

The rank of the semigroup of all order-preserving transformations on a finite fence

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Abstract

A zig-zag (or fence) order is a special partial order on a (finite) set. In this paper, we consider the semigroup \mathcal{TF}_n of all order-preserving transformations on an n -element zig-zag ordered set. We determine the rank of \mathcal{TF}_n and provide a minimal generating set for \mathcal{TF}_n . Moreover, a formula for the number of idempotents in \mathcal{TF}_n is given.

1 Introduction

Let $n \in \mathbb{N}$ and denote by \mathcal{T}_n the monoid (under composition) of all full transformations on the set $\bar{n} = \{1, \dots, n\}$ of the first n natural numbers. Let \preceq be any partial order on \bar{n} . Let $\alpha \in \mathcal{T}_n$. We say that α is an *order-preserving* transformation (with respect to \preceq) if $x \preceq y$ implies $x\alpha \preceq y\alpha$, for all $x, y \in \bar{n}$. Clearly, the subset of \mathcal{T}_n of all order-preserving transformations (with respect to a fixed partial order) forms a submonoid of \mathcal{T}_n .

A very important particular and natural case occurs when a linear order (for instance the one induced by the usual order on the natural numbers) is considered. The monoid \mathcal{O}_n of all order-preserving transformations on \bar{n} , endowed with a linear order, has been extensively studied since the early 1960s. In fact, in 1962, Aizenštat [1, 2] showed that all non-trivial congruences of \mathcal{O}_n are Rees congruences and gave a monoid presentation for \mathcal{O}_n , in terms of $2n - 2$ idempotent generators, from which it can be deduced that, for $n > 1$, \mathcal{O}_n only has one non-trivial automorphism. In 1971, Howie [13] calculated the cardinal and the number of idempotents of \mathcal{O}_n and later (1992), jointly with Gomes [11], determined its rank and idempotent rank. More recently, Fernandes et al. [9] described the endomorphisms of the semigroup \mathcal{O}_n by showing that there are three types of endomorphism: automorphisms, constants, and a certain type of endomorphism with two idempotents in the image. The monoid \mathcal{O}_n also played a main role in several other papers [3, 7, 8, 10, 12, 16, 17, 19], where the central topic concerns the problem of the decidability of the pseudovariety generated by the family $\{\mathcal{O}_n \mid n \in \mathbb{N}\}$. This question was posed by J.-E. Pin in 1987 in the “Szeged International Semigroup Colloquium” and, as far as we know, is still open.

A non-linear order (in some sense) *close* to a linear order is the so-called zig-zag order. The pair (\bar{n}, \preceq) is

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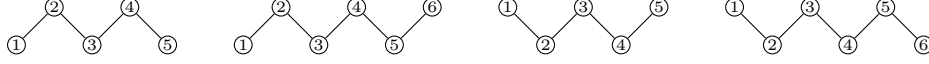
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called a *zig-zag poset* or *fence* if

$$\begin{aligned}
1 \prec 2 \succ 3 \prec \cdots \prec n-1 \succ n, & \quad \text{if } n \text{ is odd, and} \\
1 \prec 2 \succ 3 \prec \cdots \succ n-1 \prec n, & \quad \text{if } n \text{ is even, or dually} \\
1 \succ 2 \prec 3 \succ \cdots \succ n-1 \prec n, & \quad \text{if } n \text{ is odd, and} \\
1 \succ 2 \prec 3 \succ \cdots \prec n-1 \succ n, & \quad \text{if } n \text{ is even.}
\end{aligned}$$

The definition of the partial order \preceq is self-explanatory. For instance, for $n = 5$ and $n = 6$, we have the following fences (given by Hasse diagrams):



Observe that, every element in a fence is either minimal or maximal.

Order-preserving transformations of (finite) fences were first investigated by Currie and Visentin [5] and by Rutkowski [18]. In [5], by using generating functions, the authors calculate the number of order-preserving transformations of a fence with an even number of elements. On the other hand, an exact formula for the number of such transformations, for any natural number n , was given in [18].

Recently, several properties of monoids of order-preserving transformations of a fence were studied. In [4] the authors discussed the regular elements in these monoids. So-called coregular elements of this monoids were determined in [15]. On the other hand, in [6] Dimitrova and Koppitz investigated the monoid of all partial permutations preserving a zig-zag order on a set with n elements, by studying Green's relations and generating sets of this monoid.

Without loss of generality, we will assume that (\bar{n}, \preceq) is an *up-fence*, i.e.

$$1 \prec 2 \succ 3 \prec \cdots .$$

Let $x, y \in \bar{n}$. We say that x and y are *comparable* if $x \prec y$ or $x = y$ or $y \prec x$. Otherwise, x and y are said *incomparable*. Clearly, x and y are comparable if and only if $x \in \{y-1, y, y+1\}$.

Denote by \mathcal{TF}_n the submonoid of \mathcal{T}_n of all order-preserving transformations of the fence (\bar{n}, \preceq) .

In this paper, we determine the rank and count the number of idempotents of \mathcal{TF}_n .

Recall that the *rank* of a (finite) semigroup S is defined by

$$\text{rank } S = \min\{|A| \mid A \subseteq S \text{ generates } S\},$$

i.e. the rank of S is the minimal size of a generating set of S . For general background on Semigroup Theory and standard notation, we refer the reader to Howie's book [14].

We begin, in the next section, by giving a characterization of the elements of \mathcal{TF}_n . Clearly, the identity mapping id_n on \bar{n} is order-preserving. Also, all the n constant mappings are order-preserving. Moreover, for an even n , id_n is the unique permutation of \bar{n} belonging to \mathcal{TF}_n and, on the other hand, if n is odd then \mathcal{TF}_n has exactly two permutations, namely the identity mapping and the reflection

$$\gamma_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

The rest of Section 2 is dedicated to counting the idempotents of \mathcal{TF}_n . Notice that, it is easy to show that an element $\alpha \in \mathcal{T}_n$ is idempotent if and only if $\text{Im } \alpha = \{x \in \bar{n} \mid x\alpha = x\}$, i.e. the image of α coincides with the set of its fix points. In the third section of this paper, we determine the rank of \mathcal{TF}_n . In particular, we provide a minimal size generating set for \mathcal{TF}_n .

Notice that \mathcal{TF}_1 coincides with \mathcal{T}_1 and \mathcal{TF}_2 coincides with the monoid \mathcal{O}_2 of all order-preserving transformations on a two-element chain. Hence, from now on, we always consider $n \geq 3$.

2 Idempotents

The aim of this section is to provide a formula for the number of idempotents of \mathcal{TF}_n . In order to accomplish this, it is useful to know the form of the elements of \mathcal{TF}_n . We have the following characterization of a transformation in \mathcal{TF}_n .

Theorem 2.1. *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{TF}_n$ if and only if*

(i) $|x\alpha - (x+1)\alpha| \leq 1$, for all $x \in \{1, \dots, n-1\}$;

(ii) x and $x\alpha$ have the same parity or $(x-1)\alpha = x\alpha = (x+1)\alpha$, for all $x \in \{2, \dots, n-1\}$.

Proof. First, suppose that $\alpha \in \mathcal{TF}_n$. Let $x \in \{1, \dots, n-1\}$. Then x and $x+1$ are comparable, which implies that $x\alpha$ and $(x+1)\alpha$ are also comparable and so $|x\alpha - (x+1)\alpha| \leq 1$. This shows (i). Now let $x \in \{2, \dots, n-1\}$. Assume that x is even. Then $x-1 \prec x \succ x+1$ and so $(x-1)\alpha \preceq x\alpha \succeq (x+1)\alpha$. If $(x-1)\alpha \neq x\alpha$ or $x\alpha \neq (x+1)\alpha$ then $(x-1)\alpha \prec x\alpha$ or $x\alpha \succ (x+1)\alpha$, which implies in both cases that $x\alpha$ is even. Similarly, if x is odd we may deduce that $x\alpha$ is also odd or $(x-1)\alpha = x\alpha = (x+1)\alpha$. This shows (ii).

Conversely, suppose that (i) and (ii) are satisfied. Let $x, y \in \bar{n}$ be such that $x \prec y$. Then x is odd and y is even. Moreover $y \in \{x-1, x+1\}$. Admit that $x\alpha \neq y\alpha$. If $y = x-1$ then $2 \leq y \leq n-1$ and so $|y\alpha - x\alpha| = |y\alpha - (y+1)\alpha| = 1$ and y and $y\alpha$ have the same parity. If $y = x+1$ then $1 \leq x \leq n-1$ and so $|x\alpha - y\alpha| = |x\alpha - (x+1)\alpha| = 1$. Furthermore, in this last case, if $x > 1$ then x and $x\alpha$ have the same parity; otherwise $y = 2 < n$ and so y and $y\alpha$ have the same parity (since $(y-1)\alpha = x\alpha \neq y\alpha$). Therefore, we have $y\alpha \in \{x\alpha - 1, x\alpha + 1\}$ and, on the other hand, $y\alpha$ is even or $x\alpha$ is odd. Thus, in all cases, $x\alpha \prec y\alpha$, as required. \square

As a consequence of Theorem 2.1 we have that the image of a transformation in \mathcal{TF}_n is an interval of \bar{n} (with the usual order).

Corollary 2.2. *Let $\alpha \in \mathcal{TF}_n$. Then $\text{Im } \alpha = \{k, k+1, \dots, \ell\}$, for some $1 \leq k < \ell \leq n$.*

Proof. Let $k = \min \text{Im } \alpha$ and $\ell = \max \text{Im } \alpha$ (with respect to the usual order of \mathbb{N}). Assume that there exists $p \in \{k, k+1, \dots, \ell\}$ such that $p \notin \text{Im } \alpha$. Let $x = \max\{i \in \bar{n} \mid i\alpha < p\}$. If $x < n$ then $(x+1)\alpha > p$ and so $|x\alpha - (x+1)\alpha| > 1$, a contradiction. Then $y = \max\{i \in \bar{n} \mid i\alpha > p\} < n$ and $(y+1)\alpha < p$, whence $|y\alpha - (y+1)\alpha| > 1$, which again is a contradiction. Thus $\text{Im } \alpha = \{k, k+1, \dots, \ell\}$, as required. \square

Next we will give a formula for the number of idempotents in \mathcal{TF}_n . Let $m \in \bar{n}$ and $0 \leq p \leq n-m$. For $r \in \{0, \dots, m-1\}$, let

$$P(p, r) = \{(p_0, \dots, p_t) \mid t \in \mathbb{N} \cup \{0\}; p_1, \dots, p_t \in \mathbb{N}; p_0 = 0; 0 \leq \sum_{i=1}^s (-1)^{i+1} p_i \leq p, \text{ for } 1 \leq s \leq t; \sum_{i=1}^t p_i = r\}$$

and

$$K(m, r) = \{(k_0, \dots, k_r) \mid k_0 + r + 2 \sum_{i=1}^r k_i = m-1, k_0, \dots, k_r \in \mathbb{N} \cup \{0\}\}.$$

Further, define

$$A(m, p) = \sum_{r=0}^{m-1} |P(p, r)| \cdot |K(m, r)|.$$

Lemma 2.3. *Let $\alpha \in \mathcal{TF}_n$ with $\text{Im } \alpha = \{k, \dots, k+p\}$, for some $k \in \bar{n}$ and some $p \in \{0, \dots, n-k\}$. Let $a_0 \in \{k, k+p\}$ and $r \in \{0, \dots, k-1\}$. Then, there exists a bijection between the set $P(p, r)$ and the set of all sequences $a_0, a_1, \dots, a_r \in \text{Im } \alpha$ such that $|a_{i-1} - a_i| = 1$, for all $i \in \{1, \dots, r\}$, and there exists a partition $A_0 > A_1 > \dots > A_r$ of $\{1, \dots, k\}$, if $a_0 = k$, or a partition $A_0 < A_1 < \dots < A_r$ of $\{k+p, \dots, n\}$, if $a_0 = k+p$, verifying $A_i\alpha = \{a_i\}$, for $i \in \{0, \dots, r\}$.*

Proof. Fix a sequence $a_0, a_1, \dots, a_r \in \text{Im } \alpha$ verifying the conditions of the lemma. Notice that, if $r = 0$ then $P(p, 0) = \{(0)\}$ and a_0 is the only possible sequence. Then, we may admit that $r > 0$. Let $j = 1$, if $a_0 = k$, or $j = 2$, if $a_0 = k + p$. Put $p_0 = 0$ (by technical reasons).

Then, there exists $p_1 \in \{1, \dots, r\}$ such that $(-1)^{1+1}p_1 \in \{0, \dots, p\}$, $a_i = a_0 + (-1)^{j+1}i$, for $1 \leq i \leq p_1$, and either $r = p_1$ or $a_{p_1+1} = a_0 + (-1)^{j+1}p_1 + (-1)^{j+2}$.

If $r > p_1$ then there exists $p_2 \in \{1, \dots, r - p_1\}$ such that $(-1)^{1+1}p_1 + (-1)^{2+1}p_2 \in \{0, \dots, p\}$, $a_{p_1+i} = a_0 + (-1)^{j+1}p_1 + (-1)^{j+2}i$, for $1 \leq i \leq p_2$, and either $r = p_1 + p_2$ or $a_{p_1+p_2+1} = a_0 + (-1)^{j+1}p_1 + (-1)^{j+2}p_2 + (-1)^{j+3}$.

Continuing in this way, we obtain $t, p_1, \dots, p_t \in \mathbb{N}$ such that

$$\sum_{i=1}^t p_i = r, \quad \sum_{i=1}^s (-1)^{i+1} p_i \in \{0, \dots, p\}, \quad \text{for } 1 \leq s \leq t,$$

and

$$a_{i + \sum_{\ell=1}^{q-1} p_\ell} = a_0 + \sum_{\ell=1}^{q-1} (-1)^{j+\ell} p_\ell + (-1)^{j+q} i, \quad \text{for } 1 \leq i \leq p_q \text{ and } 1 \leq q \leq t.$$

Hence, the sequence a_0, a_1, \dots, a_r is uniquely determined by the t -uple (p_0, \dots, p_t) . \square

Let us denote by E_m the set of all idempotents of \mathcal{TF}_m , for all $m \geq 1$. It is clear that $E_1 = \mathcal{TF}_1 = \mathcal{T}_1 = \left\{ \binom{1}{1} \right\}$ and $E_2 = \mathcal{TF}_2 = \mathcal{T}_2 \setminus \left\{ \binom{12}{21} \right\} = \left\{ \binom{12}{12}, \binom{12}{11}, \binom{12}{22} \right\}$.

Theorem 2.4. *We have*

$$|E_n| = \sum_{k=1}^n \sum_{p=0}^{n-k} A(k, p) \cdot A(n+1 - (k+p), p).$$

Proof. Let $\alpha \in E_n$. Then, by Corollary 2.2, there exist $k \in \bar{n}$ and $p \in \{0, \dots, n-k\}$ such that

$$\text{Im } \alpha = \{k, k+1, \dots, k+p\}.$$

Since α is idempotent, we have $(k+i)\alpha = k+i$, for $i \in \{0, \dots, p\}$. Let

$$A^- = \{1, \dots, k\} \quad \text{and} \quad A^+ = \{k+p, \dots, n\}.$$

First, we consider the set A^- . By Theorem 2.1, we have $|x\alpha - (x+1)\alpha| \leq 1$ for all $x \in \{1, \dots, k-1\}$. Hence, there exist $r \in \{0, \dots, k-1\}$, a sequence $a_0, \dots, a_r \in \text{Im } \alpha$ and a partition $A_0 > A_1 > \dots > A_r$ of A^- such that $|a_{i-1} - a_i| = 1$, for $1 \leq i < r$, and $A_i\alpha = \{a_i\}$, for $0 \leq i \leq r$. Moreover, $x\alpha$ and x have the same parity or $(x-1)\alpha = x\alpha = (x+1)\alpha$, for all $x \in A^- \setminus \{1, n\}$. It follows that there exist $k_0, k_1, \dots, k_r \in \mathbb{N} \cup \{0\}$ such that $|A_i| = 1 + 2k_i$, for $0 \leq i \leq r-1$, and $|A_r| = k_r + 1$. Then $k_r + r + 2 \sum_{i=0}^{r-1} k_i = k-1$ and so the sequence

$A_0 > A_1 > \dots > A_r$ is uniquely determined by an element of $K(k, r)$.

If $r = 0$ then $A^- = A_0$ and $P(p, 0) = \{(0)\}$. On the other hand, admit that $r > 0$. Then, by Lemma 2.3 (with $a_0 = k$), we have that the sequence a_0, \dots, a_r is uniquely determined by an element of the set $P(p, r)$. Hence, $\alpha|_{A^-}$ is uniquely determined by an element of the set

$$B^-(k, p) = \bigcup_{r=0}^{k-1} K(k, r) \times P(p, r) \times \{r\}.$$

Dually, there exist $s \in \{0, \dots, n-(k+p)\}$, a sequence $a_0, \dots, a_s \in \text{Im } \alpha$ and a partition $A_0 < A_1 < \dots < A_s$ of A^+ such that $|a_{i-1} - a_i| = 1$, for $1 \leq i < s$, and $A_i\alpha = \{a_i\}$, for $0 \leq i \leq s$. Also, there exist $\ell_0, \ell_1, \dots, \ell_s \in \mathbb{N} \cup \{0\}$ such that $|A_i| = 1 + 2\ell_i$, for $0 \leq i \leq s-1$, and $|A_s| = \ell_s + 1$. Then $\ell_r + r + 2 \sum_{i=0}^{s-1} \ell_i = n - (k+p) = (n+1) - (k+p) - 1$, whence the sequence $A_0 < A_1 < \dots < A_s$ is uniquely determined by an element of $K(n+1 - (k+p), s)$.

If $s = 0$ then $A^+ = A_0$ and $P(p, 0) = \{(0)\}$. So, admit that $s > 0$. Then, by Lemma 2.3 (with $a_0 = k + p$), we have that the sequence a_0, \dots, a_s is uniquely determined by an element of the set $P(p, s)$. Consequently, $\alpha|_{A^+}$ is uniquely determined by an element of the set

$$B^+(k, p) = \bigcup_{s=0}^{n-(k+p)} K(n+1-(k+p), s) \times P(p, s) \times \{s\}.$$

Notice that, it is easy to verify that $|B^-(k, p)| = A(k, p)$ and $|B^+(k, p)| = A(n+1-(k+p), p)$. Moreover, $\alpha|_{\text{Im } \alpha}$ is the identity mapping on $\text{Im } \alpha$ and $\text{Im } \alpha$ is uniquely determined by an element k of the set \bar{n} and an element p of the set $\{0, \dots, n-k\}$. Thus, the transformation $\alpha \in E_n$ is uniquely determined by an element of the set

$$\bigcup_{k=1}^n \bigcup_{p=0}^{n-k} B^-(k, p) \times B^+(k, p) \times \{(k, p)\}.$$

Conversely, as the construction of this set clearly justifies that each of its elements determines uniquely an idempotent in \mathcal{TF}_n , we have

$$\begin{aligned} |E_n| &= \left| \bigcup_{k=1}^n \bigcup_{p=0}^{n-k} B^-(k, p) \times B^+(k, p) \times \{(k, p)\} \right| = \sum_{k=1}^n \sum_{p=0}^{n-k} |B^-(k, p) \times B^+(k, p) \times \{(k, p)\}| \\ &= \sum_{k=1}^n \sum_{p=0}^{n-k} |B^-(k, p)| \cdot |B^+(k, p)| \cdot |\{(k, p)\}| = \sum_{k=1}^n \sum_{p=0}^{n-k} A(k, p) \cdot A(n+1-(k+p), p), \end{aligned}$$

as required. □

The table below gives us an idea of the size of the monoids \mathcal{TF}_m and of their number of idempotents.

m	$ E_m $	$ \mathcal{TF}_m $	m	$ E_m $	$ \mathcal{TF}_m $
1	1	1	9	1039	6187
2	3	3	10	2243	16459
3	8	11	11	4901	44931
4	19	31	12	10591	117831
5	44	99	13	23190	315067
6	98	275	14	50335	817323
7	218	811	15	110651	2152915
8	474	2199	16	241457	5537839

These numbers were calculated by the formula of Theorem 2.4 and by the formulas given by Rutkowski [18].

3 The rank of \mathcal{TF}_n

This section is devoted to determine the rank of \mathcal{TF}_n . In the process we give an explicit minimal size set of generators of \mathcal{TF}_n . The cases n odd and n even will be treated separately.

The following general observation will be frequently used without reference.

Lemma 3.1. *Let $\alpha, \alpha' \in \mathcal{TF}_n$ be such that $\text{Ker } \alpha = \text{Ker } \alpha'$ and $\text{rank } \alpha > 1$. Then $x\alpha$ and $x\alpha'$ have the same parity, for all $x \in \bar{n}$.*

Proof. Let $x \in \bar{n}$. Since $\text{rank } \alpha > 1$, there exists $y \in x\alpha\alpha^{-1}$ such that $y+1 \in \bar{n} \setminus y\alpha\alpha^{-1}$ or $y-1 \in \bar{n} \setminus y\alpha\alpha^{-1}$. Therefore we may consider four cases. For instance, if $y+1 \in \bar{n} \setminus y\alpha\alpha^{-1}$ and $y \prec y+1$ then $x\alpha = y\alpha \prec (y+1)\alpha$ and $x\alpha' = y\alpha' \prec (y+1)\alpha'$, whence $x\alpha$ and $x\alpha'$ have the same parity. The other three cases are similar. □

Next, we define a series of transformations of \mathcal{TF}_n . Let (for any n)

$$\alpha_{1,2} = \begin{pmatrix} \overline{1,2} & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & \cdots & n \end{pmatrix},$$

$$\alpha_{k,k+2} = \begin{pmatrix} 1 & \cdots & k-1 & \overline{k,k+2} & \overline{k+1,k+3} & k+4 & \cdots & n \\ 1 & \cdots & k-1 & k & k+1 & k+2 & \cdots & n-2 \end{pmatrix}, \text{ for } 2 \leq k \leq n-4,$$

$$\alpha_{n-2,n} = \begin{pmatrix} 1 & \cdots & n-3 & \overline{n-2,n} & n-1 \\ 1 & \cdots & n-3 & n-2 & n-1 \end{pmatrix}, \text{ for } n \geq 4,$$

$$\alpha_{k,k+1,k+2} = \begin{pmatrix} 1 & \cdots & k-1 & \overline{k,k+1,k+2} & k+3 & \cdots & n \\ 1 & \cdots & k-1 & k & k+1 & \cdots & n-2 \end{pmatrix}, \text{ for } 1 \leq k \leq n-2,$$

$$\alpha_{1,2k+1} = \begin{pmatrix} k+1 & \overline{k,k+2} & \cdots & \overline{2,2k} & \overline{1,2k+1} & 2k+2 & \cdots & n \\ k+1 & k+2 & \cdots & 2k & 2k+1 & 2k+2 & \cdots & n \end{pmatrix}, \text{ for } 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor, \text{ and}$$

$$\beta_{k,m} = \begin{pmatrix} 1 & \cdots & k-1 & \overline{k,k+2m} & \overline{k+1,k+2m-1,k+2m+1} & \cdots \\ 1 & \cdots & k-1 & k & k+1 & \cdots \\ \cdots & \cdots & \overline{k+(m-1),k+2m-(m-1),k+2m+(m-1)} & \overline{k+m,k+3m} & k+3m+1 & \cdots & n \\ \cdots & \cdots & k+(m-1) & k+m & k+m+1 & \cdots & n-2m \end{pmatrix},$$

for $2 \leq k, m \leq n$ such that $k+3m \leq n-1$.

Moreover, for an odd n , recall that

$$\gamma_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix},$$

and, for an even n , let

$$\alpha_{1,2}^e = \begin{pmatrix} \overline{1,2} & 3 & 4 & \cdots & n \\ n & n-1 & n-2 & \cdots & 2 \end{pmatrix},$$

$$\alpha_{n-1,n} = \begin{pmatrix} 1 & 2 & \cdots & n-2 & \overline{n-1,n} \\ n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix},$$

$$\alpha_{2k,n} = \begin{pmatrix} 1 & \cdots & 2k-1 & \frac{n}{2}+k & \overline{\frac{n}{2}+k-1, \frac{n}{2}+k+1} & \overline{\frac{n}{2}+k-2, \frac{n}{2}+k+2} & \cdots & \overline{2k,n} \\ 1 & \cdots & 2k-1 & \frac{n}{2}+k & \frac{n}{2}+k-1 & \frac{n}{2}+k-2 & \cdots & 2k \end{pmatrix},$$

for $1 \leq k \leq \frac{n-4}{2}$, and

$$\alpha_{1,2k+1}^e = \begin{pmatrix} k+1 & \overline{k,k+2} & \cdots & \overline{2,2k} & \overline{1,2k+1} & 2k+2 & \cdots & n \\ k-1 & k & \cdots & 2k-2 & 2k-1 & 2k & \cdots & n-2 \end{pmatrix}, \text{ for } 2 \leq k \leq \frac{n-2}{2}.$$

Now, for an odd n , define

$$G_n = \{\gamma_n, \alpha_{1,2}\} \cup \{\alpha_{k,k+2} \mid 2 \leq k \leq \frac{n-3}{2}\} \cup \{\alpha_{k,k+1,k+2} \mid 1 \leq k \leq \frac{n-1}{2}\} \cup \\ \{\alpha_{1,2k+1} \mid 1 \leq k \leq \frac{n-1}{2}\} \cup \{\beta_{k,m} \mid 2 \leq k, m \leq \frac{n-1}{2} \text{ and } 2k+3m \leq n+1\}$$

and, for an even n , define

$$G_n = \{\text{id}_n, \alpha_{1,2}^e, \alpha_{1,3}, \alpha_{n-1,n}, \alpha_{n-2,n}\} \cup \{\alpha_{k,k+2} \mid 2 \leq k \leq n-4\} \cup \{\alpha_{k,k+1,k+2} \mid 2 \leq k \leq n-3\} \cup \\ \{\alpha_{1,2k+1}^e \mid 2 \leq k \leq \frac{n}{2}-1\} \cup \{\alpha_{2k,n} \mid 1 \leq k \leq \frac{n-4}{2}\} \cup \{\beta_{k,m} \mid 2 \leq k, m \leq n \text{ and } k+3m \leq n-1\}.$$

From now on, our main aim is to prove that G_n is a generating set for \mathcal{TF}_n of minimal size.

The following lemma shows that all the transformations above defined belong to the subsemigroup $\langle G_n \rangle$ of \mathcal{TF}_n generated by G_n . Frequently, we will use it without reference.

Lemma 3.2. *We have:*

(i) $\{\alpha_{k,k+1,k+2} \mid 1 \leq k \leq n-2\} \subseteq \langle G_n \rangle$;

(ii) $\{\alpha_{1,2k+1} \mid 2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\} \subseteq \langle G_n \rangle$;

(iii) $\{\alpha_{k,k+2} \mid 2 \leq k \leq n-4\} \subseteq \langle G_n \rangle$;

(iv) $\{\beta_{k,m} \mid 2 \leq k, m \leq n \text{ and } k+3m \leq n-1\} \subseteq \langle G_n \rangle$;

(v) $\alpha_{2k,n} = \begin{pmatrix} 1 & \cdots & 2k & \overline{2k+1, n} & \cdots & \overline{\frac{n-1}{2} + k, \frac{n+3}{2} + k} & \frac{n+1}{2} + k \\ 1 & \cdots & 2k & 2k+1 & \cdots & \frac{n-1}{2} + k & \frac{n+1}{2} + k \end{pmatrix} \in \langle G_n \rangle$, for n odd and $1 \leq k \leq \frac{n-5}{2}$;

(vi) $\alpha_{n-2,n} \in \langle G_n \rangle$.

Proof. (i) For n odd and $\frac{n-1}{2} < k \leq n-2$, we have $\alpha_{k,k+1,k+2} = \gamma_n \alpha_{n-k-1, n-k, n-k+1} \gamma_n \alpha_{1,2,3}$. On the other hand, for n even, we have $\alpha_{1,2,3} = \alpha_{1,2}^e \alpha_{n-1,n}$ and $\alpha_{n-2,n-1,n} = \alpha_{n-1,n} \alpha_{1,2}^e \alpha_{1,2,3}$.

(ii) For n even and $2 \leq k \leq \frac{n-2}{2}$, we have $\alpha_{1,2k+1} = \alpha_{1,2k+1}^e \alpha_{1,2}^e \alpha_{1,2,3} \alpha_{1,2}^e$.

(iii) For n odd and $\frac{n-1}{2} \leq k \leq n-4$, we have $\alpha_{k,k+2} = \gamma_n \alpha_{n-k-2, n-k} \gamma_n \alpha_{1,2,3}$.

(iv) Let n be an odd number and let $k, m \in \bar{n}$ be such that $k+3m \leq n-1$ and $2k+3m > n+1$. Then $2(n-(k+3m)+1) \leq n+1$ and we have $\beta_{k,m} = \gamma_n \beta_{n-(k+3m)+1, m} \gamma_n (\alpha_{1,2,3})^m$.

(v) For $1 \leq k \leq \frac{n-5}{2}$, we have $\alpha_{2k,n} = \gamma_n \alpha_{1,2(k+1)+1} \gamma_n$.

(vi) Finally, we have $\alpha_{n-2,n} = \gamma_n \alpha_{1,3} \gamma_n$, whenever n is odd. □

In order to prove that the set G_n generates \mathcal{TF}_n , our first step is to show that, for any transformation in \mathcal{TF}_n , there exists a transformation in $\langle G_n \rangle$ with the same kernel. For any set $A \subseteq \bar{n}$, define

$$\text{Rel}(A) = \{x \in \bar{n} \setminus A \mid x \text{ and } a \text{ are comparable, for some } a \in A\}.$$

Lemma 3.3. *For any $\alpha \in \mathcal{TF}_n$ there exists $\alpha' \in \langle G_n \rangle$ such that $\text{Ker } \alpha' = \text{Ker } \alpha$.*

Proof. Let $\alpha \in \mathcal{TF}_n$. We make the proof by induction on the rank of α .

If $\text{rank } \alpha = n$ then $\text{Ker } \alpha = \text{Ker } \text{id}_n$ and we have $\text{id}_n \in G_n$, for n even, and $\text{id}_n = \gamma_n^2 \in \langle G_n \rangle$, for n odd.

Assume that $\text{rank } \alpha = n-1$. Then, there exists $i \in \text{Im } \alpha$ such that $|i\alpha^{-1}| = 2$ and $|j\alpha^{-1}| = 1$, for all $j \in \text{Im } \alpha \setminus \{i\}$. This implies $|\text{Rel}(i\alpha^{-1})| \leq 2$, i.e. $i\alpha^{-1} = \{1, 2\}$ or $i\alpha^{-1} = \{1, 3\}$ or $i\alpha^{-1} = \{n-2, n\}$ or $i\alpha^{-1} = \{n-1, n\}$. By noticing that, for an odd n , we have $\alpha_{n-1,n} = \gamma_n \alpha_{1,2}$ and $\alpha_{n-2,n} = \gamma_n \alpha_{1,3} \gamma_n$, it follows that there exists $\alpha' \in \langle G_n \rangle$ such that $\text{Ker } \alpha' = \text{Ker } \alpha$.

Admit now that $\text{rank } \alpha = n-2$. Then, for some $i \in \text{Im } \alpha$, we have $2 \leq |i\alpha^{-1}| \leq 3$.

If $|i\alpha^{-1}| = 3$ then there exists $k \in \{1, \dots, n-2\}$ such that $i\alpha^{-1} = \{k, k+1, k+2\}$ and $|j\alpha^{-1}| = 1$, for all $j \in \text{Im } \alpha \setminus \{i\}$, i.e. $\text{Ker } \alpha = \text{Ker } \alpha_{k,k+1,k+2}$, with $\alpha_{k,k+1,k+2} \in \langle G_n \rangle$.

Now, suppose that $|i\alpha^{-1}| = 2$. Then $|j\alpha^{-1}| = 2$, for some $j \in \text{Im } \alpha \setminus \{i\}$.

Admit that $|\text{Rel}(i\alpha^{-1})| \leq 2$. Then $i\alpha^{-1} = \{1, 2\}$ or $i\alpha^{-1} = \{1, 3\}$ or $i\alpha^{-1} = \{n-2, n\}$ or $i\alpha^{-1} = \{n-1, n\}$. Since $\text{rank } \alpha = n-2$, we conclude that $|\text{Rel}(j\alpha^{-1})| \leq 2$ or $i\alpha^{-1} \subseteq \text{Rel}(j\alpha^{-1})$. So, we have $j\alpha^{-1} = \{n-2, n\}$ or $j\alpha^{-1} = \{n-1, n\}$, if $i\alpha^{-1} = \{1, 2\}$ or $i\alpha^{-1} = \{1, 3\}$, or $j\alpha^{-1} = \{2, 4\}$, if $i\alpha^{-1} = \{1, 3\}$, or $j\alpha^{-1} = \{n-3, n-1\}$, if $i\alpha^{-1} = \{n-2, n\}$. Hence, we get $\text{Ker } \alpha' = \text{Ker } \alpha$, with $\alpha' = \alpha_{1,2} \alpha_{n-1,n}$ (and $\alpha' = (\alpha_{1,2} \gamma_n)^2$, whenever n is odd) or $\alpha' = \alpha_{1,2} \alpha_{n-2,n}$ or $\alpha' = \alpha_{1,3} \alpha_{n-1,n}$ (and $\alpha' = \alpha_{1,3} \gamma_n \alpha_{1,2} \gamma_n$, whenever n is odd) or $\alpha' = \alpha_{1,3} \alpha_{n-2,n}$ or $\alpha' = \alpha_{1,3} \alpha_{1,5}$ or $\alpha' = \alpha_{n-2,n} \alpha_{n-4,n}$. Observe $\alpha_{n-4,n} = \gamma_n \alpha_{1,5} \gamma_n \in \langle G_n \rangle$, whenever n is odd (since $\alpha_{1,5} \in \langle G_n \rangle$ by Lemma 3.2), and $\alpha_{1,2} = \alpha_{1,2}^e \alpha_{1,2}^e$, whenever n is even. Since all the other transformations used belong to $\langle G_n \rangle$, we have $\alpha' \in \langle G_n \rangle$. Dually, in the case $|\text{Rel}(j\alpha^{-1})| \leq 2$, we can show that there exists $\alpha' \in \langle G_n \rangle$, with $\text{Ker } \alpha' = \text{Ker } \alpha$.

Notice that the case $|\text{Rel}(i\alpha^{-1})| \geq 4$ or $|\text{Rel}(j\alpha^{-1})| \geq 4$ is not possible since $\text{rank } \alpha = n-2$. So, it remains the case $|\text{Rel}(i\alpha^{-1})| = |\text{Rel}(j\alpha^{-1})| = 3$. This provides $i\alpha^{-1} = \{1, k\}$, for some $k \in 2\mathbb{N}+3$, or $i\alpha^{-1} = \{n-k, n\}$,

for some $k \in 2\mathbb{N} + 2$, or $i\alpha^{-1} = \{k, k + 2\}$ for some $k \in \{2, \dots, n - 3\}$. Then there are two elements in $\text{Rel}(j\alpha^{-1})$ with the same image, which is i since $\text{rank } \alpha = n - 2$. This shows that $i\alpha^{-1} \subseteq \text{Rel}(j\alpha^{-1})$. By the same argumentation, we obtain $j\alpha^{-1} \subseteq \text{Rel}(i\alpha^{-1})$.

Suppose that $i\alpha^{-1} = \{1, k\}$, for some $k \in 2\mathbb{N} + 3$. Assume that $k \geq 7$. Then $j\alpha^{-1} \subseteq \text{Rel}(i\alpha^{-1}) = \{2, k - 1, k + 1\}$ and $i\alpha^{-1} \subseteq \text{Rel}(j\alpha^{-1})$ implies $|\text{Rel}(j\alpha^{-1})| = 4$, a contradiction. Hence, we have $i\alpha^{-1} = \{1, 5\}$. Then, once again $i\alpha^{-1} \subseteq \text{Rel}(j\alpha^{-1})$ and $|\text{Rel}(j\alpha^{-1})| = 3$ implies $j\alpha^{-1} = \{2, 4\}$. Thus $\text{Ker } \alpha = \text{Ker } \alpha_{1,5}$ and $\alpha_{1,5} \in \langle G_n \rangle$. Dually, we can show the existence of $\alpha' \in \langle G_n \rangle$ with $\text{Ker } \alpha' = \text{Ker } \alpha$, if $i\alpha^{-1} = \{n - k, n\}$, for some $k \in 2\mathbb{N} + 2$. Similarly, we obtain $\alpha' \in \langle G_n \rangle$ with $\text{Ker } \alpha' = \text{Ker } \alpha$, if $j\alpha^{-1} = \{1, k\}$, for some $k \in 2\mathbb{N} + 3$, or $j\alpha^{-1} = \{n - k, n\}$, for some $k \in 2\mathbb{N} + 2$.

Finally, we consider the case $i\alpha^{-1} = \{k, k + 2\}$ and $j\alpha^{-1} = \{\ell, \ell + 2\}$, for some $k, \ell \in \{2, \dots, n - 3\}$. Notice that $\{k, k + 2\} = i\alpha^{-1} \subseteq \text{Rel}(j\alpha^{-1}) = \{\ell - 1, \ell + 1, \ell + 3\}$ and so $k = \ell - 1$ or $k = \ell + 1$. Therefore, we have $\text{Ker } \alpha = \text{Ker } \alpha_{m, m+2}$, with $m = k$, if $k = \ell - 1$, or $m = \ell$, if $k = \ell + 1$. Hence, $\text{Ker } \alpha = \text{Ker } \alpha_{m, m+2}$ and $\alpha_{m, m+2} \in \langle G_n \rangle$.

Next, we suppose that $p = \text{rank } \alpha < n - 2$ and assume that for all $\beta \in \mathcal{JF}_n$ with $\text{rank } \beta > p$, there exists $\beta' \in \langle G_n \rangle$ such that $\text{Ker } \beta' = \text{Ker } \beta$. Further, there exist a unique $m \in \bar{n}$, a sequence $a_1, \dots, a_m \in \text{Im } \alpha$ and a partition $A_1 < \dots < A_m$ of \bar{n} with $|a_i - a_{i+1}| = 1$, for $1 \leq i < m$, and $A_i \alpha = \{a_i\}$, for $1 \leq i \leq m$. Notice that the elements in the sequence a_1, \dots, a_m have not to be pairwise distinct and $\text{Im } \alpha = \{a_1, \dots, a_m\}$. Put $\chi(\alpha) = m$. Observe that this construction can be applied to any element of \mathcal{JF}_n and so we have a well defined mapping $\chi : \mathcal{JF}_n \rightarrow \bar{n}$.

Let

$$a_0 = \begin{cases} 0 & \text{if } a_1 \text{ is odd} \\ 1 & \text{if } a_1 \text{ is even} \end{cases}$$

and define

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ 1 + a_0 & 2 + a_0 & \dots & m + a_0 \end{pmatrix}.$$

It is clear that $\beta \in \mathcal{JF}_n$.

First, consider the case $m = p$ (i.e. $\text{Ker } \alpha = \text{Ker } \beta$). Take $i \in \{1, \dots, p\}$ such that $|A_i| \geq 3$ and $A_i = \{k, k + 1, \dots, k + s\}$, with $k \in \{1, \dots, n - 2\}$ and $s \in \{2, \dots, n - k\}$. Define

$$\alpha_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & k & k + 1 & \overline{k + 2 \dots k + s} & A_{i+1} & \dots & A_p \\ 1 + a_0 & \dots & i - 1 + a_0 & i + a_0 & i + 1 + a_0 & i + 2 + a_0 & i + 3 + a_0 & \dots & p + 2 + a_0 \end{pmatrix},$$

for $i > 1$, and

$$\alpha_1 = \begin{pmatrix} \overline{1 \dots k - 2 + s} & k - 1 + s & k + s & A_2 & \dots & A_p \\ 1 + a_0 & 2 + a_0 & 3 + a_0 & 4 + a_0 & \dots & p + 2 + a_0 \end{pmatrix},$$

if $i = 1$. Since $p < n - 2$, we have $p + 2 + a_0 \in \bar{n}$. By using Theorem 2.1, we can verify that $\alpha_1 \in \mathcal{JF}_n$. Since $\text{rank } \alpha_1 > p$, there is $\alpha_1^* \in \langle G_n \rangle$ with $\text{Ker } \alpha_1^* = \text{Ker } \alpha_1$. Suppose that $\text{Im } \alpha_1^* = \{a_1^*, \dots, a_{p+2}^*\}$ such that $a_j^*(\alpha_1^*)^{-1} = (j + a_0)\alpha_1^{-1}$ for $j \in \{1, \dots, p + 2\}$. Let

$$\alpha_2 = \begin{cases} \alpha_{a_i^*, a_{i+1}^*, a_{i+2}^*} & \text{if } a_i^* < a_{i+1}^* \\ \alpha_{a_{i+2}^*, a_{i+1}^*, a_i^*} & \text{if } a_{i+1}^* < a_i^*. \end{cases}$$

It is a routine matter to verify that $\text{Ker } \alpha_1 \alpha_2 = \text{Ker } \beta$ and so there exists $\alpha' \in \langle G_n \rangle$ such that $\text{Ker } \alpha' = \text{Ker } \beta = \text{Ker } \alpha$.

Now, admit that $m > p$. Then, there exist $i \in \{1, \dots, m - 1\}$ and $s \in \{i, \dots, m - i\}$ such that the elements of $\{a_i, \dots, a_{i+s}\}$ are pairwise distinct, $a_{i+2s} = a_i$ and one of the following five conditions is satisfied:

- (a) $i + a_0 = 1$;
- (b) $i + a_0 \geq 2$, $i + 2s = m$ and $a_0 + i + 2s = n$;

- (c) $i + a_0 \geq 2$, $i + 2s = m$, $a_0 + i + 2s < n$ and $n - m < i$;
- (d) $i + a_0 \geq 2$, $i + 2s = m$, $a_0 + i + 2s < n$ and $n - m \geq i$;
- (e) $a_{i+3s} = a_{i+s}$ and $i + 3s < n$.

We will define in each of these five cases transformations ρ_1 and ω_1 . Let $\rho_1 = \alpha_{1,2s+1}$, in the case (a); let $\rho_1 = \alpha_{2\lfloor \frac{n-2s}{2} \rfloor, n}$, in the case (b); let $\rho_1 = \alpha_{2\lfloor \frac{2(i+s)-n}{2} \rfloor, n}$, in the case (c), where $2(i+s) - n = i + m - n > i - i = 0$; let ρ_1^* be defined by

$$x\rho_1^* = \begin{cases} 2(i+s+a_0) - x & \text{if } 1 \leq x \leq i+s+a_0 \\ x & \text{otherwise,} \end{cases}$$

in the case (d); and let $\rho_1 = \beta_{i,s}$, in the case (e). It is easy to verify that $\rho_1 \in \langle G_n \rangle$ in the cases (a), (b), (c) and (e). In the case (d), we observe that $q = \text{rank } \rho_1^* = n - (i+s+a_0) + 1 > p$. Then there exists $\rho_1 \in \langle G_n \rangle$ such that $\text{Ker } \rho_1 = \text{Ker } \rho_1^*$. Suppose that $\text{Im } \rho_1 = \{d_1, \dots, d_q\}$ such that $j(\rho_1^*)^{-1} = d_{j-(s+i+a_0)+1}\rho_1^{-1}$ for $i+s+a_0 \leq j \leq n$. Let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{1+s} & \text{if } 1 \leq x \leq 1+s \\ a_x & \text{if } 1+s < x < m \\ a_m & \text{otherwise,} \end{cases}$$

in the case (a); let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{x-a_0} & \text{if } 1+a_0 \leq x < i+s+a_0 \\ a_{i+s} & \text{if } i+s+a_0 \leq x \leq n \\ a_1 & \text{otherwise,} \end{cases}$$

in the cases (b) and (c). Since ℓ and $a_{\ell-a_0}$ have the same parity for all $1+a_0 \leq \ell \leq m+a_0$, we conclude that $\omega_1 \in \mathcal{TF}_n$. Let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{i+s} & \text{if } 1 \leq x \leq d_1 < d_2 \text{ or } d_2 < d_1 \leq x \leq n \\ a_{i+s-\ell+1} & \text{if } x = d_\ell \text{ and } 1 \leq \ell \leq i+s \\ a_1 & \text{otherwise} \end{cases}$$

in the case (d). Let $l \in \{1, \dots, i+s\}$. Then there exists $j \in \{i+s+a_0, \dots, n\}$ such that $\ell = j - (i+a+a_0) - 1$. From $j(\rho_1^*)^{-1} = d_\ell\rho_1^{-1}$, $d_\ell\omega_1 = a_{i+s-\ell+1}$ and the fact that j and a_{j+a_0} have the same parity, we conclude that d_ℓ and $d_\ell\omega_1$ have the same parity. This shows that $\omega_1 \in \mathcal{TF}_n$. Moreover, $\text{rank } \omega_1 = \text{rank } \alpha = p$ and $\chi(\alpha) = \chi(\omega_1) + s$. Consider now the case (e) and define ω_1 by

$$x\omega_1 = \begin{cases} a_{x-a_0} & \text{if } 1+a_0 \leq x \leq i+s+a_0 \\ a_{2s+x-a_0} & \text{if } i+s+a_0+1 \leq x \leq m-2s+a_0 \\ a_m & \text{if } m-2s+a_0 < x \leq n \\ a_1 & \text{if } x = 1. \end{cases}$$

It is easy to verify that $\text{rank } \alpha = \text{rank } \omega_1$ and $\chi(\alpha) = \chi(\omega_1) + 2s$. Moreover, it is a routine matter to show that $\omega_1 \in \mathcal{TF}_n$ and $\alpha = \beta\rho_1\omega_1$.

Next, we can focus on ω_1 and end up getting a sequence $\rho_1, \dots, \rho_t \in \langle G_n \rangle$ (for a suitable $t \in \mathbb{N}$) and an element $\omega \in \mathcal{TF}_n$ such that $\text{rank } \alpha = \text{rank } \omega$, $\chi(\omega) = p$ and $\alpha = \beta\rho_1 \cdots \rho_t\omega$.

By the case $m = p$, there exists $\omega' \in \langle G_n \rangle$ such that $\text{Ker } \omega' = \text{Ker } \omega$, whence $\text{Ker } \beta\rho_1 \cdots \rho_t\omega' = \text{Ker } \alpha$.

On the other hand, since $m > p$, there exists $\mu \in \langle G_n \rangle$ such that $\text{Ker } \mu = \{A_1, \dots, A_m\}$, say

$$\mu = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ c_1 & c_2 & \cdots & c_m \end{pmatrix},$$

by our inductive assumption. Clearly, by Theorem 2.1, either $c_1 > \cdots > c_m$ or $c_1 < \cdots < c_m$. If $c_1 > \cdots > c_m$ then we take $\varepsilon_1 = \alpha_{1,2}^e$, if n is even, and we take $\varepsilon_1 = \gamma_n$, if n is odd. Since $\varepsilon_1 \in G_n$, whence $\mu\varepsilon_1 \in \langle G_n \rangle$, we

can assume that $c_1 < \cdots < c_m$. If $1 + a_0 < c_1$ then there exists $s \in \bar{n}$ such that $1 + a_0 = c_1 - 2s$. It follows that $\beta = \mu(\alpha_{1,2,3})^s$ and so $\beta \in \langle G_n \rangle$.

Altogether, we have shown that $\beta\rho_1 \cdots \rho_t w' \in \langle G_n \rangle$ and $\text{Ker } \beta\rho_1 \cdots \rho_t w' = \text{Ker } \alpha$, as required. \square

Now, we are able to prove that G_n is a generating set for \mathcal{TF}_n .

Proposition 3.4. *We have $\langle G_n \rangle = \mathcal{TF}_n$.*

Proof. Let $\alpha \in \mathcal{TF}_n$.

Admit that $\text{rank } \alpha = n$. If n is even then $\alpha = \text{id}_n \in G_n$. If n is odd then $\alpha = \text{id}_n$ or $\alpha = \gamma_n \in G_n$, with $\gamma_n \gamma_n = \text{id}_n$. Thus $\alpha \in \langle G_n \rangle$.

Suppose now that $2 \leq m = \text{rank } \alpha < n$. By Lemma 3.3, there exists $\alpha' \in \langle G_n \rangle$ such that $\text{Ker } \alpha = \text{Ker } \alpha'$. Take

$$\text{Im } \alpha = \{a_1, \dots, a_m\} \quad \text{and} \quad \text{Im } \alpha' = \{a'_1, \dots, a'_m\},$$

with $a_1 < a_2 < \cdots < a_m$ and $a'_1 < a'_2 < \cdots < a'_m$, and define $A_i = a_i \alpha^{-1}$, for $1 \leq i \leq m$. Observe that $A_i = a'_i \alpha'^{-1}$, for $1 \leq i \leq m$, or $A_i = a'_{m-i+1} \alpha'^{-1}$, for $1 \leq i \leq m$.

Let $m = n - 1$. Then $n \notin \text{Im } \alpha$ or $1 \notin \text{Im } \alpha$ as well as $n \notin \text{Im } \alpha'$ or $1 \notin \text{Im } \alpha'$.

If $A_i = a'_i \alpha'^{-1}$, for $1 \leq i \leq n - 1$ then $a_1 = a'_1$, since a_1 and a'_1 have the same parity, by Lemma 3.1. Hence, $a_i = a'_i$, for $1 \leq i \leq n - 1$, and so $\alpha = \alpha'$.

Next consider the case $A_i = a'_{m-i+1} \alpha'^{-1}$, for $1 \leq i \leq n - 1$. Let

$$k = \begin{cases} 0 & \text{if } a_1 = 1 \\ 1 & \text{if } a_1 = 2. \end{cases}$$

Then, $a_i = i + k$ and

$$a'_{m-i+1} = \begin{cases} n - k - i + 1 & \text{if } n \text{ is odd} \\ n + k - i & \text{if } n \text{ is even,} \end{cases}$$

for $i = 1, \dots, n - 1$. If n is odd, then we have

$$a_i (\alpha' \gamma_n)^{-1} = (i + k) \gamma_n^{-1} \alpha'^{-1} = (n - (i + k) + 1) \alpha'^{-1} = a'_{m-i+1} \alpha'^{-1} = A_i = a_i \alpha^{-1},$$

for $1 \leq i \leq n - 1$. Since $\text{Ker } \alpha = \text{Ker } \alpha' = \text{Ker } \alpha' \gamma_n$, this shows that $\alpha = \alpha' \gamma_n \in \langle G_n \rangle$. If n is even then put $\rho_0 = \alpha_{n-1,n} \in \langle G_n \rangle$ and $\rho_1 = \alpha_{1,2}^e \in \langle G_n \rangle$. Observe that ρ_k restricted to $\text{Im } \alpha'$ is an injection. Hence, we have $\text{Ker } \alpha = \text{Ker } \alpha' = \text{Ker } \alpha' \rho_k$ and

$$a_i (\alpha' \rho_k)^{-1} = (i + k) \rho_k^{-1} \alpha'^{-1} = (n - i + k) \alpha'^{-1} = A_i = a_i \alpha^{-1},$$

for $1 \leq i \leq n - 1$. Thus $\alpha = \alpha' \rho_k \in \langle G_n \rangle$.

Admit now that $2 \leq m \leq n - 2$ and suppose that $\beta \in \langle G_n \rangle$, for all $\beta \in \mathcal{TF}_n$ such that $\text{rank } \beta > m$.

Suppose that $A_i = a'_{m-i+1} \alpha'^{-1}$, for $1 \leq i \leq m$. Take

$$\rho = \begin{cases} \gamma_n & \text{if } n \text{ is odd} \\ \alpha_{1,2}^e & \text{if } n \text{ is even and } 1 \notin \text{Im } \alpha \\ \alpha_{n-1,n} & \text{if } n \text{ is even and } 1 \in \text{Im } \alpha. \end{cases}$$

Then, we have $\text{Ker } \alpha = \text{Ker } \alpha' = \text{Ker } \alpha' \rho$ and

$$A_i = a'_{m-i+1} \alpha'^{-1} = (a'_{m-i+1} \rho)^{-1} \alpha'^{-1} = (a'_{m-i+1} \rho) (\alpha' \rho)^{-1},$$

for $1 \leq i \leq m$, with $\alpha' \rho \in \langle G_n \rangle$ and $a'_{m-i+1} \rho < a'_{m-j+1} \rho$, for $1 \leq i < j \leq m$. Thus, we can assume that $A_i = a'_i \alpha'^{-1}$, for $1 \leq i \leq m$.

If $a_1 = a'_1 = 1$ then we immediately obtain that $a_i = a'_i$, for $1 \leq i \leq m$, i.e. $\alpha = \alpha' \in \langle G_n \rangle$.

Consider $a_1 = 1$, $a'_1 > 1$ and $a'_m \neq n$. This implies $a'_m, a_m < n$ and so we put

$$\beta_0 = \begin{pmatrix} \overline{1 \cdots a'_1} & a'_2 & \cdots & a'_m & \overline{a'_m + 1 \cdots n} \\ a_1 & a_2 & \cdots & a_m & a_m + 1 \end{pmatrix}.$$

It is easy to show that $\beta_0 \in \mathcal{TF}_n$, with $\text{rank } \beta_0 = \text{rank } \alpha + 1$, whence $\beta_0 \in \langle G_n \rangle$. For $1 \leq i \leq m$, we have

$$a_i(\alpha' \beta_0)^{-1} = a_i \beta_0^{-1} \alpha'^{-1} = a'_i \alpha'^{-1} = A_i = a_i \alpha^{-1},$$

as a_i is the unique element in $\text{Im } \alpha' \cap a_i \beta_0^{-1}$. Since the restriction of β_0 to $\text{Im } \alpha'$ is injective, we also have $\text{Ker } \alpha = \text{Ker } \alpha' = \text{Ker } \alpha' \beta_0$. Thus $\alpha = \alpha' \beta_0 \in \langle G_n \rangle$.

Next, consider $a_1 = 1$, $a'_1 > 1$ and $a'_m = n$. Then $a'_1 \geq 3$, since a_1 and a'_1 have the same parity. Further, we have $a_i = i$, for $1 \leq i \leq m$. So, we obtain

$$\beta_1 = \begin{pmatrix} \overline{1, 3} & \overline{2, 4} & 5 & \cdots & n \\ 1 & 2 & 3 & \cdots & n-2 \end{pmatrix} = \begin{cases} \alpha_{1,3} \alpha_{1,5}^e \in \langle G_n \rangle & \text{if } n \text{ is even} \\ \alpha_{1,3} \alpha_{1,5} \alpha_{1,2,3} \in \langle G_n \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, let

$$\beta_2 = \begin{pmatrix} 1 & \overline{2 \cdots a'_1 - 1} & a'_1 & \cdots & a'_{m-1} & \overline{a'_m \cdots n} \\ 1 & 2 & 3 & \cdots & m+1 & m+2 \end{pmatrix}.$$

It is easy to verify that $\beta_2 \in \mathcal{TF}_n$, with $\text{rank } \beta_2 = \text{rank } \alpha + 2 > m$, whence $\beta_2 \in \langle G_n \rangle$. Hence

$$a_1(\alpha' \beta_2 \beta_1)^{-1} = a_1 \beta_1^{-1} \beta_2^{-1} \alpha'^{-1} = 1 \beta_1^{-1} \beta_2^{-1} \alpha'^{-1} = \{1, 3\} \beta_2^{-1} \alpha'^{-1} = \{1, a'_1\} \alpha'^{-1} = a'_1 \alpha'^{-1} = A_1 = a_1 \alpha^{-1},$$

$$a_2(\alpha' \beta_2 \beta_1)^{-1} = 2 \beta_1^{-1} \beta_2^{-1} \alpha'^{-1} = \{2, 4\} \beta_2^{-1} \alpha'^{-1} = \{2, \dots, a'_1 - 1, a'_2\} \alpha'^{-1} = a'_2 \alpha'^{-1} = A_2 = a_2 \alpha^{-1}$$

and, for $3 \leq i \leq m$,

$$a_i(\alpha' \beta_2 \beta_1)^{-1} = i \beta_1^{-1} \beta_2^{-1} \alpha'^{-1} = (i+2) \beta_2^{-1} \alpha'^{-1} = a'_i \alpha'^{-1} = A_i = a_i \alpha^{-1}.$$

Notice that β_2 restricted to $\text{Im } \alpha'$ and β_1 restricted to $\text{Im } \alpha' \beta_2 = \{3, \dots, m+2\}$ are injective. It follows that $\text{Ker } \alpha = \text{Ker } \alpha' \beta_2 \beta_1$ and so $\alpha = \alpha' \beta_2 \beta_1 \in \langle G_n \rangle$.

Now, consider $a_1 > 1$. Suppose that $a'_1 = 1$. Then $a'_m < n-1$, since $\text{rank } \alpha' \leq n-2$. Take

$$\beta_3 = \begin{pmatrix} 1 & 2 & \cdots & n-3 & \overline{n-2, n-1, n} \\ 3 & 4 & \cdots & n-1 & n \end{pmatrix}.$$

If n is even then $\beta_3 = \alpha_{n-1, n} \alpha_{1,2}^e$, whence $\beta_3 \in \langle G_n \rangle$. On the other hand, if n is odd then $\beta_3 = \gamma_n \alpha_{1,2,3} \gamma_n \in \langle G_n \rangle$. Thus, we have $\alpha' \beta_3 \in \langle G_n \rangle$. Clearly, $1 \notin \text{Im } \beta_3$ and so $1 \notin \text{Im } \alpha' \beta_3$. Since $n, n-1 \notin \text{Im } \alpha'$, we have that β_3 restricted to $\text{Im } \alpha'$ is injective. Hence $\text{Ker } \alpha' = \text{Ker } \alpha' \beta_3$. Therefore, we can assume that $a'_1 > 1$. Take

$$\beta_4 = \begin{pmatrix} \overline{1 \cdots a'_1 - 1} & a'_1 & \cdots & a'_{m-1} & \overline{a'_m \cdots n} \\ a_1 - 1 & a_1 & \cdots & a_{m-1} & a_m \end{pmatrix}.$$

It is easy to verify that $\beta_4 \in \mathcal{TF}_n$, with $\text{rank } \beta_4 = \text{rank } \alpha + 1 > m$, whence $\beta_4 \in \langle G_n \rangle$. Since β_4 restricted to $\text{Im } \alpha'$ is injective, we obtain $\text{Ker } \alpha = \text{Ker } \alpha' = \text{Ker } \alpha' \beta_4$ and, for $i \in \{1, \dots, m\}$, we have

$$a_i(\alpha' \beta_4)^{-1} = a_i \beta_4^{-1} \alpha'^{-1} = a'_i \alpha'^{-1} = A_i = a_i \alpha^{-1}.$$

Thus $\alpha = \alpha' \beta_4 \in \langle G_n \rangle$.

Finally, let $m = 1$, i.e. there exists $a \in \bar{n}$ such that $i\alpha = a$, for all $i \in \bar{n}$. Without loss of generality, suppose that $a > 1$. Clearly, $\beta_5 = \begin{pmatrix} 1 & \overline{2 \cdots n} \\ 1 & 2 \end{pmatrix} \in \langle G_n \rangle$ and either $\beta_6 = \begin{pmatrix} \overline{1, 2} & \overline{3 \cdots n} \\ a & a-1 \end{pmatrix} \in \langle G_n \rangle$ (if a is even) or $\beta_6 = \begin{pmatrix} \overline{1, 2} & \overline{3 \cdots n} \\ a-1 & a \end{pmatrix} \in \langle G_n \rangle$ (if a is odd). Then $\beta_5 \beta_6$ is the constant mapping with image $\{a\}$, i.e. $\alpha = \beta_5 \beta_6 \in \langle G_n \rangle$, as required. \square

It remains to show that G_n is a generating set for \mathcal{TF}_n of minimal size. With this goal in mind, in the next two lemmas, we determine a lower bound for the minimal size of a generating set for \mathcal{TF}_n (for n odd as well as for n even) and find it coincides with the cardinality of G_n (which gives us an upper bound).

First, we consider an odd n .

Lemma 3.5. *Let n be an odd number. Then $\text{rank}(\mathcal{TF}_n) \geq \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} (\lfloor \frac{n+1-2k}{3} \rfloor - 1) = |G_n|$.*

Proof. Let A be a generating set of \mathcal{TF}_n .

Since $\{\alpha \in \mathcal{TF}_n \mid \text{rank } \alpha = n\} = \{\gamma_n, \text{id}_n\}$, we have $\gamma_n \in A$. Let $A^{(0)} = \{\gamma_n\}$. Then $|A^{(0)}| = 1$.

Let $\alpha \in \mathcal{TF}_n$ be such that $\text{rank } \alpha \leq n-1$. Then, for some natural number p , there exist $\alpha_1, \dots, \alpha_p \in A \setminus \{\text{id}_n\}$, with $\alpha_1 \neq \gamma_n$, such that $\alpha = \alpha_1 \cdots \alpha_p$ or $\alpha = \gamma_n \alpha_1 \cdots \alpha_p$. Take

$$\alpha_1^* = \begin{cases} \alpha_1 & \text{if } \alpha = \alpha_1 \cdots \alpha_p \\ \gamma_n \alpha_1 & \text{if } \alpha = \gamma_n \alpha_1 \cdots \alpha_p. \end{cases}$$

Clearly, $\text{Ker } \alpha_1^* \subseteq \text{Ker } \alpha$ and $\text{rank } \alpha_1^* \leq n-1$.

If $\alpha = \alpha_{1,2}$ then $\text{Ker } \alpha_1^* = \text{Ker } \alpha_{1,2}$ or $\text{Ker } \alpha_1^* = \text{Ker } \gamma_n \alpha_{1,2}$, i.e. there exists $\rho_{1,2} \in A$ with $\text{Ker } \rho_{1,2} = \text{Ker } \alpha_{1,2}$ or $\text{Ker } \rho_{1,2} = \text{Ker } \gamma_n \alpha_{1,2}$ (namely $\rho_{1,2} = \alpha_1$). Take $A^{(1)} = A^{(0)} \cup \{\rho_{1,2}\}$. Then $|A^{(1)}| = |A^{(0)}| + |\{\rho_{1,2}\}| = 2$. Analogously, there exists $\rho_{1,3} \in A$ with $\text{Ker } \rho_{1,3} = \text{Ker } \alpha_{1,3}$ or $\text{Ker } \rho_{1,3} = \text{Ker } \gamma_n \alpha_{1,3}$. Clearly, $\rho_{1,3} \notin A^{(1)}$ and we take $A^{(2)} = A^{(1)} \cup \{\rho_{1,3}\}$. Then $|A^{(2)}| = |A^{(1)}| + |\{\rho_{1,3}\}| = 2 + 1 = 3$.

Let $\alpha = \alpha_{k,k+2}$, for some $k \in \{2, \dots, \frac{n-3}{2}\}$. Then $(k, k+2) \in \text{Ker } \alpha_1^*$ or $(k+1, k+3) \in \text{Ker } \alpha_1^*$. From $2 \leq k \leq \frac{n-3}{2}$, it follows that $k+3 < n$. Hence, $|\text{Rel}(\{k, k+2\})| = |\text{Rel}(\{k+1, k+3\})| = 3$ and there exist $a, b \in \bar{n} \setminus \{k, k+2\}$ or $a, b \in \bar{n} \setminus \{k+1, k+3\}$ such that $(a, b) \in \text{Ker } \alpha_1^*$. But $\text{Ker } \alpha_1^* \subseteq \text{Ker } \alpha_{k,k+2}$ implies that $(a, b) \in \text{Ker } \alpha_{k,k+2}$. Since $\text{rank } \alpha_{k,k+2} = n-2$, we have $\text{Ker } \alpha_1^* = \text{Ker } \alpha_{k,k+2}$. Hence, there exists $\rho_{k,k+2} \in A$ with $\text{Ker } \rho_{k,k+2} = \text{Ker } \alpha_{k,k+2}$ or $\text{Ker } \rho_{k,k+2} = \text{Ker } \gamma_n \alpha_{k,k+2}$. Moreover, we have $\rho_{k,k+2} \notin A^{(2)}$. On the other hand, assume there exist $2 \leq k < \ell \leq \frac{n-3}{2}$ such that $\text{Ker } \alpha_{k,k+2} = \text{Ker } \gamma_n \alpha_{\ell,\ell+2}$. Then $k = n - (\ell + 3) + 1$ and so $n = k + \ell + 3 - 1 < \frac{n-3}{2} + \frac{n-3}{2} + 2 = n - 3 + 2 = n - 1$, a contradiction. Hence $\rho_{k,k+2} \neq \rho_{\ell,\ell+2}$, for $2 \leq k < \ell \leq \frac{n-3}{2}$. Take

$$B^{(3)} = \{\rho_{k,k+2} \mid k \in \{2, \dots, \frac{n-3}{2}\}\}$$

and $A^{(3)} = A^{(2)} \cup B^{(3)}$. Since $A^{(2)} \cap B^{(3)} = \emptyset$, we obtain $|A^{(3)}| = |A^{(2)}| + |B^{(3)}| = 3 + \frac{n-5}{2} = \frac{n+1}{2}$.

Let $\alpha = \alpha_{k,k+1,k+2}$, for some $k \in \{2, \dots, \frac{n-1}{2}\}$. Then $k+2 < n$ and, by Theorem 2.1, there exists no $\beta \in \mathcal{TF}_n$ with $\text{rank } \beta = n-1$ such that $\text{Ker } \beta \subseteq \text{Ker } \alpha_{k,k+1,k+2}$. Hence, $\text{Ker } \alpha_1^* = \text{Ker } \alpha_{k,k+1,k+2}$ and so there exists $\rho_{k,k+1,k+2} \in A$ with $\text{Ker } \rho_{k,k+1,k+2} = \text{Ker } \alpha_{k,k+1,k+2}$ or $\text{Ker } \rho_{k,k+1,k+2} = \text{Ker } \gamma_n \alpha_{k,k+1,k+2}$. Clearly, $\rho_{k,k+1,k+2} \notin A^{(3)}$.

Let $\alpha = \alpha_{1,2,3}$. If $\text{rank } \alpha_1^* = n-2$ then $\text{Ker } \alpha_1^* = \text{Ker } \alpha_{1,2,3}$ or $\text{Ker } \alpha_1^* = \text{Ker } \gamma_n \alpha_{1,2,3}$. Now, admit that $\text{rank } \alpha_1^* = n-1$. Then there exists $j \in \{2, \dots, p\}$ such that $\text{rank } \alpha_1^* \alpha_2 \dots \alpha_{j-1} = n-1$ and $\text{rank } \alpha_1^* \alpha_2 \dots \alpha_j = n-2$. Observe that either $\text{Im } \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{1, \dots, n-1\}$, with $\{1, 2, 3\} \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{n-2, n-1\}$, or $\text{Im } \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{2, \dots, n\}$, with $\{1, 2, 3\} \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{2, 3\}$. Suppose that $\text{Im } \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{2, \dots, n\}$. Then $\{1, 2, 3\} \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{2, 3\}$ and we conclude that $(2, 3) \in \text{Ker } \alpha_j$. By Theorem 2.1, this implies that $(1, 2) \in \text{Ker } \alpha_j$ or $(3, 4) \in \text{Ker } \alpha_j$. The case $(3, 4) \in \text{Ker } \alpha_j$ is not possible since otherwise $\text{rank } \alpha_1^* \alpha_2 \dots \alpha_j \leq n-3$, a contradiction. Thus $(1, 2) \in \text{Ker } \alpha_j$ and so $\text{Ker } \alpha_j = \text{Ker } \alpha_{1,2,3}$. If $\text{Im } \alpha_1^* \alpha_2 \dots \alpha_{j-1} = \{1, \dots, n-1\}$ then, similarly, we obtain $\text{Ker } \alpha_j = \text{Ker } \alpha_{n-2, n-1, n} = \text{Ker } \gamma_n \alpha_{1,2,3}$. Therefore, there exists $\rho_{1,2,3} \in A$ with $\text{Ker } \rho_{1,2,3} = \text{Ker } \alpha_{1,2,3}$ or $\text{Ker } \rho_{1,2,3} = \text{Ker } \gamma_n \alpha_{1,2,3}$. Clearly, $\rho_{1,2,3} \notin A^{(3)}$. Assume there exist $1 \leq k < \ell \leq \frac{n-1}{2}$ such that $\text{Ker } \alpha_{k,k+1,k+2} = \text{Ker } \gamma_n \alpha_{\ell,\ell+1,\ell+2}$. Then $k = n - (\ell + 2) + 1$ and so $n = \ell + k + 1 < \frac{n-1}{2} + \frac{n-1}{2} + 1 = n - 1 + 1 = n$, a contradiction. Hence $\rho_{k,k+1,k+2} \neq \rho_{\ell,\ell+1,\ell+2}$, for $1 \leq k < \ell \leq \frac{n-1}{2}$. Take

$$B^{(4)} = \{\rho_{k,k+1,k+2} \mid k \in \{1, \dots, \frac{n-1}{2}\}\}$$

and $A^{(4)} = A^{(3)} \cup B^{(4)}$. Since, $A^{(3)} \cap B^{(4)} = \emptyset$, we obtain $|A^{(4)}| = |A^{(3)}| + |B^{(4)}| = \frac{n+1}{2} + \frac{n-1}{2} = n$.

Let $\alpha = \alpha_{1,2k+1}$, for some $k \in \{2, \dots, \frac{n-1}{2}\}$. Then

$$\text{Ker } \alpha_{1,2k+1} = \{(1+i, 2k+1-i) \mid 0 \leq i \leq k-1\} \cup \{(x, x) \mid x \in \bar{n}\}.$$

Given $i \in \{1, \dots, k-2\}$ such that $(1+i, 2k+1-i) \in \text{Ker } \alpha_1^*$, we have

$$\text{Rel}(\{1+i, 2k+1-i\}) = \{1+i-1, 2k+1-i-1, 1+i+1, 2k+1-i+1\}.$$

Since $\text{Ker } \alpha_1^* \subseteq \text{Ker } \alpha_{1,2k+1}$, we have $(1+(i+1), 2k+1-(i+1)), (1+(i-1), 2k+1-(i-1)) \in \text{Ker } \alpha_1^*$. If $(k, k+2) \in \text{Ker } \alpha_1^*$ then $\text{Rel}(\{k, k+2\}) = \{k-1, k+1, k+3\}$ and so we have $(k-1, k+3) \in \text{Ker } \alpha_1^*$. Now, assume that $(1+i, 2k+1-i) \notin \text{Ker } \alpha_1^*$, for all $i \in \{1, \dots, k-1\}$. Then $\text{Ker } \alpha_1^* \subseteq \text{Ker } \alpha_{1,2k+1}$ implies $(1, 2k+1) \in \text{Ker } \alpha_1^*$ and $\text{rank } \alpha_1^* = n-1$, which is not possible by Theorem 2.1. Therefore, $\text{Ker } \alpha_1^* = \text{Ker } \alpha_{1,2k+1}$ and so there exists $\rho_{1,2k+1} \in A$ with $\text{Ker } \rho_{1,2k+1} = \text{Ker } \alpha_{1,2k+1}$ or $\text{Ker } \rho_{1,2k+1} = \text{Ker } \gamma_n \alpha_{1,2k+1}$. Since $(1, 2k+1) \in \text{Ker } \rho_{1,2k+1}$ or $(n, n-2k) \in \text{Ker } \rho_{1,2k+1}$, we have $\rho_{1,2k+1} \notin A^{(4)}$. For $k, l \in \{2, \dots, \frac{n-1}{2}\}$, we have $(1, 2k+1) \in \text{Ker } \alpha_{1,2k+1}$ and $(1, 2k+1) \notin \text{Ker } \gamma_n \alpha_{1,2\ell+1}$. Hence $\rho_{1,2k+1} \neq \rho_{1,2\ell+1}$, for $2 \leq k < \ell \leq \frac{n-1}{2}$. Take

$$B^{(5)} = \{\rho_{1,2k+1} \mid k \in \{2, \dots, \frac{n-1}{2}\}\}$$

and $A^{(5)} = A^{(4)} \cup B^{(5)}$. Since $A^{(4)} \cap B^{(5)} = \emptyset$, we obtain $|A^{(5)}| = |A^{(4)}| + |B^{(5)}| = n + \frac{n-3}{2} = \frac{3n-3}{2} = \frac{3}{2}(n-1)$.

Finally, let $\alpha = \beta_{k,m}$, for some $k, m \in \{2, \dots, \frac{n-1}{2}\}$ such that $2k+3m \leq n+1$. It is easy to verify that $\{k+i, k+2m-i, k+2m+i\}$, for $0 \leq i \leq m$, are all the non-singleton $\text{Ker } \beta_{k,m}$ -classes. If $i \in \{1, \dots, m-1\}$ is such that $(k+i)\alpha_1^* = (k+2m-i)\alpha_1^* = (k+2m+i)\alpha_1^*$ then

$$\text{Rel}(\{k+i, k+2m-i, k+2m+i\}) = \{k+i-1, k+2m-i-1, k+2m+i-1, k+i+1, k+2m-i+1, k+2m+i+1\}$$

implies

$$(k+(i-1))\alpha_1^* = (k+2m-(i-1))\alpha_1^* = (k+2m+(i-1))\alpha_1^*$$

and

$$(k+(i+1))\alpha_1^* = (k+2m-(i+1))\alpha_1^* = (k+2m+(i+1))\alpha_1^*,$$

since $\text{Ker } \alpha_1^* \subseteq \text{Ker } \beta_{k,m}$. If $(k, k+2m) \in \text{Ker } \alpha_1^*$ then, similarly, we have

$$(k+1)\alpha_1^* = (k+2m-1)\alpha_1^* = (k+2m+1)\alpha_1^*.$$

Moreover, we obtain

$$(k+m-1)\alpha_1^* = (k+2m-(m-1))\alpha_1^* = (k+2m+(m-1))\alpha_1^*,$$

whenever $(k+m, k+3m) \in \text{Ker } \alpha_1^*$. Therefore $\text{Ker } \alpha_1^* = \text{Ker } \beta_{k,m}$ and so there exists $\delta_{k,m} \in A$ with $\text{Ker } \delta_{k,m} = \text{Ker } \beta_{k,m}$ or $\text{Ker } \delta_{k,m} = \text{Ker } \gamma_n \beta_{k,m}$. Moreover, it is easy to verify that $\delta_{k,m} \notin A^{(5)}$. Take

$$B^{(6)} = \{\delta_{k,m} \mid k, m \in \{2, \dots, \frac{n-1}{2}\} \text{ and } 2k+3m \leq n+1\}.$$

Assume there exist $k, m, p, q \in \{2, \dots, \frac{n-1}{2}\}$ such that $\beta_{k,m} = \gamma_n \beta_{p,q}$, with $2k+3m, 2p+3q \leq n+1$ and $k \neq p$ or $m \neq q$. Then $k = n - (p+3q) + 1$. If $k < p$ then $n = k + p + 3q - 1 < 2p + 3q - 1 \leq n + 1 - 1 = n$, a contradiction. Admit that $p < k$. From $\beta_{k,m} = \gamma_n \beta_{p,q}$ it follows that $\beta_{p,q} = \gamma_n \beta_{k,m}$ and so $p = n - (k+3m) + 1$. This provides again $n < n$, as in the previous case. Suppose now that $p = k$. Then $q \neq m$ and we have $p = n - (p+3m) + 1 \neq n - (p+3q) + 1 = k$, i.e. $p \neq k$, a contradiction. This allows us to conclude that $\delta_{k,m} \neq \delta_{p,q}$, whenever $k, m, p, q \in \{2, \dots, \frac{n-1}{2}\}$, with $2k+3m, 2p+3q \leq n+1$ and $k \neq p$ or $m \neq q$. Thus

$$|B^{(6)}| = \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1 \right).$$

Take $A^{(6)} = A^{(5)} \cup B^{(6)}$. Since $A^{(5)} \cap B^{(6)} = \emptyset$, we obtain

$$|A^{(6)}| = |A^{(5)}| + |B^{(6)}| = \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} (\lfloor \frac{n+1-2k}{3} \rfloor - 1) = |G_n|.$$

Since $A^{(6)} \subseteq A$, we have $|A| \geq |A^{(6)}| = \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} (\lfloor \frac{n+1-2k}{3} \rfloor - 1)$, which allows us to deduce that

$$\text{rank}(\mathcal{TF}_n) \geq \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} (\lfloor \frac{n+1-2k}{3} \rfloor - 1) = |G_n|, \text{ as required.} \quad \square$$

Next, we consider the even case.

Lemma 3.6. *Let n be an even number. Then $\text{rank}(\mathcal{TF}_n) \geq 3n - 8 + \sum_{k=2}^{\frac{n-7}{2}} (\lfloor \frac{n-1-k}{3} \rfloor - 1) = |G_n|$.*

Proof. Let A be a generating set of \mathcal{TF}_n .

Since $\{\alpha \in \mathcal{TF}_n \mid \text{rank } \alpha = n\} = \{\text{id}_n\}$, we have $\text{id}_n \in A$. Let $A^{(0)} = \{\text{id}_n\}$. Then $|A^{(0)}| = 1$.

Let $\alpha \in \mathcal{TF}_n$ be such that $\text{rank } \alpha \leq n-1$. Then, there exist $\alpha_1, \dots, \alpha_p \in A \setminus \{\text{id}_n\}$ such that $\alpha = \alpha_1 \dots \alpha_p$, for some natural number p . Clearly, $\text{Ker } \alpha_1 \subseteq \text{Ker } \alpha$ and $\text{rank } \alpha_1 \leq n-1$.

If $\alpha \in B^{(1)} = \{\alpha_{1,2}, \alpha_{1,3}, \alpha_{n-1,n}, \alpha_{n-2,n}\}$ then it is easy to verify that $\alpha = \alpha_1$. Hence $B^{(1)} \subseteq A$ and we define $A^{(1)} = A^{(0)} \cup B^{(1)}$. We have $|A^{(1)}| = |A^{(0)}| + |B^{(1)}| = 1 + 4 = 5$.

Let $\alpha = \alpha_{k,k+2}$, for some $2 \leq k \leq n-4$. Then $(k, k+2) \in \text{Ker } \alpha_1$ or $(k+1, k+3) \in \text{Ker } \alpha_1$. Since $2 \leq k < n-3$, we have $\text{Rel}(\{k, k+2\}) = \{k-1, k+1, k+3\} \subseteq \bar{n}$ or $\text{Rel}(\{k+1, k+3\}) = \{k, k+2, k+4\} \subseteq \bar{n}$, respectively. Since $\text{Ker } \alpha_1 \subseteq \text{Ker } \alpha_{k,k+2}$, we obtain $\text{Ker } \alpha_1 = \text{Ker } \alpha_{k,k+2}$. Hence, there exists $\rho_{k,k+2} \in A$ such that $\text{Ker } \rho_{k,k+2} = \text{Ker } \alpha_{k,k+2}$. Thus, being

$$B^{(2)} = \{\rho_{k,k+2} \mid k \in \{2, \dots, n-4\}\},$$

we have $|B^{(2)}| = n-5$. Take $A^{(2)} = A^{(1)} \cup B^{(2)}$. Since $\text{rank } \rho_{k,k+2} = n-2$, it follows that $\rho_{k,k+2} \notin A^{(1)}$. Then $|A^{(2)}| = |A^{(1)}| + |B^{(2)}| = 5 + n - 5 = n$.

Let $\alpha = \alpha_{k,k+1,k+2}^e$, for some $k \in \{2, \dots, n-3\}$. Then there is no $\beta \in \mathcal{TF}_n$ such that $\text{rank } \beta = n-1$ and $\text{Ker } \beta \subseteq \text{Ker } \alpha_{k,k+1,k+2}^e$. Thus, there exists $\rho_{k,k+1,k+2} \in A$ with $\text{Ker } \rho_{k,k+1,k+2} = \text{Ker } \alpha_{k,k+1,k+2}^e$. Clearly, $\rho_{k,k+1,k+2} \notin A^{(2)}$. Take

$$B^{(3)} = \{\rho_{k,k+1,k+2} \mid k \in \{2, \dots, n-3\}\}.$$

Then $|B^{(3)}| = n-4$. Furthermore, being $A^{(3)} = A^{(2)} \cup B^{(3)}$, we have $|A^{(3)}| = |A^{(2)}| + |B^{(3)}| = n + n - 4 = 2n - 4$.

Let $\alpha = \alpha_{1,2k+1}$, for some $k \in \{2, \dots, \frac{n}{2} - 1\}$. It is clear that

$$\text{Ker } \alpha_{1,2k+1} = \{(1+i, 2k+1-i) : 0 \leq i \leq k-1\} \cup \{(x, x) : x \in \bar{n}\}.$$

If $i \in \{1, \dots, k-2\}$ is such that $(1+i, 2k+1-i) \in \text{Ker } \alpha_1$ then

$$\text{Rel}(\{1+i, 2k+1-i\}) = \{1+i-1, 2k+1-i-1, 1+i+1, 2k+1-i+1\}$$

and, as $\text{Ker } \alpha_1 \subseteq \text{Ker } \alpha_{1,2k+1}$, it follows $(1+(i+1), 2k+1-(i+1)) \in \text{Ker } \alpha_1$ and $(1+(i-1), 2k+1-(i-1)) \in \text{Ker } \alpha_1$. If $(k, k+2) \in \text{Ker } \alpha_1$ then $\text{Rel}(\{k, k+2\}) = \{k-1, k+1, k+3\}$, whence $(k-1, k+3) \in \text{Ker } \alpha_1$ (since $\text{Ker } \alpha_1 \subseteq \text{Ker } \alpha_{1,2k+1}$). If $(1, 2k+1) \in \text{Ker } \alpha_1$ then $\text{Rel}(\{1, 2k+1\}) = \{2, 2k, 2k+2\} \subseteq \bar{n}$ (note that $k \leq \frac{n}{2} - 1$ implies $2k+2 \leq n$) and, since $\text{Ker } \alpha_1 \subseteq \text{Ker } \alpha_{1,2k+1}$, we have $(2, 2k) \in \text{Ker } \alpha_1$. Therefore $\text{Ker } \alpha_1 = \text{Ker } \alpha_{1,2k+1}$ and there exists $\rho_{1,2k+1} \in A$ with $\text{Ker } \rho_{1,2k+1} = \text{Ker } \alpha_{1,2k+1}$. Clearly, $\rho_{1,2k+1} \notin A^{(3)}$.

Let $\alpha = \alpha_{2m,n}$, for some $m \in \{1, \dots, \frac{n-4}{2}\}$. Analogously, we can show there exists $\rho_{2m,n} \in A$ with $\text{Ker } \rho_{2m,n} = \text{Ker } \alpha_{2m,n}$. Moreover, it is easy to verify that $\rho_{2m,n} \notin A^{(3)}$ and $\rho_{2m,n} \neq \rho_{1,2k+1}$, since $(2m, n) \in \text{Ker } \rho_{2m,n}$ and $(2m, n) \notin \text{Ker } \rho_{1,2k+1}$, for $k \in \{2, \dots, \frac{n}{2} - 1\}$.

Take

$$B^{(4)} = \{\rho_{1,2k+1} \mid k \in \{2, \dots, \frac{n}{2} - 1\}\} \cup \{\rho_{2m,n} \mid m \in \{1, \dots, \frac{n-4}{2}\}\}.$$

Then $|B^{(4)}| = \frac{n-4}{2} + \frac{n-4}{2} = n - 4$. Furthermore, define $A^{(4)} = A^{(3)} \cup B^{(4)}$. Since $A^{(3)} \cap B^{(4)} = \emptyset$, it follows that $|A^{(4)}| = |A^{(3)}| + |B^{(4)}| = 2n - 4 + n - 4 = 3n - 8$.

Let $\alpha = \beta_{k,m}$, for some $k, m \in \{2, \dots, n\}$ such that $k + 3m \leq n - 1$. Similarly to the proof of Lemma 3.5, we can prove the existence of an element $\delta_{k,m} \in A$ such that $\text{Ker } \delta_{k,m} = \text{Ker } \beta_{k,m}$. Clearly, we also have $\delta_{k,m} \notin A^{(4)}$. Take

$$B^{(5)} = \{\delta_{k,m} \mid k, m \in \{2, \dots, n\} \text{ and } k + 3m \leq n - 1\}.$$

Then $|B^{(5)}| = \sum_{k=2}^{n-7} (\lfloor \frac{n-1-k}{3} \rfloor - 1)$. Moreover, being $A^{(5)} = A^{(4)} \cup B^{(5)}$, since $A^{(4)} \cap B^{(5)} = \emptyset$, we obtain

$$|A^{(5)}| = |A^{(4)}| + |B^{(5)}| = 3n - 8 + \sum_{k=2}^{n-7} (\lfloor \frac{n-1-k}{3} \rfloor - 1) = |G_n|.$$

Since $A^{(5)} \subseteq A$, we have $|A| \geq |A^{(5)}| = 3n - 8 + \sum_{k=2}^{n-7} (\lfloor \frac{n-1-k}{3} \rfloor - 1)$, which allows us to conclude that $\text{rank}(\mathcal{TF}_n) \geq 3n - 8 + \sum_{k=2}^{n-7} (\lfloor \frac{n-1-k}{3} \rfloor - 1) = |G_n|$, as required. \square

As an immediate consequence of Proposition 3.4 and Lemmas 3.5 and 3.6, we can state our main result.

Theorem 3.7. *We have*

$$\text{rank}(\mathcal{TF}_n) = \begin{cases} \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} (\lfloor \frac{n+1-2k}{3} \rfloor - 1) & \text{if } n \text{ is odd} \\ 3n - 8 + \sum_{k=2}^{n-7} (\lfloor \frac{n-1-k}{3} \rfloor - 1) & \text{if } n \text{ is even.} \end{cases}$$

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References

- [1] A.Ya. Aizenštat, The defining relations of the endomorphism semigroup of a finite linearly ordered set, *Sibirsk. Mat.* 3 (1962), 161–169 (Russian).
- [2] A.Ya. Aizenštat, Homomorphisms of semigroups of endomorphisms of ordered sets, *Uch. Zap., Leningr. Gos. Pedagog. Inst.* 238 (1962), 38–48 (Russian).
- [3] J. Almeida and M.V. Volkov, The gap between partial and full, *Int. J. Algebra Comput.* 8 (1998), 399–430.
- [4] R. Chinram, R. Srithus and R. Tanyawong, Regular subsemigroups of the semigroups of transformations preserving a fence, *Asian-European Journal of Mathematics* 9 (2016), 1650003.
- [5] J.D. Currie and T.I. Visentin, The number of order-preserving maps of fences and crowns, *Order* 8 (1991), 133–142.
- [6] I. Dimitrova and J. Koppitz, On the semigroup of all partial fence preserving injections on a finite set, *Journal of algebra and its applications* 16 (11) (2017), 1750223.

- [7] V.H. Fernandes, Semigroups of order-preserving mappings on a finite chain: a new class of divisors, *Semigroup Forum* 54 (1997), 230–236.
- [8] V.H. Fernandes, Semigroups of order-preserving mappings on a finite chain: another class of divisors, *Izv. Vyssh. Uchebn. Zaved. Mat.*, 2002, Number 3, 51–59 (Russian).
- [9] V.H. Fernandes, M.M. Jesus, V. Maltcev and J.D. Mitchell, Endomorphisms of the semigroup of order-preserving mappings, *Semigroup Forum* 81 (2010), 277–285.
- [10] V.H. Fernandes and M.V. Volkov, On divisors of semigroups of order-preserving mappings of a finite chain, *Semigroup Forum* 81 (2010), 551–554.
- [11] G.M.S. Gomes and J.M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* 45 (1992), 272–282.
- [12] P.M. Higgins, Divisors of semigroups of order-preserving mappings on a finite chain, *Int. J. Algebra Comput.* 5 (1995), 725–742.
- [13] J.M. Howie, Products of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* 17 (2) (1971), 223–236.
- [14] J.M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.
- [15] K. Jendana and R. Srithus, Coregularity of order-preserving self-mapping semigroups of fences, *Commun. Korean Math. Soc.* 30 (2015), 349–361.
- [16] V.B. Repnitskiĭ and A. Vernitskii, Semigroups of order preserving mappings, *Comm. in Algebra* 28 (8) (2000), 3635–3641.
- [17] V.B. Repnitskiĭ and M.V. Volkov, The finite basis problem for the pseudovariety \mathcal{O} , *Proc. R. Soc. Edinb., Sect. A, Math.* 128 (1998), 661–669.
- [18] A. Rutkowski, The formula for the number of order-preserving self-mappings of a fence, *Order* 9 (2) (1992), 127–137.
- [19] A. Vernitskii and M.V. Volkov, A proof and generalisation of Higgins’ division theorem for semigroups of order-preserving mappings, *Izv. Vyssh. Uchebn. Zaved. Mat.*, Number 1 (1995), 38–44 (Russian).

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