# On a conjecture concerning the Bruhat order


Rosário Fernandes[*], Henrique F. da Cruz[†], Domingos Salomão[‡]

[*] CMA and Departamento de Matemática da Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829-516 Caparica, Portugal.

[†] CMA and Departamento de Matemática da Universidade da Beira Interior, Rua Marquês D'Avila e Bolama, 6201-001 Covilhã, Portugal.

[‡] Universidade Mandume ya Ndemufayo, Escola Superior Pedagógica do Namibe, Angola.


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**Abstract**

Let $R$ and $S$ be two sequences of positive integers in nonincreasing order having the same sum. Let $\mathcal{A}(R,S)$ be the class of all $(0,1)$-matrices with row sum vector $R$ and column sum vector $S$. If $\mathcal{A}(R,S)$ is nonempty, an inversion in $A \in \mathcal{A}(R,S)$ consists of two entries of $A$ equal to 1, one of them is located to the top-right of the other. Let $\nu(A)$ be the total number of inversions in $A$. The Bruhat order is a partial order defined on $\mathcal{A}(R,S)$ and denoted by $\preceq_B$. In this paper, we prove the conjecture:

- "If $A, C \in \mathcal{A}(R,S)$, $A \neq C$ and $A \preceq_B C$ then $\nu(A) < \nu(C)$".



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[*]Email: mrff@fct.unl.pt.


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[†]Email: hcruz@ubi.pt.


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[‡]Email: d.salomao@ubi.pt.

## 1 Introduction

Let $R = (r_1, \ldots, r_m)$, and $S = (s_1, \ldots, s_n)$ be two sequences of positive integers in nonincreasing order having the same sum, $r_1 + \ldots + r_m = s_1 + \ldots + s_n$, that is, $R$ and $S$ are partitions of the same weight. Let $\mathcal{A}(R, S)$ be the class of all $m$-by-$n$ $(0, 1)$-matrices with row sum vector $R$ and column sum vector $S$. The class of all $n$-by-$n$ $(0, 1)$-matrices and constant row and column sums $k$ is denoted by $\mathcal{A}(n, k)$.

In the literature there are two partial orders defined on each nonempty class $\mathcal{A}(R, S)$: the Bruhat order and the secondary Bruhat order, [2].

The Bruhat order on $\mathcal{A}(R, S)$ is defined using other matrix: for each $A = [a_{ij}] \in \mathcal{A}(R, S)$, let $\Sigma(A)$ be the $m$-by-$n$ matrix whose $(r, s)$-entry is

$$\sigma_{r,s}(A) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{ij}, \;\; 1 \leq r \leq m, \, 1 \leq s \leq n. \qquad (1)$$

So, if $A, C \in \mathcal{A}(R, S)$ then $A$ precedes $C$ in the Bruhat order, written $A \preceq_B C$, provided that $\Sigma(A) \geq \Sigma(C)$ (by the entrywise order).

Since it is possible to construct the matrix $\Sigma(X)$, for all real matrix $X$, the Bruhat order was extended to other classes of matrices in a natural way, [4, 8]: let $A, C$ be $m$-by-$n$ real matrices with the same row sum vectors and the same column sum vectors then $A$ precedes $C$ in the Bruhat order, written $A \preceq_B C$, provided that $\Sigma(A) \geq \Sigma(C)$ (by the entrywise order).

The secondary Bruhat order on $\mathcal{A}(R, S)$ is based on the notion of interchanges. An interchange consists of replacing one of the following two submatrices by the other, the 2-by-2 matrix

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and the identity matrix of order 2, } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $A_1 \in \mathcal{A}(R, S)$ and $A_2$ is the matrix obtained from $A_1$ replacing a 2-by-2 submatrix of $A_1$ equal to $L_2$ (respectively, $I_2$) by $I_2$ (respectively, $L_2$) then we say that $A_2$ is obtained from $A_1$ by an $L_2 \to I_2$ (respectively, $I_2 \to L_2$) interchange. Sometimes we omit the kind of the interchange.

So, if $A, C \in \mathcal{A}(R, S)$ then $A$ precedes $C$ in the secondary Bruhat order, written $A \preceq_{\widehat{B}} C$, if $A$ can be obtained from $C$ by a finite sequence of $L_2 \to I_2$ interchanges.

It is straightforward to verify that if $A \preceq_{\widehat{B}} C$ then $A \preceq_B C$, that is, the Bruhat order is a refinement of the secondary Bruhat order. Consequently, the minimal (respectively, maximal) matrices for the Bruhat order on $\mathcal{A}(R, S)$ are minimal (respectively, maximal) matrices for the secondary

Bruhat order on $\mathcal{A}(R,S)$. It was conjectured in [1] that the converse is true. However, this conjecture was shown to be false in [3]. Since in the counterexample exhibited in [3] the matrices belong to $\mathcal{A}(R,S)$, with $R \neq S$, this conjecture is still open in $\mathcal{A}(n,k)$.

Many researchers have tried to prove this conjecture and consequently, other notions arose linked to the Bruhat orders, [5, 6, 7, 8, 9, 10]. One of them was the notion of an *inversion* in a $(0,1)$-matrix. This notion arose linked to the construction of chains in the Bruhat orders, [9]. An inversion in $A = [a_{ij}] \in \mathcal{A}(R,S)$ consists of two entries $a_{ij} = a_{kl} = 1$ such that $(i-k)(j-l) < 0$. The total number of inversions in $A$ is denoted by $\nu(A)$.

It was proved in [9] that if $A, C \in \mathcal{A}(R,S)$, $A \neq C$ and $A \preceq_{\widehat{B}} C$ then $\nu(A) < \nu(C)$. Moreover, it was conjectured that the same happens for the Bruhat order on $\mathcal{A}(R,S)$.

In this paper we will prove that conjecture. To prove this fact, we generalize the notion of an interchange from a $(0,1)$-matrix to a real matrix. Note that this generalization to a doubly stochastic matrix can be seen in [4].

The paper is organized as follows. In Section 2 we define an interchange in a real matrix. We also give some general properties of this kind of interchange. We introduce the notion of an inversion in a real matrix. This definition is the matter of Section 3 and extends the notion of an inversion in a $(0,1)$-matrix. In Section 4 we show that if $A, C \in \mathcal{A}(R,S)$, $A \neq C$ and $A \preceq_B C$ then $\nu(A) < \nu(C)$. Finally, in Section 5 we give some concluding remarks.

Here and throughout, we will use the following notation: given two $m$-by-$n$ matrices $A$ and $C$,

- we denote the principal submatrix of $A$ indexed by rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_s$ by $A[\{i_1, \ldots, i_p\}; \{j_1, \ldots, j_s\}]$.
- if $A \preceq_B C$ but $A \neq C$ then we write $A \prec_B C$.

# 2 Interchanges

In this section we present the definition of an interchange in a real matrix and we state some important results for the next sections. As we mentioned before, the definition of an interchange in a doubly stochastic matrix can be seen in [4]. So, we just rewrite this definition.

**Definition 1** *Let $A = [a_{ij}]$ be an $m$-by-$n$ real matrix, with $m, n \geq 2$, and $b$ be a real number. Let $1 \leq k < l \leq m$, $1 \leq p < r \leq n$ be integers and*

$E^{(b)}_{\{k,l;p,r\}} = [e_{ij}]$ *be the* $m$*-by-*$n$ *real matrix with all entries equal to zero, except*

$$E^{(b)}_{\{k,l;p,r\}}[\{k,l\};\{p,r\}] = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}.$$

*We say that the* $m$*-by-*$n$ *real matrix* $D$ *is obtained from* $A$ *by the* $b$*-interchange in the submatrix* $A[\{k,l\};\{p,r\}]$ *if*

$$D - A = E^{(b)}_{\{k,l;p,r\}}.$$

In Definition 1

- when the location of the $b$-interchange is not important or it is obvious, we omit the submatrix where the interchange is.
- sometimes we omit the real number $b$ and we just write the interchange.

**Remark 2** *1. In Definition 1 the matrix* $D$ *is obtained from* $A$ *replacing the submatrix*

$$A[\{k,l\};\{p,r\}] = \begin{bmatrix} a_{kp} & a_{kr} \\ a_{lp} & a_{lr} \end{bmatrix}$$

*by*

$$D[\{k,l\};\{p,r\}] = \begin{bmatrix} a_{kp} - b & a_{kr} + b \\ a_{lp} + b & a_{lr} - b \end{bmatrix}.$$

*2. If* $D$ *is obtained from* $A$ *by an interchange then the two matrices have the same row sum vectors and the same column sum vectors.*

**Remark 3** *Let* $A$ *and* $D$ *be two* $(0,1)$*-matrices with the same row sum vectors and the same column sum vectors.*

1. *If* $D$ *is obtained from* $A$ *by an* $L_2 \to I_2$ *interchange then* $D$ *is obtained from* $A$ *by a* $(-1)$*-interchange.*
2. *If* $D$ *is obtained from* $A$ *by an* $I_2 \to L_2$ *interchange then* $D$ *is obtained from* $A$ *by an* $1$*-interchange.*

It is easy to connect an interchange between two matrices and the Bruhat order.

**Proposition 4** *Let* $A$ *be an* $m$*-by-*$n$ *real matrix,* $b$ *be a real number and* $1 \leq k < l \leq n$*,* $1 \leq p < r \leq n$ *be integers. Let* $D$ *be the matrix obtained* *from $A$ by the $b$-interchange in the submatrix $A[\{k,l\};\{p,r\}]$. Let $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$. Then*

$$\sigma_{i,j}(D) = \begin{cases} \sigma_{ij}(A) - b & \text{if} \quad (i,j) \in \{k,\ldots,l-1\} \times \{p,\ldots,r-1\} \\ \sigma_{ij}(A) & \text{otherwise} \end{cases}.$$

**Remark 5** *Let $D$ be the matrix obtained from a real matrix $A$ by a $b$-interchange.*

- *If $b > 0$ then $A \prec_B D$.*
- *if $b < 0$ then $D \prec_B A$.*
- *if $b = 0$ then $D = A$.*

Using last remark we obtain the next result.

**Proposition 6** *Let $A$ and $D$ be two real matrices with the same row sum vectors and the same column sum vectors. If $D$ is obtained from $A$ by a sequence of nonnegative interchanges then $A \preceq_B D$.*

# 3 Inversions in real matrices

Let $R = (r_1,\ldots,r_m)$ and $S = s_1,\ldots,s_n)$ be two partitions of the same weight such that $\mathcal{A}(R,S)$ is nonempty. As we mentioned in the introduction, an inversion in $A = [a_{ij}] \in \mathcal{A}(R,S)$ consists of two entries $a_{ij} = a_{kl} = 1$ such that $(i-k)(j-l) < 0$. The total number of inversions in $A$ is denoted by $\nu(A)$. In this section we define a new function on the class of real matrices whose restriction to $\mathcal{A}(R,S)$ gives us the total number of inversions in a $(0,1)$-matrix.

**Definition 7** *Let $A = [a_{ij}]$ be an $m$-by-$n$ real matrix with $m,n \geq 2$. We denote by $\xi(A)$ the real number*

$$\xi(A) = \sum_{i=1}^{m-1} \sum_{j=2}^{n} (\sigma_{m,j-1}(A) - \sigma_{i,j-1}(A)) a_{ij},$$

*where the number $\sigma_{ij}(A)$ is defined in (1).*

**Example 8** *Consider the* 2*-by-*3 *real matrix*

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & -1 & 3 \end{bmatrix}.$$

*Then*

$$\begin{aligned}
\xi(A) &= \sum_{i=1}^{1}\sum_{j=2}^{3}(\sigma_{2,j-1}(A) - \sigma_{i,j-1}(A))a_{ij} \\
&= (\sigma_{2,1}(A) - \sigma_{1,1}(A))a_{12} + (\sigma_{2,2}(A) - \sigma_{1,2}(A))a_{13} \\
&= (1-1)2 + (2-3)\tfrac{1}{2} = -\tfrac{1}{2}.
\end{aligned}$$

**Lemma 9** *Let $A \in \mathcal{A}(R, S)$. Then $\xi(A) = \nu(A)$.*

**Proof.** Assume that $A = [a_{ij}]$ and let $(p, q) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ such that $a_{pq} = 1$. By definition,

$$(\sigma_{m,q-1}(A) - \sigma_{p,q-1}(A))a_{pq},$$

counts the total number of $1's$ in $A[\{p+1, \ldots, m\}; \{1, \ldots, q-1\}]$, that is, it counts the number of $1's$ located down-left of $a_{pq}$. Therefore, $(\sigma_{m,q-1}(A) - \sigma_{p,q-1}(A))a_{pq}$ counts the number of inversions with $a_{pq}$ as their top-right 1. ■

**Lemma 10** *Let $A = [a_{ij}]$ be an $m$-by-$n$ real matrix and $b$ be a real number. Let $k, p, r$ be positive integers with $1 \leq k \leq m-1$, $1 \leq p < r \leq n$. Let $x = \sum_{j=p+1}^{r} a_{kj}$ and $z = \sum_{j=p}^{r-1} a_{k+1,j}$. Let $D$ be the matrix obtained from $A$ by the $b$-interchange in the submatrix $A[\{k, k+1\}; \{p, r\}]$. Then*

$$\xi(D) = \xi(A) + (x + z + b)b.$$

**Proof.** By definition we have

$$\xi(D) = \sum_{i=1}^{m-1}\sum_{j=2}^{n}(\sigma_{m,j-1}(D) - \sigma_{i,j-1}(D))d_{ij}.$$

To simplify the proof we consider that

$$\sigma_{l,0}(D) = \sigma_{l,0}(A) = 0, \qquad \text{for } 1 \leq l \leq m.$$

So,

$$\xi(D) = \sum_{i=1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(D) - \sigma_{i,j-1}(D))d_{ij} =$$

$$\sum_{i=1,i\neq k,i\neq k+1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(D)-\sigma_{i,j-1}(D))d_{ij}+\sum_{j=1}^{n}(\sigma_{m,j-1}(D)-\sigma_{k,j-1}(D))d_{kj}+$$

$$\sum_{j=1}^{n}(\sigma_{m,j-1}(D) - \sigma_{k+1,j-1}(D))d_{k+1,j} =$$

Using Proposition 4 and Definition 7

$$\sum_{i=1,i\neq k,i\neq k+1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(A) - \sigma_{i,j-1}(A))a_{ij} - \sum_{j=1}^{n}\sigma_{k,j-1}(D)d_{kj}+$$

$$\sum_{j=1}^{n}[\sigma_{m,j-1}(D)d_{kj} + (\sigma_{m,j-1}(D) - \sigma_{k+1,j-1}(D))d_{k+1,j}] =$$

$$\sum_{i=1,i\neq k,i\neq k+1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(A) - \sigma_{i,j-1}(A))a_{ij}-$$

$$\left(\sum_{j=1,j\notin\{p,\ldots,r\}}^{n}\sigma_{k,j-1}(D)d_{kj} + \sum_{j\in\{p,\ldots,r\}}\sigma_{k,j-1}(D)d_{kj}\right) +$$

$$\sum_{j=1,j\neq p,j\neq r}^{n}[\sigma_{m,j-1}(D)d_{kj} + (\sigma_{m,j-1}(D) - \sigma_{k+1,j-1}(D))d_{k+1,j}]+$$

$$[\sigma_{m,p-1}(D)d_{kp} + (\sigma_{m,p-1}(D) - \sigma_{k+1,p-1}(D))d_{k+1,p}]+$$

$$[\sigma_{m,r-1}(D)d_{kr} + (\sigma_{m,r-1}(D) - \sigma_{k+1,r-1}(D))d_{k+1,r}] =$$

Using Proposition 4 and Definition 7

$$\sum_{i=1,i\neq k,i\neq k+1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(A) - \sigma_{i,j-1}(A))a_{ij}-$$

$$\left(\sum_{j=1,j\notin\{p,\ldots,r\}}^{n}\sigma_{k,j-1}(A)a_{kj} + \sigma_{k,p-1}(A)(a_{kp} - b)\right) -$$

$$\left(\sum_{j=p+1}^{r-1}(\sigma_{k,j-1}(A)-b)a_{k,j}+(\sigma_{k,r-1}(A)-b)(a_{k,r}+b)\right)+$$

$$\sum_{j=1,j\neq p,j\neq r}^{n}[\sigma_{m,j-1}(A)a_{kj}+(\sigma_{m,j-1}(A)-\sigma_{k+1,j-1}(A))a_{k+1,j}]+$$

$$[\sigma_{m,p-1}(A)(a_{kp}-b)+(\sigma_{m,p-1}(A)-\sigma_{k+1,p-1}(A))(a_{k+1,p}+b)]+$$

$$[\sigma_{m,r-1}(A)(a_{kr}+b)+(\sigma_{m,r-1}(A)-\sigma_{k+1,r-1}(A))(a_{k+1,r}-b)]=$$

Using the definition of $\sigma$ (see (1)) we get

$$\sum_{i=1}^{m-1}\sum_{j=1}^{n}(\sigma_{m,j-1}(A)-\sigma_{i,j-1}(A))a_{ij}+\sum_{j=p+1}^{r}a_{k,j}b+\sum_{j=p}^{r-1}a_{k+1,j}b+b^2=$$

$$\sum_{i=1}^{m-1}\sum_{j=2}^{n}(\sigma_{m,j-1}(A)-\sigma_{i,j-1}(A))a_{ij}+(x+z+b)b.$$

So, we get the result. ■

**Remark 11** *In the conditions of the previous proposition we conclude that if $A$ is a real matrix, $x\geq 0$, $z\geq 0$ and $b>0$ then $\xi(D)>\xi(A)$.*

## 4 Conjecture about inversions in $\mathcal{A}(R,S)$

The main result of this section is to prove the conjecture made in [9]. Throughout this section we consider $R=(r_1,\ldots,r_m)$ and $S=(s_1,\ldots,s_n)$ two partitions of the same weight, with $m,n\geq 2$, such that $\mathcal{A}(R,S)$ is nonempty. Using two matrices of $\mathcal{A}(R,S)$, we begin the section with a new construction of an integer matrix whose row sum vector is $R$ and column sum vector is $S$.

**Definition 12** *Let $Y=[y_{ij}]$, $W=[w_{ij}]\in\mathcal{A}(R,S)$ and $\alpha$ be a positive integer with $1\leq\alpha\leq m$. The $(\alpha,Y,W)$-matrix is the $m$-by-$n$ matrix $Z=[z_{ij}]$ such that*

- *for $\alpha=1$, $Z=W$.*
- *for $\alpha=m$, $Z=Y$.*

- *for* $1 < \alpha < m$ *and* $1 \leq j \leq n$,

$$z_{ij} = \begin{cases} y_{ij} & \text{if} \quad i < \alpha \\ s_j - \left( \sum_{l=1}^{\alpha-1} y_{lj} + \sum_{l=\alpha+1}^{m} w_{lj} \right) & \text{if} \quad i = \alpha \\ w_{ij} & \text{if} \quad i > \alpha \end{cases} .$$

**Remark 13** *In the conditions of last definition, if* $1 < \alpha < m$ *and* $Z$ *is the* $(\alpha, Y, W)$*-matrix then*

- *the* $i$*-row of* $Z$ *is the* $i$*-row of* $Y$*, for* $1 \leq i < \alpha$.
- *the* $i$*-row of* $Z$ *is the* $i$*-row of* $W$*, for* $\alpha < i \leq m$.

*So, for* $1 \leq i \leq m$ *and* $i \neq \alpha$*, the* $i$*-row sum of* $Z$ *is* $r_i$*. The* $\alpha$*-row sum of* $Z$ *is*

$$\sum_{j=1}^{n} \left[ s_j - \left( \sum_{l=1}^{\alpha-1} y_{lj} + \sum_{l=\alpha+1}^{m} w_{lj} \right) \right] = \sum_{j=1}^{n} s_j - \sum_{i=1, i \neq \alpha}^{m} r_i = r_\alpha.$$

*For* $1 \leq j \leq n$*, the* $j$*-column sum of* $Z$ *is*

$$\sum_{l=1}^{\alpha-1} y_{lj} + \left( s_j - \left( \sum_{l=1}^{\alpha-1} y_{lj} + \sum_{l=\alpha+1}^{m} w_{lj} \right) \right) + \sum_{l=\alpha+1}^{m} w_{lj} = s_j.$$

**Remark 14** *In the conditions of last definition, if* $1 \leq \alpha < m$*,* $Z$ *is the* $(\alpha, Y, W)$*-matrix,* $X$ *is the* $(\alpha+1, Y, W)$*-matrix and* $Z \neq X$ *then the nonzero entries of* $Z - X$ *are in the submatrix*

$$(Z - X)[\{\alpha, \alpha + 1\}; \{1, \ldots, n\}].$$

*Moreover, as the* $j$*-column sum of* $Z$ *and* $X$ *is* $s_j$*, for* $1 \leq j \leq n$*, then the entry* $(\alpha, j)$ *of* $Z - X$ *is nonzero if and only if the entry* $(\alpha + 1, j)$ *of* $Z - X$ *is nonzero.*

The next proposition is an easy consequence of Definition 12.

**Proposition 15** *Let* $Y = [y_{ij}]$*,* $W = [w_{ij}] \in \mathcal{A}(R, S)$ *and* $\alpha$ *be a positive integer with* $1 \leq \alpha \leq m$*. If* $Z$ *is the* $(\alpha, Y, W)$*-matrix then for* $1 \leq j \leq n$*,*

$$\sigma_{ij}(Z) = \begin{cases} \sigma_{ij}(Y) & \text{if} \quad i < \alpha \\ \sigma_{ij}(W) & \text{if} \quad i \geq \alpha \end{cases} ,$$

*where* $\sigma_{ij}$ *is defined in* (1).

Now we can prove the main result of this paper.

**Theorem 16** *Let $A, C \in \mathcal{A}(R,S)$. If $A \prec_B C$, then $\nu(A) < \nu(C)$.*

**Proof.** Let $A = [a_{ij}]$ and $C = [c_{ij}]$. Let $\beta$ be the minimum integer such that $1 \le \beta \le m-1$ and the $\beta$-row of $A$ is different from the $\beta$-row of $C$.

Let $D = [d_{ij}]$ be the $(\beta, C, A)$-matrix and let $E = [e_{ij}]$ be the $(\beta+1, C, A)$-matrix. By Proposition 15, we get $D \prec_B E \preceq_B C$.

Note that if $\beta$ is the minimum row index where $A$ and $C$ differ then $D = A$.

**Step 1:** Using Remark 14 and the fact that $D \prec_B C$, let $v$ be the minimum integer such that $d_{\beta,v} \neq e_{\beta,v}$. So, $d_{\beta,v} > e_{\beta,v}$.

Let $u$ be the maximum integer such that $d_{\beta,u} \neq e_{\beta,u}$ (this integer exists because the $k$-row sum of $D$ is equal to the $k$-row sum of $E$). Since $D \prec_B E$ and $d_{ij} = e_{ij}$, for $1 \le i < \beta$ and $1 \le j \le n$, then

$$0 \le \sigma_{\beta,j}(D-E) = \begin{cases} \sum_{\gamma=v}^{j}(d_{\beta,\gamma} - e_{\beta,\gamma}) & \text{if } v \le j < u \\ 0 & \text{otherwise} \end{cases}.$$

Consequently, $d_{\beta,u} < e_{\beta,u}$.

Let $\tau = u - v$. For $1 \le t \le \tau + 1$, let

$$b_t = d_{\beta,v+t-1} - e_{\beta,v+t-1}.$$

**Claim 1** The following conditions hold:

- $b_1 > 0$.
- $\displaystyle\sum_{t=1}^{\tau+1} b_t = 0$.
- For $1 < \pi < \tau$, then $\displaystyle\sum_{t=1}^{\pi} b_t \ge 0$.
- $\displaystyle\sum_{t=1}^{\tau} b_t > 0$.

**Proof.** Since $b_1 = d_{\beta,v} - e_{\beta,v}$ and $d_{\beta,v} > e_{\beta,v}$ then $b_1 > 0$. On the other hand, for $1 < \pi \le \tau+1$, $\displaystyle\sum_{t=1}^{\pi} b_t = \sum_{t=1}^{\pi}(d_{\beta,v+t-1} - e_{\beta,v+t-1}) = \sigma_{\beta,v+\pi-1}(D-E)$. As $D \prec_B E$ and $d_{\beta,u} < e_{\beta,u}$, we get the claim. ∎

For $1 \leq t \leq \tau$, let

$$g_t = \sum_{\rho=t}^{\tau} d_{\beta,v+\rho} + \sum_{\rho=1}^{t} b_\rho + \sum_{\rho=t-1}^{\tau-1} d_{\beta+1,v+\rho}.$$

**Claim 2** We have:

- For $1 \leq t \leq \tau$, then $g_t = \sum_{\rho=t}^{\tau} e_{\beta,v+\rho} + \sum_{\rho=t-1}^{\tau-1} d_{\tau+1,v+\rho} \geq e_{\beta,u} \geq 0$.
- If $\tau > 1$, for $1 \leq t \leq \tau - 1$, then $g_t = g_{t+1} + e_{\beta,v+t} + d_{\beta+1,v+t-1} \geq g_{t+1}$.

**Proof.** Since $\sum_{\rho=1}^{t} b_\rho = \sum_{\rho=1}^{t} (d_{\beta,v+\rho-1} - e_{\beta,v+\rho-1})$, we have

$$g_t = \sum_{\rho=0}^{\tau} d_{\beta,v+\rho} - \sum_{\rho=0}^{t-1} e_{\beta,v+\rho} + \sum_{\rho=t-1}^{\tau-1} d_{\beta+1,v+\rho}.$$

As $\sum_{\rho=0}^{\tau} (d_{\beta,v+\rho} - e_{\beta,v+\rho}) = 0$ (see Claim 1), then

$$g_t = \sum_{\rho=t}^{\tau} e_{\beta,v+\rho} + \sum_{\rho=t-1}^{\tau-1} d_{\beta+1,v+\rho}.$$

Using the fact that the $\beta$-row of $E$ (recall that the $\beta$-row of $E$ is the $\beta$-row of $C$) and the $(\beta+1)$-row of $D$ (recall that the $(\beta+1)$-row of $D$ is the $(\beta+1)$-row of $A$) are vectors whose coordinates are zeros and ones and the fact that when $\rho = \tau$ then $v + \rho = u$, we conclude that $g_t \geq e_{\beta,u} \geq 0$, for $1 \leq t \leq \tau$. Moreover, if $\tau > 1$ and $1 \leq t \leq \tau - 1$ then $g_t = g_{t+1} + e_{\beta,v+t} + d_{\beta+1,v+t-1} \geq g_{t+1}$. ■

**Claim 3** If $\epsilon$ is an integer and $1 \leq \epsilon \leq \tau$ then there is an integer $h$, with $1 \leq h \leq \varepsilon$ such that

$$\sum_{t=1}^{\epsilon} g_t b_t \geq g_h \left( \sum_{t=1}^{\epsilon} b_t \right) \geq 0.$$

**Proof.** For $\epsilon = 1$, by Claims 1 and 2 we get $b_1 g_1 \geq 0$. Suppose that $\tau > 1$, $\epsilon$ is an integer such that $1 \leq \epsilon < \tau - 1$ and there is an integer $f$, with

$1 \leq f \leq \epsilon$, such that

$$\sum_{t=1}^{\epsilon} g_t b_t \geq g_f \left( \sum_{t=1}^{\epsilon} b_t \right) \geq 0.$$

By Claim 1, $\sum_{t=1}^{\epsilon} b_t \geq 0$. Now we consider two cases:

- Suppose that $b_{\epsilon+1} \geq 0$. By Claim 2 we get $g_f \left( \sum_{t=1}^{\epsilon} b_t \right) \geq g_{\epsilon+1} \left( \sum_{t=1}^{\epsilon} b_t \right)$. So, $\sum_{t=1}^{\epsilon} g_t b_t \geq g_f \left( \sum_{t=1}^{\epsilon} b_t \right) \geq g_{\epsilon+1} \left( \sum_{t=1}^{\epsilon} b_t \right)$. Consequently,

  $$\sum_{t=1}^{\epsilon+1} g_t b_t = \sum_{t=1}^{\epsilon} g_t b_t + g_{\epsilon+1} b_{\epsilon+1} \geq g_{\epsilon+1} \left( \sum_{t=1}^{\epsilon} b_t \right) + g_{\epsilon+1} b_{\epsilon+1} = g_{\epsilon+1} \left( \sum_{t=1}^{\epsilon+1} b_t \right).$$

  By Claims 1 and 2, $\sum_{t=1}^{\epsilon+1} b_t \geq 0$ and $g_{\epsilon+1} \geq 0$. Thus, $g_{\epsilon+1} \left( \sum_{t=1}^{\epsilon+1} b_t \right) \geq 0$.

- Suppose that $b_{\epsilon+1} < 0$. By Claim 2 we get $g_{\epsilon+1} b_{\epsilon+1} \geq g_f b_{\epsilon+1}$. Consequently,

  $$\sum_{t=1}^{\epsilon+1} g_t b_t = \sum_{t=1}^{\epsilon} g_t b_t + g_{\epsilon+1} b_{\epsilon+1} \geq g_f \left( \sum_{t=1}^{\epsilon} b_t \right) + g_f b_{\epsilon+1} = g_f \left( \sum_{t=1}^{\epsilon+1} b_t \right).$$

  By Claims 1 and 2, $\sum_{t=1}^{\epsilon+1} b_t \geq 0$ and $g_f \geq 0$. Thus, $g_f \left( \sum_{t=1}^{\epsilon+1} b_t \right) \geq 0$.

Therefore, we conclude the result. ■

**Claim 4** If $D = A$ then the following statements hold:

- For $1 < t \leq \tau$, then $g_t > 0$.
- $\sum_{t=1}^{\tau} g_t b_t > 0$.

**Proof.** By Claim 2, $g_t = \sum_{\rho=t}^{\tau} e_{\beta,v+\rho} + \sum_{\rho=t-1}^{\tau-1} d_{\beta+1,v+\rho} \geq e_{\beta,u}$. Using the fact that $e_{\beta,u} > d_{\beta,u}$, $D = A$ and the $\beta$-rows of $E$ and $D$ are vectors whose coordinates are zeros and ones, we conclude $e_{\beta,u} = 1$. Using Claim 1 we have $b_1 + \ldots + b_\tau > 0$. By Claim 3, the result follows. ■

Denote by $X_0$ the matrix $D$ and for $1 \leq i \leq \tau$, let $X_i$ be the matrix obtained from $X_{i-1}$ by the $b_i$-interchange in the submatrix

$$X_{i-1}[\{\beta, \beta+1\}; \{v+i-1, u\}].$$

So,

$$X_0[\{\beta, \beta+1\}; \{v, u\}] = D[\{\beta, \beta+1\}; \{v, u\}] = \begin{bmatrix} d_{\beta,v} & d_{\beta,u} \\ d_{\beta+1,v} & d_{\beta+1,u} \end{bmatrix}$$

and

$$X_1[\{\beta, \beta+1\}; \{v, u\}] = \begin{bmatrix} d_{\beta,v} - b_1 & d_{\beta,u} + b_1 \\ d_{\beta+1,v} + b_1 & d_{\beta+1,u} - b_1 \end{bmatrix}.$$

As $b_1 = d_{\beta,v} - e_{\beta,v}$ then the entries $(\beta, v)$ of $X_1$ and of $E$ are equal.

If $\tau = 1$ , as

$$\sum_{\rho=0}^{\tau}(d_{\beta,v+\rho} - e_{\beta,v+\rho}) = 0 = \sum_{\rho=0}^{1}(d_{\beta,v+\rho} - e_{\beta,v+\rho}) = (d_{\beta,v} - e_{\beta,v}) + (d_{\beta,u} - e_{\beta,u})$$

then $d_{\beta,u} + b_1 = e_{\beta,u}$. Therefore, the $\beta$-row of $X_1$ is the $\beta$-row of $E$. Using Remark 14 we get $X_1 = E$.

If $\tau > 1$ then

$$X_2[\{\beta, \beta+1\}; \{v+1, u\}] = \begin{bmatrix} d_{\beta,v+1} - b_2 & d_{\beta,u} + b_1 + b_2 \\ d_{\beta+1,v+1} + b_2 & d_{\beta+1,u} - b_1 - b_2 \end{bmatrix}.$$

As $b_2 = d_{\beta,v+1} - e_{\beta,v+1}$ then the entries $(\beta, v+1)$ of $X_2$ and of $E$ are equal.

If $\tau = 2$ , as

$$\sum_{\rho=0}^{\tau}(d_{\beta,v+\rho} - e_{\beta,v+\rho}) = 0 = \sum_{\rho=0}^{2}(d_{\beta,v+\rho} - e_{\beta,v+\rho}) =$$

$$(d_{\beta,v} - e_{\beta,v}) + (d_{\beta,v+1} - e_{\beta,v+1}) + (d_{\beta,u} - e_{\beta,u})$$

then $d_{\beta,u} + b_1 + b_2 = e_{\beta,u}$. Therefore, the $\beta$-row of $X_2$ is the $\beta$-row of $E$. Using Remark 14 we get $X_2 = E$.

If $\tau > 2$ then we repeat this argument.

Since $\sum_{\rho=0}^{\tau}(d_{\beta,v+\rho} - e_{\beta,v+\rho}) = 0$ then $X_\tau$ is the matrix $E$.

By Lemma 10,

$$\xi(X_1) = \xi(X_0) + (\sum_{\rho=1}^{\tau} d_{\beta,v+\rho} + \sum_{\rho=0}^{\tau-1} d_{\beta,v+\rho} + b_1)b_1 = \xi(D) + g_1 b_1.$$

If $\tau > 1$, by Lemma 10 we get

$$\xi(X_2) = \xi(X_1) + (\sum_{\rho=2}^{\tau} d_{\beta,v+\rho} + b_1 + \sum_{\rho=1}^{\tau-1} d_{\beta,v+\rho} + b_2)b_2 = \xi(D) + \sum_{t=1}^{2} g_t b_t.$$

Consequently, by Lemma 10 we have

$$\xi(E) = \xi(D) + \sum_{t=1}^{\tau} g_t b_t.$$

By Claim 3 we conclude that

$$\xi(D) \leq \xi(E).$$

Note that, when $D = A$ then by Claim 4 and Lemma 9,

$$\nu(A) = \xi(D) < \xi(E).$$

If $E = C$ we get the result. If $E \neq C$, let $\delta$ be the minimum integer such that the $\delta$-rows of $E$ and $C$ are different. Since the first $\beta$-rows of $E$ and $C$ are equal then $\delta > \beta$. Let $Q = [q_{ij}]$ be the $m$-by-$n$ $(\delta + 1, C, A)$-matrix. By Proposition 15, we get $E \prec_B Q \preceq_B C$. We repeat the proof starting at step 1, with $E$ instead of $D$, $Q$ instead of $E$ and $\delta$ instead of $\beta$, and we get $\xi(E) \leq \xi(Q)$ (note that Claim 4 is not valid because $E \neq A$). By Lemma 9 we get

$$\nu(A) = \xi(A) < \xi(E) \leq \xi(Q).$$

If $Q = C$ we get the result. If $Q \neq C$ we repeat the proof starting at step 1 with other two matrices and a new row index greater than $\delta$.

As each time we start the step 1 we use a row index greater than the row index used in the previous time then this process stops. Therefore, the result follows. ■

# 5 Conclusions

We have defined $b$-interchanges in a real matrix. In particular, we have shown that this definition is a generalization of the one on the class of $(0,1)$-matrices with fixed row sum and column sum vectors.

We have defined a new function from the class of real matrices in the set of real numbers whose restriction to the class of $(0,1)$-matrices gives us the total number of inversions in each matrix.

Using two matrices of $\mathcal{A}(R,S)$, we have defined a new construction of an integer matrix whose row sum vector is $R$ and column sum vector is $S$.

We have proved a conjecture that involves the Bruhat order, the number of inversions and two $(0,1)$-matrices. More precisely, we have proved that if two $(0,1)$-matrices, with the same row sum vectors and column sum vectors, have the same total number of inversions then they are incomparable in the Bruhat order.

The converse of this conjecture is not valid. In fact, there are matrices that are incomparable in the Bruhat order and have different total number of inversions. For instance, consider the minimal matrices for the Bruhat order in $\mathcal{A}(7,3)$, [2],

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then, $\nu(A) = 20$ and $\nu(C) = 23$.

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