct. So, $t_1, t_2, \cdots t_{r-1}, t_r, s_2, \cdots, s_z$ is a path between i and j. Since G is a tree, all vertices of this path are distinct or there is a first vertex, say t_h, in the fixed ordered path $t_1, t_2, \cdots t_{r-1}, t_r$, that appears in $s_2, \cdots s_{z-1}, s_z$, say $s_v = t_h$. In the second case, $t_1, t_2, \cdots t_{h-1}, t_h, s_{v+1}, \cdots, s_z$ is the only path between i and j with all vertices distinct. Moreover, $t_r, t_{r-1}, \cdots, t_h$ is the path $s_1, \cdots, s_v$, implying that

$$a_{t_h, t_{h+1}} \cdots a_{t_{r-1}, t_r} a_{s_1, s_2} \cdots a_{s_{v-1}, s_v} = 1.$$

Consequently,

$$a_{ik} a_{kj} = a_{ij}.$$

Moreover, $a_{ji} = \frac{1}{a_{ij}}$. Thus, the matrix A is unique. ■

Corollary 12 *There is a unique transitive matrix with one of the following sets of nonzero entries prescribed:*

- *the nondiagonal entries of an arbitrary row,*
- *the nondiagonal entries of an arbitrary column,*
- *the upper diagonal entries,*
- *the lower diagonal entries.*

We next describe the maximum number of equal entries, and distinct from ± 1, in a transitive matrix. We will use the following auxiliary results, the first of which is a well known theorem in graph theory, obtained by König [7].

Theorem 13 *A graph is bipartite if and only if it has no odd cycle.*

Lemma 14 *The maximum number of edges among all bipartite graphs with n vertices is*

$$\frac{n^2}{4} \text{ if } n \text{ is even}$$

and

$$\frac{n^2 - 1}{4} \text{ if } n \text{ is odd.}$$

Proof. Let $G = (X_1 \cup X_2, U)$ be a bipartite graph with n vertices and classes X_1 and X_2 of vertices, where $|X_1| = p$. Then the maximum number of edges of G is $f(p) = (n-p)p$ (this number occurs when G is the complete bipartite graph).

Since $0 = f'(p) = n - 2p$ implies $p = \frac{n}{2}$, then, if n is even, the maximum number of edges among all bipartite graphs with n vertices is $f(\frac{n}{2}) = \frac{n^2}{4}$.

On the other hand, if n is odd, the maximum number of edges among all bipartite graphs with n vertices is $f(\frac{n+1}{2}) = f(\frac{n-1}{2}) = \frac{n^2-1}{4}$. ■

Theorem 15 *Let A be an n-by-n transitive matrix and let $x \in \mathbb{R} \backslash \{-1, 0, 1\}$. Then the number of entries of A equal to x is at most $\frac{n^2}{4}$ if n is even and $\frac{n^2-1}{4}$ if n is odd.*

Proof. Let $G = (X, U)$ be a undirected graph where $X = \{1, \ldots, n\}$ and there is an edge between vertices i and j if and only if $a_{ij} = x$.

Suppose that there is an odd cycle in G, $i_1, i_2, \ldots, i_p, i_1$. Then p is odd. Since $A = [a_{ij}]$ is transitive,

$$a_{i_1,i_p} = a_{i_1,i_2} a_{i_2,i_3} \ldots a_{i_{p-1} i_p},$$

where $a_{i_l,i_{l+1}}$ is x or $\frac{1}{x}$. This product has $p-1$ factors and $p-1$ is even, so the product is never equal to x or $\frac{1}{x}$, a contradiction since there is an edge in G between i_1 and i_p. Therefore, G has no odd cycle and, by Theorem 13, it is a bipartite graph. By Lemma 14, G has at most $\frac{n^2}{4}$ edges if n is even or $\frac{n^2-1}{4}$ edges if n is odd. ■

Recall that e_m is the m-by-1 column vector with all entries equal to 1. Also, for a positive real number a, we denote by $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the largest integer less than or equal to a (resp. the smallest integer greater than or equal to a).

Remark 16 *The upper bound given in Theorem 15 for the number of entries equal to a given $x \in \mathbb{R} \backslash \{-1, 0, 1\}$ of a transitive matrix is attained for example by the matrix $ww^{(-1)}$, where $w^{(-1)} = \left[\begin{array}{cc} e^T_{\lfloor \frac{n+1}{2} \rfloor} & xe^T_{\lceil \frac{n-1}{2} \rceil} \end{array} \right]$.*

From Remark 16, we obtain the following result.

Corollary 17 *Let $1 \leq l \leq n$ and $x \in \mathbb{R} \backslash \{-1, 0, 1\}$. Then there is an n-by-n transitive matrix A with exactly k_l entries equal to x, where*

$$k_l = \begin{cases} \frac{l^2}{4} \text{ if } l \text{ is even} \\ \\ \frac{l^2-1}{4} \text{ if } l \text{ is odd} \end{cases}.$$

Proof. Let, for example, $w^{(-1)} = \left[\begin{array}{ccc} e^T_{\lfloor \frac{l+1}{2} \rfloor} & xe^T_{\lceil \frac{l-1}{2} \rceil} & x^3 e^T_{n-l} \end{array} \right]$. Then $ww^{(-1)}$ is a transitive matrix with exactly k_l entries equal to x. ■

Another interesting corollary of our results is the following.

Corollary 18 *Let $x_1, \ldots, x_s$ be positive integers such that $x_1 + \cdots + x_s = n$. Then*

$$\sum_{i=1}^{s-1} x_i x_{i+1} \leq \begin{cases} \frac{n^2}{4} \text{ if } n \text{ is even} \\ \\ \frac{n^2-1}{4} \text{ if } n \text{ is odd} \end{cases}.$$

Proof. Let $w^{(-1)} = \left[\begin{array}{cccc} e^T_{x_1} & ye^T_{x_2} & \cdots & y^s e^T_{x_s} \end{array} \right]$ with $y > 0$ and $y \neq 1$. Then $ww^{(-1)}$ is a transitive matrix with exactly $\sum_{i=1}^{s-1} x_i x_{i+1}$ entries equal to y. By Theorem 15, the result follows. ■

5 The Toeplitz matrix $C(n, x)$

In this section we consider the n-by-n Toeplitz matrix

$$C(n,x) = \begin{bmatrix} 1 & x & \cdots & x & x \\ \frac{1}{x} & 1 & \cdots & x & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{x} & \frac{1}{x} & \cdots & 1 & x \\ \frac{1}{x} & \frac{1}{x} & \cdots & \frac{1}{x} & 1 \end{bmatrix},$$

where $x > 0$, and study some approximations of this matrix by a transitive matrix. Note that $C(n, x)$ is a positive reciprocal matrix. Since, for $n = 2$, $C(n, x)$ is transitive, from now on we assume $n > 2$. In this case $C(n, x)$ is transitive if and only if $x = 1$. The consistency index CI of $C(n, x)$, defined as

$$CI = \frac{\lambda_0 - n}{n-1},$$

where λ_0 is the Perron eigenvalue of $C(n, x)$, was studied in [3, 19].

5.1 Spectral properties of $C(n, x)$

In this section we describe the eigenvalues of $C(n, x)$ when $x \in \mathbb{R}$ (not necessarily positive). In [19] the Perron eigenvalue of $C(n, x)$, with $x > 0$, was deduced.

It can be easily verified that the eigenvalues of $C(n, x)$ are 1, with multiplicity n, if $x = 0$; n, with multiplicity 1, and 0, with multiplicity $n-1$, if $x = 1$; $2-n$, with multiplicity 1, and 2, with multiplicity $n-1$, if $x = -1$. Next, we assume that $x \in \mathbb{R}\backslash\{0, 1, -1\}$.

Given $a \in \mathbb{R}\backslash\{0\}$, we denote by $a^{\frac{p}{q}}$, where p and q are integers and p is even, the real positive qth root of a^p.

Theorem 19 *Let $x \in \mathbb{R}\backslash\{0, 1, -1\}$. Then, the matrix $C(n, x)$ has n distinct eigenvalues which are the n complex solutions of the equation*

$$\left(\frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}\right)^n = \frac{1}{x^2},$$

that is, if $s_k = x^{\frac{2}{n}} e^{\frac{i2k\pi}{n}}$, $k = 1, \ldots, n$, where $i = \sqrt{-1}$, then

$$\lambda_k = \frac{(x-1)(x+s_k)}{x(s_k - 1)}.$$

The Perron eigenvalue of $C(n, x)$ is λ_n. Moreover, the eigenvectors associated with λ_k are proportional to $w^{(k)} = \begin{bmatrix} w_1^{(k)} & \cdots & w_n^{(k)} \end{bmatrix}^T$, with

$$w_j^{(k)} = \left(\frac{x^{-1} - 1 + \lambda_k}{x - 1 + \lambda_k}\right)^{j-1} = \left(\frac{1}{s_k}\right)^{j-1} = x^{\frac{-2(j-1)}{n}} e^{-\frac{i2k\pi(j-1)}{n}}.$$

Proof. We will use arguments similar to those in [19]. Let λ be an eigenvalue of $C(n,x)$ and $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ be an associated eigenvector. The equation $C(n,x)w = \lambda w$ is equivalent to

$$x^{-1}(w_1 + \cdots + w_{k-1}) + w_k + x(w_{k+1} + \cdots + w_n) = \lambda w_k,$$

$k = 1, \ldots, n$. Subtracting equation $k-1$ from equation k, $k = 2, \ldots, n$, we obtain

$$\left\{ \begin{array}{l} (1-\lambda)w_1 + x(w_2 + \cdots + w_n) = 0 \\ (x^{-1} - 1 + \lambda)w_{k-1} + (1 - x - \lambda)w_k = 0, \ k = 2, \ldots, n. \end{array} \right. . \tag{2}$$

Note that $\lambda \neq 1-x$. In fact, if $\lambda = 1-x$ we would have $x^{-1} - 1 + \lambda \neq 0$ implying $w_{k-1} = 0$ for $k = 2, \ldots, n$. Then, from the first equation, as w is nonzero, we would have $x = 0$, a contradiction. Thus (2) is equivalent to

$$\left\{ \begin{array}{l} (1-\lambda)w_1 + x(w_2 + \cdots + w_n) = 0 \\ w_k = \left(\frac{x^{-1}-1+\lambda}{x-1+\lambda} \right)^{k-1} w_1 \end{array} \right. .$$

For $w_1 = 1$ and noting that $w_2 \neq 1$, substituting the expressions for w_k into the first equation, we get

$$\lambda = 1 + x\frac{w_2^n - w_2}{w_2 - 1}. \tag{3}$$

We also have

$$\begin{aligned} w_2 &= \frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda} \\ \Leftrightarrow \lambda &= \frac{xw_2 - w_2 + 1 - x^{-1}}{1 - w_2} = \frac{(x-1)(xw_2 + 1)}{x(1 - w_2)}. \end{aligned} \tag{4}$$

From (3) and (4), we obtain

$$w_2^n = \frac{1}{x^2}.$$

Now the claim follows easily taking into account that $w_2^n = \frac{1}{s_k^n}$, $k = 1, \ldots, n$. ∎

From Theorem 19, we obtain the following result.

Corollary 20 *Let $x \in \mathbb{R}\backslash\{0, 1, -1\}$. If n is even, $C(n,x)$ has exactly two real eigenvalues, while if n is odd it has one real eigenvalue. Moreover, $rank(C(n,x)) = n$.*

5.2 Perron transitive matrix associated with $C(n,x)$

In this section we show that the left and right Perron transitive matrices associated with $C(n,x)$, with $x > 0$ and $x \neq 1$, coincide and are Toeplitz matrices.

Let $x > 0$ and $x \neq 1$. From Theorem 19, the Perron eigenvalue of $C(n,x)$ is

$$\lambda = \frac{(x-1)(x + x^{\frac{2}{n}})}{x(x^{\frac{2}{n}} - 1)}$$

and the principal eigenvector is $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ with

$$w_j = \left(\frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}\right)^{j-1} = x^{-\frac{2(j-1)}{n}}.$$

Since a left Perron eigenvector of $C(n,x)$ is a right Perron eigenvector of $C^T(n,x)$, the left principal eigenvector of $C(n,x)$, say w', is obtained from w by replacing x with x^{-1}, that is, $w' = \begin{bmatrix} w_1' & \cdots & w_n' \end{bmatrix}^T$ with

$$w_j' = x^{\frac{2(j-1)}{n}}.$$

Note that the Hadamard product $w \circ w'$ is a constant vector. Thus, using the notation introduced in Section 2, we have $ww^{(-1)} = (w'^{(-1)})^T w'^T$, that is, the left and right Perron transitive matrices associated with $C(n,x)$ coincide.

The transitive matrix $ww^{(-1)} = [s_{ij}]$ is defined by

$$s_{ij} = \frac{\left(\frac{x^{-1}-1+\lambda}{x-1+\lambda}\right)^{i-1}}{\left(\frac{x^{-1}-1+\lambda}{x-1+\lambda}\right)^{j-1}} = \frac{x^{-\frac{2(i-1)}{n}}}{x^{-\frac{2(j-1)}{n}}} = x^{\frac{2(j-i)}{n}},$$

and is a Toeplitz matrix. We will denote this matrix by $P(n,x)$. We then have

$$||C(n,x) - P(n,x)||_F^2 = \sum_{j=1}^{n-1} (n-j)((x - x^{\frac{2j}{n}})^2 + (x^{-1} - x^{-\frac{2j}{n}})^2). \qquad (5)$$

5.3 Transitive matrix associated with $C(n,x)$ preserving the maximum number of entries

In this section we give a positive transitive matrix that preserves the maximum possible number of entries of $C(n,x)$, according to Theorem 15, and that has distance to $C(n,x)$, measured in the Frobenius matrix norm, smaller than the one of the Perron transitive matrix associated with $C(n,x)$. Here and throughout, we assume, without loss of generality, that $x > 1$, as the case $0 < x < 1$ reduces to the previous one by considering the transpose matrix $C^T(n,x)$.

We will use the following technical auxiliary result.

Lemma 21 *Let $n > 2$ be an integer and $x \in \mathbb{R}$ with $x > 1$. Then*

$$(x - x^{\frac{2(n-1)}{n}})^2 + (x^{-1} - x^{-\frac{2}{n}})^2 > \frac{1}{2}\left((x-1)^2 + (x^{-1} - 1)^2\right).$$

Proof. A calculation shows that

$$\begin{aligned} &((x - x^{\frac{2(n-1)}{n}})^2 + (x^{-1} - x^{-\frac{2}{n}})^2) - \frac{1}{2}\left((x-1)^2 + (x^{-1}-1)^2\right) \\ &= \left(x^{-\frac{4}{n}} + x^{-1} - 2x^{-1-\frac{2}{n}}\right) + \left(x^{4-\frac{4}{n}} - 2x^{3-\frac{2}{n}}\right) + (x-1) + \frac{x^{-2}}{2} + \frac{1}{2}x^2. \end{aligned}$$

We have

$$x^{4-\frac{4}{n}} - 2x^{3-\frac{2}{n}} > 0 \Leftrightarrow x^{1-\frac{2}{n}} > 2 \Leftrightarrow x > 2^{\frac{n}{n-2}}.$$

Clearly,

$$x^{-\frac{4}{n}} + x^{-1} - 2x^{-1-\frac{2}{n}} = \left(x^{-\frac{4}{n}} - x^{-1-\frac{2}{n}}\right) + \left(x^{-1} - x^{-1-\frac{2}{n}}\right) > 0.$$

Thus, if $x > 2^{\frac{n}{n-2}}$, the claim follows. Now suppose that $x \leq 2^{\frac{n}{n-2}}$. Since $\frac{n}{n-2} \leq 3$, we have $x \leq 8$ and

$$\begin{aligned}
&(x - x^{\frac{2(n-1)}{n}})^2 - \frac{1}{2}((x-1)^2 + (x^{-1}-1)^2) \\
&> (x-64)^2 - \frac{1}{2}((x-1)^2 + (x^{-1}-1)^2) \\
&= \frac{1}{2x^2}\left(x^4 - 254x^3 + 8190x^2 + 2x - 1\right) \\
&> \frac{1}{2x^2}\left(x^4 - 2032x^2 + 8190x^2 + 2x - 1\right) > 0.
\end{aligned}$$

Since $(x^{-1} - x^{-\frac{2}{n}})^2 > 0$, the claim follows. ■

Recall that e_m is the $m \times 1$ vector with all entries equal to 1.

If n is even, let

$$Q(n,x) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} e_{\frac{n}{2}} \\ \frac{1}{x} \\ \vdots \\ \frac{1}{x} \end{bmatrix} \begin{bmatrix} e_{\frac{n}{2}} & x & \cdots & x \end{bmatrix}, \tag{6}$$

where Q_{11} and Q_{22} have size $\frac{n}{2}$ and all entries equal to 1, Q_{12} has all entries equal to x and Q_{21} has all entries equal to $\frac{1}{x}$.

If n is odd, for $b \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$, consider the n-by-n matrix

$$Q_b(n,x) = \begin{bmatrix} e_b \\ \frac{1}{x} \\ \vdots \\ \frac{1}{x} \end{bmatrix} \begin{bmatrix} e_b^T & x & \cdots & x \end{bmatrix}.$$

According to Theorem 15, the positive transitive matrix $Q(n,x)$, if n is even, or $Q_b(n,x)$, if n is odd, preserves the maximum possible number of entries of $C(n,x)$. We next show that this matrix is closer to $C(n,x)$ than the Perron transitive matrix $P(n,x)$ associated with $C(n,x)$.

Theorem 22 *Let $x \in \mathbb{R}$ with $x > 1$. Let $P(n,x)$ be the Perron transitive matrix associated with $C(n,x)$. Let $Q(n,x)$ be the matrix (6), if n is even, or $Q_b(n,x)$ for some $b \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$, if n is odd. Then*

$$||C(n,x) - Q(n,x)||_F^2 < ||C(n,x) - P(n,x)||_F^2.$$

Proof. Using (5), we have

$$\begin{aligned}||C(n,x)-P(n,x)||_F^2 &= \sum_{j=1}^{n-1}(n-j)((x-x^{\frac{2j}{n}})^2+(x^{-1}-x^{-\frac{2j}{n}})^2)\\ &> \sum_{j=1}^{n-1}(n-j)((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2)\\ &= \frac{n}{2}\,(n-1)\,((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2).\end{aligned}$$

Case 1: Suppose that n is even. We have

$$\begin{aligned}||C(n,x)-Q(n,x)||_F^2 &= 2\sum_{i=1}^{n/2}\sum_{j=i+1}^{n/2}((x-1)^2+(x^{-1}-1)^2)\\ &= \frac{n}{2}(\frac{n}{2}-1)((x-1)^2+(x^{-1}-1)^2).\end{aligned}$$

Taking into account Lemma 21, we have

$$\begin{aligned}(n-1)\,((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2) &> (n-2)\,((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2)\\ &> \frac{n-2}{2}((x-1)^2+(x^{-1}-1)^2),\end{aligned}$$

implying the claim.

Case 2: Suppose that n is odd. We have

$$\begin{aligned}||C(n,x)-Q(n,x)||_F^2 &= \frac{n-1}{2}((x-1)^2+(x^{-1}-1)^2)+2\sum_{i=1}^{(n-1)/2}\sum_{j=i+1}^{(n-1)/2}((x-1)^2+(x^{-1}-1)^2)\\ &= \frac{1}{4}\,(n-1)^2\,((x-1)^2+(x^{-1}-1)^2).\end{aligned}$$

Taking into account Lemma 21, we have

$$\begin{aligned}n((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2) &> (n-1)((x-x^{\frac{2(n-1)}{n}})^2+(x^{-1}-x^{-\frac{2}{n}})^2)\\ &> \frac{n-1}{2}((x-1)^2+(x^{-1}-1)^2),\end{aligned}$$

implying the result. ■

5.4 Closest Toeplitz transitive matrix associated with $C(n,x)$

In many practical problems it is known a priory that the comparison matrix is a Toeplitz matrix. Thus, when finding a transitive matrix close to a given Toeplitz reciprocal matrix, it makes sense to find one that preserves the Toeplitz structure.

In Section 5.3 we have shown that there are positive transitive matrices closer to $C(n,x)$ than the Perron transitive matrix $P(n,x)$ associated with $C(n,x)$

(which is a Toeplitz matrix). However, the exhibited matrices do not have a Toeplitz structure. In this section, we show that there are positive transitive Toeplitz matrices closer to $C(n,x)$ than $P(n,x)$.

The next characterization of positive transitive Toeplitz matrices can be easily shown.

Lemma 23 *Let A be an n-by-n positive transitive matrix. Then A is Toeplitz if and only if, for some $a>0$, we have $A=ww^{(-1)}$, with $w=\begin{bmatrix} 1 & a & \cdots & a^{n-1}\end{bmatrix}^T$.*

We denote the matrix $ww^{(-1)}$ in Lemma 23 by $T(n,\frac{1}{a})$. Note that the entry in position i,j of $T(n,\frac{1}{a})$ is a^{i-j}. In particular, the entry in position $1,2$ is $1/a$.

The following technical proposition will be used in the proofs of the next results and can be easily shown.

Proposition 24 *Let $x\in\mathbb{R}$ with $x>1$. For $y\in\mathbb{R}$ with $y>0$, let $f(y)=(x-y)^2+(x^{-1}-y^{-1})^2$. Then, the derivative of f is given by $f'(y)=\frac{2}{xy}g(y)$, where*

$$g(y)=\frac{(y-x)\left(xy^3+1\right)}{y^2}.$$

For $b\in\mathbb{R}$ positive, define $h(b)=||C(n,x)-T(n,b)||_F^2$. We then have

$$h(b)=\sum_{j=1}^{n-1}(n-j)((x-b^j)^2+(x^{-1}-b^{-j})^2)=\sum_{j=1}^{n-1}(n-j)f(b^j). \tag{7}$$

The next lemma shows that any local minimum of h occurs in the open interval $]x^{\frac{1}{n-1}},x[$.

Lemma 25 *Let $x\in\mathbb{R}$ with $x>1$ and $n>2$ be an integer. We have $h'(b)>0$ if $b\geq x$ and $h'(b)<0$ if $0<b\leq x^{\frac{1}{n-1}}$.*

Proof. We next use the functions f and g defined in Proposition 24. From (7) and taking into account Proposition 24, we have

$$h'(b)=\sum_{j=1}^{n-1}(n-j)jb^{j-1}f'(b^j)=\frac{2}{xb}\sum_{j=1}^{n-1}(n-j)jg(b^j).$$

If $b>x$, then we have $b^j>x$ for $j=1,\ldots,n-1$. Thus, $h'(b)>0$. If $b^{n-1}<x$, then $b^j<x$ for $j=1,\ldots,n-1$ and $h'(b)<0$. On the other hand, $h'(x)>0$ and $h'(x^{\frac{1}{n-1}})<0$. ■

Observe that the Perron transitive matrix $P(n,x)$ associated with $C(n,x)$ is $T(n,x^{\frac{2}{n}})$. Next we show, in particular, that there is a positive transitive Toeplitz matrix closer to $C(n,x)$ than $T(n,x^{\frac{2}{n}})$.

Lemma 26 *Let $x\in\mathbb{R}$ with $x>1$ and $n>2$ be an integer. We have $h'(x^{\frac{2}{n}})>0$.*

Proof. We next use the functions f and g defined in Proposition 24. We have

$$h'(b) = \frac{2}{xb}\sum_{j=1}^{n-1}(n-j)jg(b^j)$$

$$= \begin{cases} \frac{2}{xb}\sum_{j=1}^{\frac{n-1}{2}}(n-j)j\left(g(b^j)+g(b^{n-j})\right) & \text{if } n \text{ is odd} \\ \frac{2}{xb}\left(\frac{n^2}{4}g(b^{\frac{n}{2}})+\sum_{j=1}^{\frac{n}{2}-1}(n-j)j\left(g(b^j)+g(b^{n-j})\right)\right) & \text{if } n \text{ is even} \end{cases}.$$

For $j = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, we have

$$g(x^{\frac{2j}{n}}) + g(x^{\frac{2(n-j)}{n}}) = \frac{\left(x^{\frac{2j}{n}} - x\right)\left(x^{1+\frac{6j}{n}}+1\right)}{x^{\frac{4j}{n}}} + \frac{\left(x^{2-\frac{2j}{n}} - x\right)\left(x^{7-\frac{6j}{n}}+1\right)}{x^{4-\frac{4j}{n}}}$$

$$= x^{-\frac{4j+3n}{n}}\left(x^4-1\right)\left(x - x^{\frac{2j}{n}}\right)^2\left(x^{1+\frac{2j}{n}} + x^{\frac{4j}{n}} + x^2\right) > 0.$$

Also, if n is even,

$$g(x^{\frac{2}{n}})^{\frac{n}{2}} = g(x) = 0.$$

Thus, we conclude that $h'(x^{\frac{2}{n}}) > 0$. ■

Observe that from Lemma 26 we conclude that h increases in a neighborhood of $x^{\frac{2}{n}}$ and therefore h does not attain a minimum at $x^{\frac{2}{n}}$. Moreover, for any $z < x^{\frac{2}{n}}$ sufficiently close to $x^{\frac{2}{n}}$, we have $h(z) < h(x^{\frac{2}{n}})$.

Theorem 27 *Let $x \in \mathbb{R}$ with $x > 1$ and let $n > 2$ be an integer. Then there is $b \in]x^{\frac{1}{n-1}}, x^{\frac{2}{n}}[$ such that*

$$||C(n,x) - T(n,b)||_F < \min\{||C(n,x) - T(n,x^{\frac{2}{n}})||_F, ||C(n,x) - T(n,x^{\frac{1}{n-1}})||_F\}.$$

Proof. By Lemma 26, $h'(x^{\frac{2}{n}}) > 0$. On the other hand, by Lemma 25, we have $h'(x^{\frac{1}{n-1}}) < 0$. Thus, we conclude, using the continuity of $h'(y)$, that $h'(y)$ has a root in $]x^{\frac{1}{n-1}}, x^{\frac{2}{n}}[$ where h attains a minimum. ■

Note that the amplitude of the interval $]x^{\frac{1}{n-1}}, x^{\frac{2}{n}}[$ approaches 0 as $n \to \infty$.

6 Numerical experiments

In this section we give some numerical examples that illustrate the results in Section 5.

Consider the matrix $C(3,2)$. Its Perron eigenvalue is $\lambda = 3.0536$ and its principal eigenvector is

$$w = \begin{bmatrix} 1 \\ 0.63 \\ 0.63^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.63 \\ 0.3969 \end{bmatrix}.$$

Thus, the Perron transitive matrix associated with $C(3,2)$ is

$$P(3,2) = ww^{(-1)} = \begin{bmatrix} 1 & 1.5874 & 2.5195 \\ 0.63 & 1 & 1.5874 \\ 0.3969 & 0.63 & 1 \end{bmatrix},$$

also denoted by $T(3,\ 1.5874)$, according to the notation introduced in Section 5.4. Note that $2^{\frac{2}{3}} = 1.5874$. A calculation shows that $||C(3,2) - P(3,2)||_F^2 = 0.6551$.

For $b > 0$, define $h(b) = ||C(3,2) - T(3,b)||_F^2$. Then,

$$h(b) = 2\,(b-2)^2 + \left(b^2 - 2\right)^2 + 2\left(\frac{1}{b} - \frac{1}{2}\right)^2 + \left(\frac{1}{b^2} - \frac{1}{2}\right)^2.$$

The next table gives the values of $h(b)$ for some b's close to 1.5874 :

b	1.4	$\sqrt{2}$	1.48	1.5	1.52	1.54	1.56	1.57	1.58	1.59	$\sqrt{3}$
$h(b)$	0.814	0.772	0.641	0.621	0.612	0.612	0.623	0.632	0.644	0.659	1.183

As follows from Lemma 26 and is confirmed by our numerical results, h is strictly increasing in a neighborhood of $b = 1.5874$. From Theorem 27, $h(b)$ has a local minimum attained in $]2^{\frac{1}{2}}, 2^{\frac{2}{3}}[=]1.4142, 1.5874[$. From the table, we can guess that this minimum occurs at $b_0 \in [1.5, 1.54]$. In fact, a calculation shows that $b_0 = 1.52902$ and $h(b_0) = 0.6105$.

In the next table we explicitly compare the distance, measured in the Frobenius matrix norm, to $C(3,2)$ of the transitive matrices $T(3,b)$ for b on the boundary of the interval given in Theorem 27 and for two values of b in the interior of this interval. For reference, the minimum distance attained in this interval is also included.

b	$2^{2/3}$	$2^{1/2}$	$2^{\frac{\frac{2}{3}+\frac{1}{2}}{2}}$	$\frac{2^{\frac{2}{3}}+2^{\frac{1}{2}}}{2}$	b_0
$h(b)$	0.655	0.772	0.622	0.621	0.611

We can observe that when $b \in \{2^{\frac{\frac{2}{3}+\frac{1}{2}}{2}}, \frac{2^{\frac{2}{3}}+2^{\frac{1}{2}}}{2}\}$, $T(3,b)$ is one of the closest matrices to $C(3,2)$.

In [13], the authors propose the transitive matrix $(w_1^{(-1)})^T w_1^T$, where w_1 is an eigenvector of the positive definite matrix P defined below associated with the smallest eigenvalue (or, equivalently, the principal eigenvector of P^{-1}):

$$P = \begin{bmatrix} 9+3 & 0 & 0 \\ 0 & \frac{21}{4}+3 & 0 \\ 0 & 0 & \frac{3}{2}+3 \end{bmatrix} - \left(\begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T \right)$$

$$= \begin{bmatrix} 10 & -\frac{5}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{25}{4} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{2} & \frac{5}{2} \end{bmatrix}.$$

We observe that, contrarily to what is claimed in Lemma 2 in [13], the matrix P in this lemma is not always positive definite implying that the proposed method to obtain a transitive matrix cannot always be applied.

A calculation shows that

$$w_1 = \begin{bmatrix} 0.3253 \\ 0.4662 \\ 0.8227 \end{bmatrix}$$

and the Perron eigenvalue of P^{-1} is 10.574. Thus,

$$(w_1^T)^{(-1)} w_1^T = \begin{bmatrix} (0.32531)^{-1} \\ (0.46625)^{-1} \\ (0.82267)^{-1} \end{bmatrix} \begin{bmatrix} 0.32531 \\ 0.46625 \\ 0.82267 \end{bmatrix}^T = \begin{bmatrix} 1 & 1.4332 & 2.5289 \\ 0.6977 & 1 & 1.7644 \\ 0.3954 & 0.5668 & 1 \end{bmatrix}.$$

Moreover, $||C(3,2) - (w_1^{(-1)})^T w_1^T||_F^2 = 0.7110$. Thus, $(w_1^{(-1)})^T w_1^T$ has a larger distance to $C(3,2)$, when compared with most of the matrices $T(3,b)$ considered in the previous table.

In Figures 1-4, we compare the distance of $T(n,b)$ to $C(n,x)$ for $n = 3$ and different values of $x > 1$ (Figures 1 and 2), and for $x = 2$ and different values of $n > 2$ (Figure 3 and 4). Bearing in mind Theorem 27, we consider

$$b \in \{x^{\frac{1}{n-1}},\ x^{\frac{2}{n}},\ x^{\frac{\frac{1}{n-1}+\frac{2}{n}}{2}},\ \frac{x^{\frac{1}{n-1}} + x^{\frac{2}{n}}}{2}\}.$$

For completeness, we also include for comparison the distance of $Q(n,x)$ to $C(n,x)$, where $Q(n,x)$ is as in Theorem 22. The vertical axis in the figures represents the value of $h(n,x) = ||M - C(n,x)||_F^2$, where M is either $Q(n,x)$ or one of the matrices $T(n,b)$ mentioned above. The red, green, blue and black lines correspond to $T(n,b)$ with $b = x^{\frac{2}{n}}$, $b = x^{\frac{1}{n-1}}$, $b = x^{\frac{\frac{1}{n-1}+\frac{2}{n}}{2}}$ and $b = \frac{x^{\frac{1}{n-1}}+x^{\frac{2}{n}}}{2}$, respectively. The magenta line corresponds to $Q(n,x)$. Figure 2 (resp. Figure 4) is a zoom of the left part of Figure 1 (resp. Figure 3).

We observe that, for $n = 3$ and x large, the Perron transitive matrix $T(3, x^{\frac{2}{n}})$ is the one with largest distance to $C(3,x)$. On the other hand, $T(x^{\frac{1}{n-1}})$ is the one with smallest distance to $C(3,x)$. When x is close to 1, all matrices $T(3,b)$ have similar distance to $C(3,x)$, as expected.

For $x = 2$ fixed, we conclude that $T(n, x^{\frac{1}{n-1}})$ and $Q(n, x)$ are the matrices with largest distance to $C(n, x)$, while all the remaining matrices have approximately the same distance to $C(n, x)$.

Note that in Figures 2-4 the blue and black lines almost coincide.

As a conclusion, it seems that the matrices $T(n, x^{\frac{\frac{1}{n-1}+\frac{2}{n}}{2}})$ and $T(n, \frac{x^{\frac{1}{n-1}}+x^{\frac{2}{n}}}{2})$ have a good behavior in terms of distance to $C(n, x)$ when compared with $T(n, x^{\frac{1}{n-1}})$, $T(n, x^{\frac{2}{n}})$ and $Q(n, x)$.

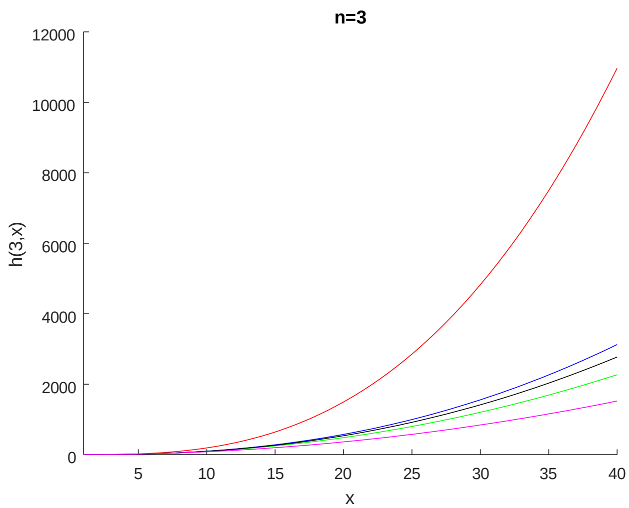

Figure 1

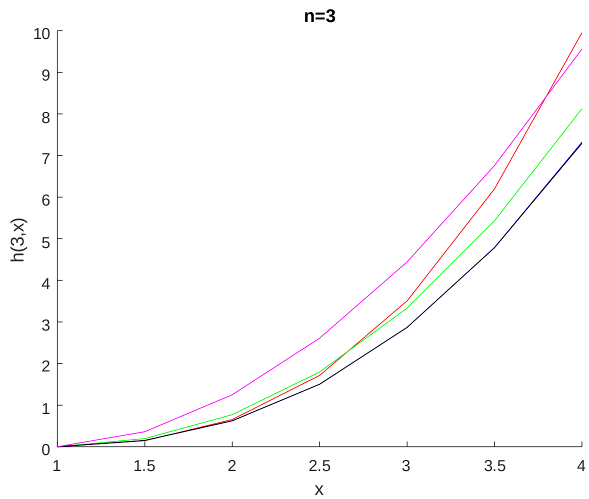

Figure 2

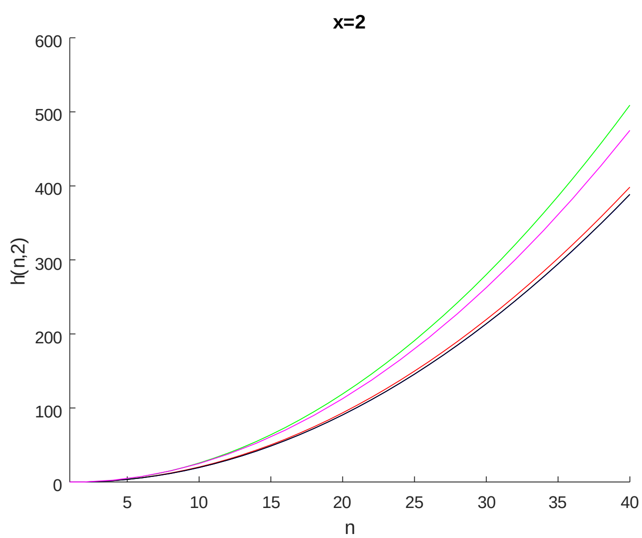

Figure 3

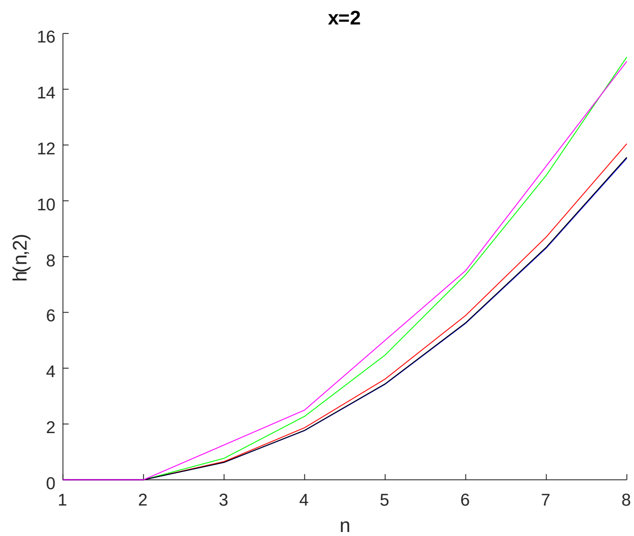

Figure 4

7 Conclusions

We have shown that an n-by-n reciprocal matrix can have any rank from 1 to n, except 2, and have given a way of constructing a (positive) reciprocal matrix with the rank and the off-diagonal entries of a row (column) prescribed. We used techniques from graph theory to describe the maximum possible number of equal entries, distinct from ± 1, in a transitive matrix. We then have considered the matrix $C(n,x)$ and have given a transitive matrix $Q(n,x)$ that preserves the maximum possible number of entries of $C(n,x)$. We have shown that this matrix is closer, with the distance measured in the Frobenius matrix norm, to $C(n,x)$ than the Perron transitive matrix $P(n,x)$ associated with $C(n,x)$, proposed by Saaty. Since $C(n,x)$ is a Toeplitz matrix, we also focused on Toeplitz transitive matrices close to $C(n,x)$. We have shown that there are Toeplitz matrices closer to $C(n,x)$ than $P(n,x)$. So, in terms of distance to $C(n,x)$, we have shown that there are matrices closer to $C(n,x)$ than $P(n,x)$ which in addition preserve other properties of $C(n,x)$. Of course, these matrices do not have the same Perron eigenvector as $C(n,x)$, contrarily to what happens with $P(n,x)$. Finally, we have given some numerical examples that show the behavior, in terms of distance to $C(n,x)$, of some transitive matrices suggested by our theoretical results.

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