HARDY-LITTLEWOOD MAXIMAL OPERATOR ON THE ASSOCIATE SPACE OF A BANACH FUNCTION SPACE

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Abstract. Let $\mathcal{E}(X, d, \mu)$ be a Banach function space over a space of homogeneous type (X, d, μ). We show that if the Hardy-Littlewood maximal operator M is bounded on the space $\mathcal{E}(X, d, \mu)$, then its boundedness on the associate space $\mathcal{E}'(X, d, \mu)$ is equivalent to a certain condition $\mathcal{A}_\infty$. This result extends a theorem by Andrei Lerner from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

1. Introduction.

We begin with the definition of a space of homogeneous type (see, e.g., [4]). Given a set X and a function $d : X \times X \to [0, \infty)$, one says that (X, d) is a quasi-metric space if the following axioms hold:

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) for all $x, y, z \in X$ and some constant $\varkappa \geq 1$,

$$d(x, y) \leq \varkappa(d(x, y) + d(y, z)). \tag{1.1}$$

For $x \in X$ and $r > 0$, consider the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ centered at x of radius r. Given a quasi-metric space (X, d) and a positive measure μ that is defined on the σ-algebra generated by quasi-metric balls, one says that (X, d, μ) is a space of homogeneous type if there exists a constant $C_\mu \geq 1$ such that for any $x \in X$ and any $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)). \tag{1.2}$$

To avoid trivial measures, we will always assume that $0 < \mu(B) < \infty$ for every ball B. Consequently, μ is a σ-finite measure.

Given a complex-valued function $f \in L^1_{\mathrm{loc}}(X, d, \mu)$, we define its Hardy-Littlewood maximal function Mf by

$$(Mf)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(x)|\, d\mu(x), \quad x \in X,$$

where the supremum is taken over all balls $B \subset X$ containing $x \in X$. The Hardy-Littlewood maximal operator M is a sublinear operator acting by the rule $f \mapsto Mf$. The aim of this paper is the studying the relations between the boundedness of the

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operator M on a so-called Banach function space $\mathcal{E}(X, d, \mu)$ and on its associate space $\mathcal{E}'(X, d, \mu)$ in the setting of general spaces of homogeneous type (X, d, μ).

Let us recall the definition of a Banach function space (see, e.g., [3, Chap. 1, Definition 1.1]). Let $L^0(X, d, \mu)$ denote the set of all complex-valued measurable functions on X and let $L^0_+(X, d, \mu)$ be the set of all non-negative measurable functions on X. The characteristic function of a set $E \subset X$ is denoted by χ_E. A mapping $\rho : L^0_+(X, d, \mu) \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+(X, d, \mu)$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets E of X, the following properties hold:

$$\begin{aligned}
&\text{(A1)} \qquad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \ \rho(af) = a\rho(f), \ \rho(f + g) \leq \rho(f) + \rho(g), \\
&\text{(A2)} \qquad 0 \leq g \leq f \text{ a.e.} \ \Rightarrow \ \rho(g) \leq \rho(f) \quad \text{(the lattice property)}, \\
&\text{(A3)} \qquad 0 \leq f_n \uparrow f \text{ a.e.} \ \Rightarrow \ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)}, \\
&\text{(A4)} \qquad \mu(E) < \infty \ \Rightarrow \ \rho(\chi_E) < \infty, \\
&\text{(A5)} \qquad \int_E f(x)\, d\mu(x) \leq C_E \rho(f)
\end{aligned}$$

with a constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set $\mathcal{E}(X, d, \mu)$ of all functions $f \in L^0(X, d, \mu)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in \mathcal{E}(X, d, \mu)$, the norm of f is defined by

$$\|f\|_{\mathcal{E}} := \rho(|f|).$$

The set $\mathcal{E}(X, d, \mu)$ under the natural linear space operations and under this norm becomes a Banach space (see [3, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $L^0_+(X, d, \mu)$ by

$$\rho'(g) := \sup \left\{ \int_X f(x) g(x)\, d\mu(x) \ : \ f \in L^0_+(X, d, \mu), \ \rho(f) \leq 1 \right\}.$$

It is a Banach function norm itself [3, Chap. 1, Theorem 2.2]. The Banach function space $\mathcal{E}'(X, d, \mu)$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $\mathcal{E}(X, d, \mu)$.

Hytönen and Kairema [10], developing further ideas of Christ [4], show that a space of homogeneous type (X, d, μ) can be equipped with a finite system of adjacent dyadic grids $\{\mathcal{D}^t : t = 1, \dots, K\}$, each of which consists of sets Q, called dyadic cubes, that resemble properties of usual dyadic cubes in $\mathbb{R}^n$. We postpone precise formulations of these results until Section 2.

Given a dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$, a sparse family $S \subset \mathcal{D}$ is a collection of dyadic cubes $Q \in \mathcal{D}$ for which there exists a collection of sets $\{E(Q)\}_{Q \in S}$ such that the sets $E(Q)$ are pairwise disjoint, $E(Q) \subset Q$, and

$$\mu(Q) \leq 2\mu(E(Q)).$$

Definition 1 (The condition $\mathcal{A}_\infty$). Following [11], we say that a Banach function space $\mathcal{E}(X, d, \mu)$ over a space of homogeneous type (X, d, μ) satisfies the condition $\mathcal{A}_\infty$ if there exist constants $C, \gamma > 0$ such that for every dyadic grid $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$, every finite sparse family $S \subset \mathcal{D}$, every collection of non-negative numbers $\{\alpha_Q\}_{Q \in S}$, and every collection of pairwise disjoint measurable sets $\{G_Q\}_{Q \in S}$

such that $G_Q \subset Q$, one has

$$\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_{\mathcal{E}} \le C \left(\max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\gamma} \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{\mathcal{E}}. \tag{1.3}$$

The main aim of the present paper is to provide a self-contained proof of the following generalization of [11, Theorem 3.1] from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

Theorem 2 (Main result)**.** *Let $\mathcal{E}(X,d,\mu)$ be a Banach function space over a space of homogeneous type (X,d,μ) and let $\mathcal{E}'(X,d,\mu)$ be its associate space.*

(a) *If the Hardy-Littlewood maximal operator M is bounded on the space $\mathcal{E}'(X,d,\mu)$, then the space $\mathcal{E}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$.*
(b) *If the Hardy-Littlewood maximal operator M is bounded on the space $\mathcal{E}(X,d,\mu)$ and the space $\mathcal{E}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$, then the operator M is bounded on the space $\mathcal{E}'(X,d,\mu)$.*

The paper is organized as follows. In Section 2, we formulate results of Hytönen and Kairema [10] on a construction of a system of adjacent dyadic grids $\{\mathcal{D}^t : t = 1, \dots, K\}$ on the underlying space of homogeneous type (X,d,μ). In Section 3, we prove that if a weight w belongs to the dyadic class $A_1^{\mathcal{D}}$ with $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$, then it satisfies a reverse Hölder inequality. Section 4 contains a proof of a version of the Calderón-Zygmund decomposition of spaces of homogeneous type. Armed with these auxiliary results, following ideas of Lerner [11, Theorem 3.1], we give a self-contained proof of Theorem 2 in Sections 5 and 6.

We conjecture that reflexive variable Lebesgue spaces (see, e.g., [6, 8, 9, 11] for definitions) over spaces of homogeneous type satisfy the condition $\mathcal{A}_\infty$. If this conjecture is true, then in view of Theorem 2, we can affirm that the Hardy-Littlewood maximal operator is bounded on a reflexive variable Lebesgue space over a space of homogeneous type if and only if it is bounded on its associate space. Note that in the Euclidean setting of $\mathbb{R}^n$, this result was proved by Diening [9, Theorem 8.1] (see also [11, Theorem 1.1]). We are going to embark on the proof of the above conjecture in a forthcoming paper.

2. Dyadic decomposition of spaces of homogeneous type.

Let (X,d,μ) be a space of homogeneous type. The doubling property of μ implies the following geometric doubling property of the quasi-metric d: any ball $B(x,r)$ can be covered by at most $N := N(C_\mu, \varkappa)$ balls of radius $r/2$. It is not difficult to show that $N \le C_\mu^{6+3\log_2 \varkappa}$.

An important tool for our proofs is the concepts of an adjacent system of dyadic grids $\mathcal{D}^t$, $t \in \{1,\dots,K\}$, on a space of homogeneous type (X,d,μ). Christ [4, Theorem 11] (see also [5, Chap. VI, Theorem 14]) constructed a system of sets on (X,d,μ), which satisfy many of the properties of a system of dyadic cubes on the Euclidean space. His construction was further refined by Hytönen and Kairema [10, Theorem 2.2]. We will use the version from [2, Theorem 4.1].

Theorem 3. *Let (X,d,μ) be a space of homogeneous type with the constant $\varkappa \ge 1$ in inequality* (1.1) *and the geometric doubling constant N. Suppose the parameter $\delta \in (0,1)$ satisfies $96\varkappa^2\delta \le 1$. Then there exist an integer number $K = K(\varkappa, N, \delta)$,* *a countable set of points* $\{z_\alpha^{k,t} : \alpha \in \mathcal{A}_k\}$ *with* $k \in \mathbb{Z}$ *and* $t \in \{1, \dots, K\}$*, and a finite number of dyadic grids*

$$\mathcal{D}^t := \{Q_\alpha^{k,t} : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\},$$

such that the following properties are fulfilled:

(a) *for every* $t \in \{1, \dots, K\}$ *and* $k \in \mathbb{Z}$ *one has*
  (i) $X = \bigcup_{\alpha \in \mathcal{A}_k} Q_\alpha^{k,t}$ *(disjoint union);*
  (ii) *if* $Q, P \in \mathcal{D}^t$*, then* $Q \cap P \in \{\emptyset, Q, P\}$*;*
  (iii) *if* $Q_\alpha^{k,t} \in \mathcal{D}^t$*, then*

$$B(z_\alpha^{k,t}, c_1\delta^k) \subset Q_\alpha^{k,t} \subset B(z_\alpha^{k,t}, C_1\delta^k), \tag{2.1}$$

  where $c_1 = (12\varkappa^4)^{-1}$ *and* $C_1 := 4\varkappa^2$*;*

(b) *for every* $t \in \{1, \dots, K\}$ *and every* $k \in \mathbb{Z}$*, if* $Q_\alpha^{k,t} \in \mathcal{D}^t$*, then there exists at least one* $Q_\beta^{k+1,t} \in \mathcal{D}^t$*, which is called a child of* $Q_\alpha^{k,t}$*, such that* $Q_\beta^{k+1,t} \subset Q_\alpha^{k,t}$*, and there exists exactly one* $Q_\gamma^{k-1,t} \in \mathcal{D}^t$*, which is called the parent of* $Q_\alpha^{k,t}$*, such that* $Q_\alpha^{k,t} \subset Q_\gamma^{k-1,t}$*;*

(c) *for every ball* $B = B(x, r)$*, there exists*

$$Q_B \in \bigcup_{t=1}^{K} \mathcal{D}^t$$

  such that $B \subset Q_B$ *and* $Q_B = Q_\alpha^{k-1,t}$ *for some indices* $\alpha \in \mathcal{A}_k$ *and* $t \in \{1, \dots, K\}$*, where* k *is the unique integer such that* $\delta^{k+1} < r \le \delta^k$*.*

The collections $\mathcal{D}^t$, $t \in \{1, \dots, K\}$, are called dyadic grids on X. The sets $Q_\alpha^{k,t} \in \mathcal{D}^t$ are referred to as dyadic cubes with center $z_\alpha^{k,t}$ and sidelength δ^k, see (2.1). The sidelength of a cube $Q \in \mathcal{D}^t$ will be denoted by $\ell(Q)$. We should emphasize that these sets are not cubes in the standard sense even if the underlying space is $\mathbb{R}^n$. Parts (a) and (b) of the above theorem describe dyadic grids $\mathcal{D}^t$, with $t \in \{1, \dots, K\}$, individually. In particular, (2.1) permits a comparison between a dyadic cube and quasi-metric balls. Part (c) guarantees the existence of a finite family of dyadic grids such that an arbitrary quasi-metric ball is contained in a dyadic cube in one of these grids. Such a finite family of dyadic grids is referred to as an adjacent system of dyadic grids.

Let $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ be a fixed dyadic grid. One can define the dyadic maximal function $M^{\mathcal{D}} f$ of a function $f \in L^1_{\rm loc}(X, d, \mu)$ by

$$(M^{\mathcal{D}} f)(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x), \quad x \in X,$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{D}$ containing x.

The following important result is proved by Hytönen and Kairema [10, Proposition 7.9].

Theorem 4. *Let* (X, d, μ) *be a space of homogeneous type and let* $\bigcup_{t=1}^K \mathcal{D}^t$ *be the adjacent system of dyadic grids given by Theorem 3. There exist a constant* $C_{HK}(X) \ge 1$ *depending only* (X, d, μ) *such that for every* $f \in L^1_{\rm loc}(X, d, \mu)$ *and* *a.e.* $x \in X$, *one has*

$$(M^{\mathcal{D}^t} f)(x) \le C_{HK}(X)(Mf)(x), \quad t \in \{1, \dots, K\},$$
$$(Mf)(x) \le C_{HK}(X) \sum_{t=1}^{K} (M^{\mathcal{D}^t} f)(x).$$

3. Reverse Hölder inequality.

A measurable non-negative locally integrable function w on X is said to be a weight. Given a weight w and a measurable set $E \subset X$, denote

$$w(E) := \int_E w(x)\, d\mu(x).$$

Fix a dyadic grid $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$. A weight $w : X \to [0, \infty]$ is said to belong to the dyadic class $A_1^{\mathcal{D}}$ if there exists a constant $c > 0$ such that for a.e. $x \in X$,

$$(M^{\mathcal{D}} w)(x) \le c w(x).$$

The smallest constant c in this inequality is denoted by $[w]_{A_1^{\mathcal{D}}}$.

Following [2, Definition 4.4], a generalized dyadic parent (gdp) of a cube Q is any cube Q^* such that $\ell(Q^*) = \frac{1}{\delta^2} \ell(Q)$ and for every $Q' \in \mathcal{D}$ such that $Q' \cap Q \ne \emptyset$ and $\ell(Q') = \ell(Q)$, one has $Q' \subset Q^*$. According to [2, Lemma 4.5], every cube $Q \in \mathcal{D}$ possesses at least one gdp.

For every $x \in X$ and $Q \in \mathcal{D}$, put

$$\mathcal{Q}_Q := \{Q' \in \mathcal{D} \ : \ Q' \cap Q \ne \emptyset,\ \ell(Q') \le \ell(Q)\},\ \mathcal{Q}_Q^x := \{Q' \in \mathcal{Q}_Q \ : \ x \in Q'\}.$$

It follows immediately that if $Q' \in \mathcal{Q}_Q$, then $Q' \subset Q^*$. For every $Q \in \mathcal{D}$, the localized dyadic maximal operator M_Q is defined by

$$(M_Q f)(x) = \begin{cases} \sup\limits_{Q' \in \mathcal{Q}_Q^x} \dfrac{1}{\mu(Q')} \displaystyle\int_{Q'} |f(y)|\, d\mu(y) & \text{if} \quad \mathcal{Q}_Q^x \ne \emptyset, \\ 0, & \text{if} \quad \mathcal{Q}_Q^x = \emptyset. \end{cases}$$

Following [2, Definition 4.7], one says that a weight $w : X \to [0, \infty]$ belongs to the dyadic class $A_\infty^{\mathcal{D}}$ if

$$[w]_\infty^{\mathcal{D}} := \sup_{Q \in \mathcal{D}} \inf_{Q^*} \frac{1}{w(Q^*)} \int_X (M_Q w)(x)\, d\mu(x) < \infty.$$

Let $C_{\mathcal{D}} \ge 1$ be a constant such that for all cubes $Q \in \mathcal{D}$ and $Q' \in \mathcal{Q}_Q$ satisfying $\ell(Q) = \ell(Q')$, one has

$$\mu(Q^*) \le C_{\mathcal{D}} \mu(Q'). \tag{3.1}$$

Lemma 5. *If* $w \in A_1^{\mathcal{D}}$, *then* $w \in A_\infty^{\mathcal{D}}$ *and* $[w]_{A_\infty^{\mathcal{D}}} \le [w]_{A_1^{\mathcal{D}}}$.

Proof. Fix a cube $Q \in \mathcal{D}$ and one of its gdp's Q^*. It follows immediately from the definition of $\mathcal{Q}_Q$ that if $Q' \in \mathcal{Q}_Q$, then $Q' \subset Q^*$. Take any $x \in X$. If $\mathcal{Q}_Q^x \ne \emptyset$, then there exists $Q' \in \mathcal{Q}_Q^x$ such that $x \in Q' \subset Q^*$. Therefore, if $x \notin Q^*$, then $(M_Q w)(x) = 0$. Thus, for a.e. $x \in Q$,

$$(M_Q w)(x) = (M_Q w)(x) \chi_{Q^*}(x) \le (M^{\mathcal{D}} w)(x) \chi_{Q^*}(x) \le [w]_{A_1^{\mathcal{D}}} w(x) \chi_{Q^*}(x),$$

whence

$$\begin{aligned}[w]_{A_\infty^{\mathcal{D}}} &= \sup_{Q\in\mathcal{D}} \inf_{Q^*} \frac{1}{w(Q^*)} \int_X (M_Q w)(x)\, d\mu(x) \\ &\le [w]_{A_1^{\mathcal{D}}} \sup_{Q\in\mathcal{D}} \inf_{Q^*} \frac{1}{w(Q^*)} \int_{Q^*} w(x)\, d\mu(x) = [w]_{A_1^{\mathcal{D}}},\end{aligned}$$

which completes the proof. □

The following result is an easy consequence of the weak reverse Hölder inequality for weights in $A_\infty^{\mathcal{D}}$ obtained recently by Anderson, Hytönen, and Tapiola [2, Theorem 5.4].

Lemma 6. *Let K be the constant from Theorem 3 and $C_{\mathcal{D}}$ be the constant defined in* (3.1)*. If $w \in A_1^{\mathcal{D}}$, then for every η satisfying*

$$0 < \eta \le \frac{1}{2C_{\mathcal{D}}^2 K [w]_{A_1^{\mathcal{D}}}} \tag{3.2}$$

and every $Q \in \mathcal{D}$, one has

$$\left(\frac{1}{2\mu(Q)} \int_Q w^{1+\eta}(x)\, d\mu(x)\right)^{\frac{1}{1+\eta}} \le C_{\mathcal{D}} [w]_{A_1^{\mathcal{D}}} \frac{1}{\mu(Q)} \int_Q w(x)\, d\mu(x). \tag{3.3}$$

Proof. We know from Lemma 5 that $w \in A_\infty^{\mathcal{D}}$ and $[w]_{A_\infty^{\mathcal{D}}} \le [w]_{A_1^{\mathcal{D}}}$. Then, by [2, Theorem 5.4], for every η satisfying (3.2), one has

$$\left(\frac{1}{2\mu(Q)} \int_Q w^{1+\eta}(x)\, d\mu(x)\right)^{\frac{1}{1+\eta}} \le C_{\mathcal{D}} \frac{1}{\mu(Q^*)} \int_{Q^*} w(x)\, d\mu(x). \tag{3.4}$$

Since $w \in A_1^{\mathcal{D}}$, for a.e. $x \in Q \subset Q^*$, one has

$$\frac{1}{\mu(Q^*)} \int_{Q^*} w(y)\, d\mu(y) \le (M^{\mathcal{D}} w)(x) \le [w]_{A_1^{\mathcal{D}}} w(x).$$

Integrating this inequality over Q, we obtain

$$\frac{\mu(Q)}{\mu(Q^*)} \int_{Q^*} w(y)\, d\mu(y) \le [w]_{A_1^{\mathcal{D}}} \int_Q w(x)\, d\mu(x). \tag{3.5}$$

Combining inequalities (3.4) and (3.5), we immediately arrive at inequality (3.3). □

The main result of this section is the following reverse Hölder inequality.

Theorem 7. *Let K be the constant from Theorem 3 and $C_{\mathcal{D}}$ be the constant defined in* (3.1)*. If $w \in A_1^{\mathcal{D}}$, then for every η satisfying* (3.2)*, every cube $Q \in \mathcal{D}$, and every measurable subset $E \subset Q$, one has*

$$\frac{w(E)}{w(Q)} \le 2^{\frac{1}{1+\eta}} C_{\mathcal{D}} [w]_{A_1^{\mathcal{D}}} \left(\frac{\mu(E)}{\mu(Q)}\right)^{\frac{\eta}{1+\eta}}. \tag{3.6}$$

Proof. By Hölder's inequality and reverse Hölder's inequality (3.3),

$$\begin{aligned} w(E) &= \int_Q w(x)\chi_E(x)\,d\mu(x) \\ &\le \left(\int_Q w^{1+\eta}(x)\,d\mu(x)\right)^{\frac{1}{1+\eta}} \big(\mu(E)\big)^{\frac{\eta}{1+\eta}} \\ &\le 2^{\frac{1}{1+\eta}} C_{\mathcal{D}}[w]_{A_1^{\mathcal{D}}} \frac{\big(\mu(Q)\big)^{\frac{1}{1+\eta}}}{\mu(Q)} w(Q)\big(\mu(E)\big)^{\frac{\eta}{1+\eta}} \\ &= 2^{\frac{1}{1+\eta}} C_{\mathcal{D}}[w]_{A_1^{\mathcal{D}}} w(Q) \left(\frac{\mu(E)}{\mu(Q)}\right)^{\frac{\eta}{1+\eta}}, \end{aligned}$$

which immediately implies (3.6). □

4. Calderón-Zygmund decomposition.

We start this section with the following important observation.

Lemma 8. *Suppose (X, d, μ) is a space of homogeneous type with the constants $\varkappa \ge 1$ in inequality (1.1) and $C_\mu \ge 1$ in inequality (1.2). Let (X, d, μ) be equipped with an adjacent system of dyadic grids $\{\mathcal{D}^t, t = 1, \dots, K\}$ and let $\delta \in (0, 1)$ be chosen as in Theorem 3. Then there is an $\varepsilon = \varepsilon(\varkappa, C_\mu, \delta) \in (0, 1)$ such that for every $t \in \{1, \dots, K\}$ and all $Q, P \in \mathcal{D}^t$, if Q is a child of P, then*

$$\mu(Q) \ge \varepsilon\mu(P). \tag{4.1}$$

This result is certainly known. For the construction of Christ, we refer to [5, Chap. VI, Theorem 14], where it is stated without proof (see also [4, Theorem 11], where it is implicit). In [1, Theorem 2.1] and [8, Theorem 2.5] it is stated without proof and attributed to Hytönen and Kairema [10], although it is only implicit in the latter paper. For the convenience of the readers, we provide its proof.

Proof. Let $Q = Q_\beta^{k+1,t}$ be a child of $P = Q_\alpha^{k,t}$ for some $t \in \{1, \dots, K\}$, $k \in \mathbb{Z}$, and $\alpha \in \mathcal{A}_k$, $\beta \in \mathcal{A}_{k+1}$. It follows from Theorem 3(a), part (iii), that $P \subset B(z_\alpha^{k,t}, 4\varkappa^2\delta^k)$ and $B(z_\beta^{k+1,t}, (12\varkappa^4)^{-1}\delta^{k+1}) \subset Q$, whence

$$\mu(P) \le \mu\big(B(z_\alpha^{k,t}, 4\varkappa^2\delta^k)\big), \quad \mu\big(B(z_\beta^{k+1,t}, (12\varkappa^4)^{-1}\delta^{k+1})\big) \le \mu(Q). \tag{4.2}$$

It follows from [10, Lemma 2.10] with $C_0 = 2\varkappa$ (cf. [10, Lemma 4.10]) that if $Q_\beta^{k+1,t}$ is a child of $Q_\alpha^{k,t}$, then

$$d(z_\alpha^{k,t}, z_\beta^{k+1,t}) < 2\varkappa\delta^k. \tag{4.3}$$

If $x \in B(z_\alpha^{k,t}, 4\varkappa^2\delta^k)$, then

$$d(x, z_\alpha^{k,t}) < 4\varkappa^2\delta^k. \tag{4.4}$$

Combining (1.1) with (4.3)–(4.4), we get

$$\begin{aligned} d(x, z_\beta^{k+1,t}) &\le \varkappa\big(d(x, z_\alpha^{k,t}) + d(z_\beta^{k+1,t}, z_\alpha^{k,t})\big) \\ &< \varkappa(4\varkappa^2\delta^k + 2\varkappa\delta^k) = \varkappa^2(4\varkappa + 2)\delta^k, \end{aligned}$$

whence $x \in B(z_\beta^{k+1,t}, \varkappa^2(4\varkappa + 2)\delta^k)$. Therefore

$$B(z_\alpha^{k,t}, 4\varkappa^2\delta^k) \subset B(z_\beta^{k+1,t}, \varkappa^2(4\varkappa + 2)\delta^k).$$

This inclusion immediately implies that

$$\mu\big(B(z_\alpha^{k,t}, 4\varkappa^2\delta^k)\big) \le \mu\big(B(z_\beta^{k+1,t}, \varkappa^2(4\varkappa+2)\delta^k)\big). \tag{4.5}$$

Let s be the smallest natural number satisfying

$$\log_2(12\varkappa^6(4\varkappa+2)\delta^{-1}) \le s.$$

Then $\varkappa^2(4\varkappa+2)\delta^k \le 2^s(12\varkappa^4)^{-1}\delta^{k+1}$ and, therefore,

$$\mu\big(B(z_\beta^{k+1,t}, \varkappa^2(4\varkappa+2)\delta^k)\big) \le \mu\big(B(z_\beta^{k+1,t}, 2^s(12\varkappa^4)^{-1}\delta^{k+1})\big). \tag{4.6}$$

Applying inequality (1.2) s times, one gets

$$\mu\big(B(z_\beta^{k+1,t}, 2^s(12\varkappa^4)^{-1}\delta^{k+1})\big) \le C_\mu^s \mu\big(B(z_\beta^{k+1,t}, (12\varkappa^4)^{-1}\delta^{k+1})\big). \tag{4.7}$$

Combining inequalities (4.2) with (4.5)–(4.7), we arrive at

$$\begin{aligned}\mu(P) &\le \mu\big(B(z_\beta^{k+1,t}, \varkappa^2(4\varkappa+2)\delta^k\big) \\ &\le C_\mu^s \mu\big(B(z_\beta^{k+1,t}(12\varkappa^4)^{-1}\delta^{k+1}\big) \le C_\mu^s \mu(Q),\end{aligned}$$

which implies inequality (4.1) with $\varepsilon = C_\mu^{-s}$. □

Once Lemma 8 is available, one can prove the following version of the Calderón-Zygmund decomposition for spaces of homogeneous type.

Theorem 9. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}_t$ be a dyadic grid. Suppose that $\varepsilon \in (0,1)$ is the same as in Lemma 8 and $f \in L^1(X, d, \mu)$.*

(a) *If*

$$\lambda > \begin{cases} 0 & \text{if} \quad \mu(X) = \infty, \\ \dfrac{1}{\mu(X)} \displaystyle\int_X |f(x)|\, d\mu(x) & \text{if} \quad \mu(X) < \infty, \end{cases}$$

and the set

$$\Omega_\lambda := \{x \in X \ : \ (M^{\mathcal{D}} f)(x) > \lambda\}$$

is nonempty, then there exists a collection $\{Q_j\} \subset \mathcal{D}$ that is pairwise disjoint, maximal with respect to inclusion, and such that

$$\Omega_\lambda = \bigcup_j Q_j. \tag{4.8}$$

Moreover, for every j,

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\, d\mu(x) \le \frac{\lambda}{\varepsilon}. \tag{4.9}$$

(b) *Let $a > 2/\varepsilon$ and, for $k \in \mathbb{Z}$ satisfying*

$$a^k > \begin{cases} 0 & \text{if} \quad \mu(X) = \infty, \\ \dfrac{1}{\mu(X)} \displaystyle\int_X |f(x)|\, d\mu(x) & \text{if} \quad \mu(X) < \infty, \end{cases} \tag{4.10}$$

let

$$\Omega_k := \{x \in X \ : \ (M^{\mathcal{D}} f)(x) > a^k\}. \tag{4.11}$$

If $\Omega_k \neq \emptyset$, then there exists a collection $\{Q_j^k\}_{j\in J_k}$ (as in part (a)*) such that it is pairwise disjoint, maximal with respect to inclusion, and*

$$\Omega_k = \bigcup_{j\in J_k} Q_j^k. \tag{4.12}$$

The collection of cubes

$$S = \{Q_j^k \ : \ \Omega_k \neq \emptyset, \ j \in J_k\}$$

is sparse, and for all j and k, the sets

$$E(Q_j^k) := Q_j^k \setminus \Omega_{k+1}$$

satisfy

$$\mu(Q_j^k) \le 2\mu(E(Q_j^k)). \tag{4.13}$$

Proof. The proof is analogous to the proof of [7, Proposition A.1]. For the convenience of the reader, we provide the proof in the case of $\mu(X) = \infty$. For $\mu(X) < \infty$, the proof is similar.

(a) Let Λ_λ be the family of dyadic cubes $Q \in \mathcal{D}$ such that

$$\lambda < \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x). \tag{4.14}$$

Notice that Λ_λ is nonempty because $\Omega_\lambda \neq \emptyset$. For each $Q \in \Lambda_\lambda$ there exists a maximal cube $Q' \in \Lambda_\lambda$ with $Q \subset Q'$, since

$$0 \le \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x) \le \frac{1}{\mu(Q)} \int_X |f(x)|\, d\mu(x) \to 0 \quad \text{as} \quad \mu(Q) \to \infty.$$

Let $\{Q_j\} \subset \Lambda_\lambda$ denote the family of such maximal cubes. By the maximality, the cubes in $\{Q_j\}$ are pairwise disjoint. If $\widetilde{Q}_j$ is the dyadic parent of Q_j, then $Q_j \subset \widetilde{Q}_j$ and $\widetilde{Q}_j$ does not belong to $\{Q_j\}$ in view of the maximality of the cubes in $\{Q_j\}$. Hence, taking into account Lemma 8, we see that

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\, d\mu(x) \le \frac{1}{\varepsilon\mu(\widetilde{Q}_j)} \int_{\widetilde{Q}_j} |f(x)|\, d\mu(x) \le \frac{\lambda}{\varepsilon},$$

which completes the proof of (4.9).

If $x \in \Omega_\lambda$, then it follows from the definition of $M^{\mathcal{D}}f$ that there exists a cube $Q \in \mathcal{D}$ such that $x \in Q$ and (4.14) is fulfilled. Hence $Q \subset Q_j$ for some j. Therefore, $\Omega_\lambda \subset \bigcup_j Q_j$.

Conversely, since

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\, d\mu(x),$$

if $x \in Q_j$, then $(M^{\mathcal{D}}f)(x) > \lambda$. This means that $x \in \Omega_\lambda$. Therefore, $\bigcup_j Q_j \subset \Omega_\lambda$. Thus (4.8) holds. Part (a) is proved.

(b) Equality (4.12) follows from part (a). Since $\Omega_{k+1} \subset \Omega_k$ and for each fixed k, the cubes Q_j^k are pairwise disjoint, it is clear that the sets $E(Q_j^k)$ are pairwise disjoint for all j and k. If $Q_j^k \cap Q_i^{k+1} \neq \emptyset$, then by the maximality of the cubes in

$\{Q_j^k\}_{j\in J_k}$ and the hypothesis $a > 2/\varepsilon$, we have $Q_i^{k+1} \subsetneqq Q_j^k$. In view of part (a),

$$\begin{aligned}\mu(Q_j^k \cap \Omega_{k+1}) &= \sum_{\{i:\ Q_i^{k+1}\subsetneqq Q_j^k\}} \mu(Q_i^{k+1}) \\ &\le \sum_{\{i:\ Q_i^{k+1}\subsetneqq Q_j^k\}} \frac{1}{a^{k+1}} \int_{Q_i^{k+1}} |f(x)|\,d\mu(x) \\ &\le \frac{1}{a^{k+1}} \int_{Q_j^k} |f(x)|\,d\mu(x) \\ &\le \frac{1}{a^{k+1}} \cdot \frac{a^k \mu(Q_j^k)}{\varepsilon} = \frac{\mu(Q_j^k)}{a\varepsilon}.\end{aligned}$$

Then

$$\begin{aligned}\mu(E(Q_j^k)) &= \mu(Q_j^k \setminus \Omega_{k+1}) = \mu(Q_j^k) - \mu(Q_j^k \cap \Omega_{k+1}) \\ &\ge \left(1 - \frac{1}{a\varepsilon}\right)\mu(Q_j^k) > \left(1 - \frac{1}{2}\right)\mu(Q_j^k),\end{aligned}$$

whence $\mu(Q_j^k) \le 2\mu(E(Q_j^k))$ for all j and k, which completes the proof of (4.13). □

Corollary 10. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ be a dyadic grid on X. Suppose $\varepsilon \in (0, 1)$ is the same as in Lemma 8 and $a > 2/\varepsilon$. For a non-negative function $f \in L^1(X, d, \mu)$ and $k \in \mathbb{Z}$ satisfying (4.10), let the sets Ω_k be given by (4.11). For all $\ell \in \mathbb{Z}_+$ and all j such that $Q_j^k \subset \Omega_k$,*

$$\mu(Q_j^k \cap \Omega_{k+\ell}) \le \frac{\mu(Q_j^k)}{a^\ell \varepsilon}. \tag{4.15}$$

Proof. The proof is analogous to the proof in the Euclidean setting of $\mathbb{R}^n$ given in [11, Lemma 2.4]. Since the cubes of $\Omega_{k+\ell}$ are pairwise disjoint and maximal, it follows from Theorem 9(a) that

$$\begin{aligned}\mu(Q_j^k \cap \Omega_{k+\ell}) &= \sum_{\{i: Q_i^{k+\ell}\subsetneqq Q_j^k\}} \mu(Q_i^{k+\ell}) \\ &< \frac{1}{a^{k+\ell}} \sum_{\{i: Q_i^{k+\ell}\subsetneqq Q_j^k\}} \int_{Q_i^{k+\ell}} f(x)\,d\mu(x) \\ &\le \frac{1}{a^{k+\ell}} \int_{Q_j^k} f(x)\,d\mu(x) \\ &\le \frac{\mu(Q_j^k)}{a^{k+\ell}} \cdot \frac{a^k}{\varepsilon} = \frac{\mu(Q_j^k)}{a^\ell \varepsilon},\end{aligned}$$

which completes the proof of (4.15). □

The following lemma in the Euclidean setting of $\mathbb{R}^n$ was proved in [11, Lemma 2.6].

Lemma 11. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}^t$ be a dyadic grid on X. Suppose $\varepsilon \in (0, 1)$ is the same as in Lemma 8 and $a > 2/\varepsilon$. For every non-negative function $f \in L^1(X, d, \mu)$ there exists a sparse family $S \subset \mathcal{D}$*

(depending on f) such that for all $x \in X$,

$$(M^{\mathcal{D}} f)(x) \le a \sum_{Q \in S} \left(\frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y) \right) \chi_{E(Q)}(x).$$

Proof. The proof is, actually, contained in the proof of [1, Theorem 3.1, p. 30]. We reproduce it here for completeness.

Let $\mathbb{K}$ denote the set of all $k \in \mathbb{Z}$ satisfying (4.10). Then

$$X = \bigcup_{k \in \mathbb{K}} \Omega_k \setminus \Omega_{k+1}. \tag{4.16}$$

Let S be the sparse family given by Theorem 9(b). For $k \in \mathbb{K}$ and a given $x \in \Omega_k \setminus \Omega_{k+1}$, there exists a cube $Q_j^k \in S$ such that $x \in Q_j^k \setminus \Omega_{k+1}$ and

$$(M^{\mathcal{D}} f)(x) \le a^{k+1} < \frac{a}{\mu(Q_j^k)} \int_{Q_j^k} f(y)\, d\mu(y). \tag{4.17}$$

Taking into account that by Theorem 9(b),

$$\Omega_k \setminus \Omega_{k+1} = \left(\bigcup_{j \in J_k} Q_j^k \right) \setminus \Omega_{k+1} = \bigcup_{j \in J_k} E(Q_j^k), \tag{4.18}$$

we obtain from (4.16)–(4.18) for all $x \in X$,

$$\begin{aligned}
(M^{\mathcal{D}} f)(x) &= \sum_{k \in \mathbb{K}} (M^{\mathcal{D}} f)(x) \chi_{\Omega_k \setminus \Omega_{k+1}}(x) \\
&\le \sum_{k \in \mathbb{K}} \sum_{j \in J_k} \left(\frac{a}{\mu(Q_j^k)} \int_{Q_j^k} f(y)\, d\mu(y) \right) \chi_{E(Q_j^k)}(x) \\
&= a \sum_{Q \in S} \left(\frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y) \right) \chi_{E(Q)}(x),
\end{aligned}$$

which completes the proof. □

5. Proof of part (a) of Theorem 2.

The scheme of the proof is borrowed from the proof of the necessity portion of [11, Theorem 3.1].

For a bounded sublinear operator on a Banach function space $\mathcal{E}'(X, d, \mu)$, let $\|T\|_{\mathcal{B}(\mathcal{E}')}$ denote its norm.

Fix $\mathcal{D} \in \bigcup_{t=1}^{K} \mathcal{D}^t$. It follows from the boundedness of the Hardy-Littlewood maximal operator M on $\mathcal{E}'(X, d, \mu)$ in view of Theorem 4 and the lattice property (axiom (A2) in the definition of a Banach function space) that the dyadic maximal operator $M^{\mathcal{D}}$ is bounded on the space $\mathcal{E}'(X, d, \mu)$ and

$$\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')} \le C_{HK}(X) \|M\|_{\mathcal{B}(\mathcal{E}')}. \tag{5.1}$$

Let $g \in L^0_+(X, d, \mu)$ with $\|g\|_{\mathcal{E}'} \le 1$. Using an idea of Rubio de Francia [12] (see also [7, Section 2.1]), put

$$(Rg)(x) := \sum_{k=0}^{\infty} \frac{\big((M^{\mathcal{D}})^k g\big)(x)}{\big(2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}\big)^k}, \quad x \in X,$$

where $(M^{\mathcal{D}})^k$ denotes the k-th iteration of $M^{\mathcal{D}}$ and $(M^{\mathcal{D}})^0 g := g$. Then

$$\|Rg\|_{\mathcal{E}'} \le 2 \tag{5.2}$$

and

$$g(x) \le (Rg)(x) \quad \text{for a.e.} \quad x \in X. \tag{5.3}$$

Since $M^{\mathcal{D}}$ is sublinear, we have

$$\begin{aligned}
(M^{\mathcal{D}}Rg)(x) &\le \sum_{k=0}^{\infty} \frac{\big((M^{\mathcal{D}})^{k+1}g\big)(x)}{\big(2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}\big)^k} \\
&= 2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')} \sum_{k=0}^{\infty} \frac{\big((M^{\mathcal{D}})^{k+1}g\big)(x)}{\big(2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}\big)^{k+1}} \\
&\le 2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')} \sum_{k=0}^{\infty} \frac{\big((M^{\mathcal{D}})^{k}g\big)(x)}{\big(2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}\big)^{k}} \\
&= 2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}(Rg)(x),
\end{aligned}$$

whence $Rg \in A_1^{\mathcal{D}}$ with

$$[Rg]_{A_1^{\mathcal{D}}} \le 2\|M^{\mathcal{D}}\|_{\mathcal{B}(\mathcal{E}')}. \tag{5.4}$$

Let the constants $C_{\mathcal{D}^t} \ge 1$ be defined for each $t \in \{1, \dots, K\}$ by (3.1). Take η and γ such that

$$0 < \eta \le \left(\left(\max_{1 \le t \le K} C^2_{\mathcal{D}^t} \right) K C_{HK}(X) \|M\|_{\mathcal{B}(\mathcal{E}')} \right)^{-1}, \quad \gamma = \frac{\eta}{1+\eta}.$$

Inequalities (5.4) and (5.1) imply that

$$[Rg]_{A_1^{\mathcal{D}}} \le 2C_{HK}(X)\|M\|_{\mathcal{B}(\mathcal{E}')}, \tag{5.5}$$

whence η satisfies (3.2). Since $Rg \in A_1^{\mathcal{D}}$, it follows from Theorem 7 and inequality (5.5) that, for every cube $Q \in \mathcal{D}$ and every measurable subset $G_Q \subset Q$, one has

$$\begin{aligned}
\int_{G_Q} (Rg)(x)\, d\mu(x) &\le 2^{1-\gamma} C_{\mathcal{D}} [Rg]_{A_1^{\mathcal{D}}} \left(\frac{\mu(G_Q)}{\mu(Q)} \right)^{\gamma} \int_Q (Rg)(x)\, d\mu(x) \\
&\le \frac{C}{2} \left(\frac{\mu(G_Q)}{\mu(Q)} \right)^{\gamma} \int_Q (Rg)(x)\, d\mu(x),
\end{aligned} \tag{5.6}$$

where

$$C := 2^{3-\gamma} \left(\max_{1 \le t \le K} C_{\mathcal{D}^t} \right) C_{HK}(X) \|M\|_{\mathcal{B}(\mathcal{E}')}.$$

Taking into account inequalities (5.3), (5.6), Hölder's inequality for Banach function spaces (see [3, Chap. 1, Theorem 2.4]), and inequality (5.2), we deduce that, for every finite sparse family $S \subset \mathcal{D}$, every collection of non-negative numbers $\{\alpha_Q\}_{Q \in S}$, every collection of pairwise disjoint measurable subsets $G_Q \subset Q$, and every $g \in L^0_+(X,d,\mu)$ satisfying $\|g\|_{\mathcal{E}'} \le 1$, one has

$$\begin{aligned}
&\int_X \left(\sum_{Q\in S} \alpha_Q \chi_{G_Q}(x)\right) g(x)\,d\mu(x) \\
&\quad \le \sum_{Q\in S} \alpha_Q \int_{G_Q} g(x)\,d\mu(x) \\
&\quad \le \sum_{Q\in S} \alpha_Q \int_{G_Q} (Rg)(x)\,d\mu(x) \\
&\quad \le \frac{C}{2} \sum_{Q\in S} \alpha_Q \left(\frac{\mu(G_Q)}{\mu(Q)}\right)^\gamma \int_Q (Rg)(x)\,d\mu(x) \\
&\quad \le \frac{C}{2} \left(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^\gamma \int_X \left(\sum_{Q\in S} \alpha_Q \chi_Q(x)\right) (Rg)(x)\,d\mu(x) \\
&\quad \le \frac{C}{2} \left(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^\gamma \left\|\sum_{Q\in S} \alpha_Q \chi_Q\right\|_{\mathcal{E}} \|Rg\|_{\mathcal{E}'} \\
&\quad \le C \left(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^\gamma \left\|\sum_{Q\in S} \alpha_Q \chi_Q\right\|_{\mathcal{E}}.
\end{aligned}$$

Then, in view of the Lorentz-Luxemburg theorem (see [3, Chap. 1, Theorem 2.7]),

$$\begin{aligned}
&\left\|\sum_{Q\in S} \alpha_Q \chi_{G_Q}\right\|_{\mathcal{E}} = \left\|\sum_{Q\in S} \alpha_Q \chi_{G_Q}\right\|_{\mathcal{E}''} \\
&\quad = \sup\left\{\int_X \left(\sum_{Q\in S} \alpha_Q \chi_{G_Q}(x)\right) g(x)\,d\mu(x) : g \in L^0_+(X,d,\mu),\ \|g\|_{\mathcal{E}'} \le 1\right\} \\
&\quad \le C \left(\max_{Q\in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^\gamma \left\|\sum_{Q\in S} \alpha_Q \chi_Q\right\|_{\mathcal{E}},
\end{aligned}$$

that is, the space $\mathcal{E}(X,d,\mu)$ satisfies the condition $\mathcal{A}_\infty$, which completes the proof of part (a) of Theorem 2. □

6. Proof of part (b) of Theorem 2.

We follow the proof of the sufficiency portion of the proof of [11, Theorem 3.1]. Let $\varepsilon \in (0,1)$ be the same as in Lemma 8. Take $a > 2/\varepsilon$. Assume that $f \in L^1(X,d,\mu) \cap \mathcal{E}'(X,d,\mu)$ is a nonnegative function and fix any dyadic grid $\mathcal{D} \in \bigcup_{t=1}^k \mathcal{D}^t$. By Lemma 11, there exists a sparse family $S \subset \mathcal{D}$ (not necessarily finite) such that for all $x \in X$,

$$(M^{\mathcal{D}} f)(x) \le a \sum_{Q\in S} \left(\frac{1}{\mu(Q)} \int_Q f(y)\,d\mu(y)\right) \chi_{E(Q)}(x). \tag{6.1}$$

For every subfamily $S' \subset S$, put

$$(A_{S'} f)(x) = \sum_{Q\in S'} \left(\frac{1}{\mu(Q)} \int_Q f(y)\,d\mu(y)\right) \chi_{E(Q)}(x).$$

Let $\{S_t\}_{t\in\mathbb{N}}$ be a sequence of subfamilies of S such that each subfamily S_t is finite, $S_t \subset S_n$ if $t < n$, and $A_{S_t} f \uparrow A_S f$ a.e. on X as $t \to \infty$. By the Fatou property (axiom (A3) in the definition of a Banach function space),

$$\lim_{t\to\infty} \|A_{S_t} f\|_{\mathcal{E}'} = \|A_S f\|_{\mathcal{E}'}. \tag{6.2}$$

By the Fubini theorem, for every $g \in \mathcal{E}(X,d,\mu)$ and every $t \in \mathbb{N}$, one has

$$\begin{aligned}
&\int_X (A_{S_t} f)(x) g(x)\, d\mu(x) \\
&\quad = \int_X \int_X \sum_{Q\in S_t} \frac{1}{\mu(Q)} f(y)\chi_Q(y)\chi_{E(Q)}(x) g(x)\, d\mu(y)\, d\mu(x) \\
&\quad = \int_X \int_X \sum_{Q\in S_t} \frac{1}{\mu(Q)} f(x)\chi_Q(x)\chi_{E(Q)}(y) g(y)\, d\mu(y)\, d\mu(x) \\
&\quad = \int_X \sum_{Q\in S_t} \left(\frac{1}{\mu(Q)} \int_{E(Q)} g(y)\, d\mu(y) \right) \chi_Q(x) f(x)\, d\mu(x) \\
&\quad = \int_X f(x)(A^*_{S_t} g)(x)\, d\mu(x),
\end{aligned} \tag{6.3}$$

where

$$(A^*_{S_t} g)(x) := \sum_{Q\in S_t} \left(\frac{1}{\mu(Q)} \int_{E(Q)} g(y)\, d\mu(y) \right) \chi_Q(x), \quad x \in X.$$

It follows from (6.3) and Hölder's inequality for Banach function spaces (see [3, Chap. 1, Theorem 2.4]) that

$$\left| \int_X (A_{S_t} f)(x) g(x)\, d\mu(x) \right| \le \|A^*_{S_t} g\|_{\mathcal{E}} \|f\|_{\mathcal{E}'}. \tag{6.4}$$

Let $C, \gamma > 0$ be as in Definition 1. Since $a > 2/\varepsilon > 2$, there exists $\nu \in \mathbb{N}$ such that

$$C\varepsilon^{-\gamma} \sum_{\ell=\nu}^{\infty} a^{-\ell\gamma} \le \frac{1}{2}. \tag{6.5}$$

For $Q \in S_t$, let

$$\alpha_Q := \frac{1}{\mu(Q)} \int_{E(Q)} |g(x)|\, d\mu(x).$$

Then for all $x \in X$,

$$\begin{aligned}
|(A^*_{S_t} g)(x)| &\le \sum_{Q\in S_t} \alpha_Q \chi_Q(x) \\
&= \sum_{\{j,k:Q_j^k\in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k}(x) \\
&= \sum_{\{j,k:Q_j^k\in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k\setminus\Omega_{k+\nu}}(x) + \sum_{\{j,k:Q_j^k\in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k\cap\Omega_{k+\nu}}(x) \\
&=: \Sigma_1(x) + \Sigma_2(x),
\end{aligned} \tag{6.6}$$

where the sets Ω_k are defined by (4.11) for all $k \in \mathbb{Z}$ satisfying (4.10).

Let $\mathbb{K}$ be the set of all those $k \in \mathbb{Z}$ that satisfy (4.10). It is easy to see that for $k \in \mathbb{K}$ and $\nu \in \mathbb{N}$,

$$\Omega_k \setminus \Omega_{k+\nu} \subset \bigcup_{i=0}^{\nu-1} \Omega_{k+i} \setminus \Omega_{k+i+1}. \tag{6.7}$$

It is also easy to see that if $k \in \mathbb{K}$ and $x \in Q_j^k$, then

$$\alpha_{Q_j^k} \le \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} |g(x)|\, d\mu(x) \le (M^{\mathcal{D}} g)(x). \tag{6.8}$$

Combining (6.7) and (6.8), we get for $x \in X$,

$$\begin{aligned} \Sigma_1(x) &= \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \setminus \Omega_{k+\nu}}(x) \\ &\le (M^{\mathcal{D}} g)(x) \sum_{k \in \mathbb{K}} \chi_{\Omega_k \setminus \Omega_{k+\nu}}(x) \\ &\le (M^{\mathcal{D}} g)(x) \sum_{i=0}^{\nu-1} \sum_{k \in \mathbb{K}} \chi_{\Omega_{k+i} \setminus \Omega_{k+i+1}}(x) \\ &= \nu (M^{\mathcal{D}} g)(x). \end{aligned} \tag{6.9}$$

On the other hand, for $x \in X$, we have

$$\begin{aligned} \Sigma_2(x) &= \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap \Omega_{k+\nu}}(x) \\ &= \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \sum_{\ell=\nu}^{\infty} \chi_{Q_j^k \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})}(x) \\ &= \sum_{\ell=\nu}^{\infty} \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})}(x). \end{aligned} \tag{6.10}$$

Since S_t is a finite sparse family, applying inequality (1.3) of Definition 1, we obtain for all $\ell \ge \nu$,

$$\begin{aligned} &\left\| \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})} \right\|_{\mathcal{E}} \\ &\quad \le C \left(\max_{\{j,k: Q_j^k \in S_t\}} \frac{\mu\big(Q_j^k \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})\big)}{\mu(Q_j^k)} \right)^{\gamma} \|A_{S_t}^* g\|_{\mathcal{E}}. \end{aligned} \tag{6.11}$$

By Corollary 10, we get for all $\ell \ge \nu$,

$$\begin{aligned} \max_{\{j,k: Q_j^k \in S_t\}} \frac{\mu\big(Q_j^k \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})\big)}{\mu(Q_j^k)} &\le \max_{\{j,k: Q_j^k \in S_t\}} \frac{\mu(Q_j^k \cap \Omega_{k+\ell})}{\mu(Q_j^k)} \\ &\le \max_{\{j,k: Q_j^k \in S_t\}} \frac{\mu(Q_j^k)}{a^{\ell} \varepsilon \mu(Q_j^k)} = \frac{1}{a^{\ell} \varepsilon}. \end{aligned} \tag{6.12}$$

It follows from (6.10)–(6.12) and (6.5) that

$$\|\Sigma_2\|_{\mathcal{E}} \le \sum_{\ell=\nu}^{\infty} \left\| \sum_{\{j,k:Q_j^k\in S_t\}} \alpha_{Q_j^k}\chi_{Q_j^k\cap(\Omega_{k+\ell}\setminus\Omega_{k+\ell+1})} \right\|_{\mathcal{E}} \le C\sum_{\ell=\nu}^{\infty}\left(\frac{1}{a^\ell\varepsilon}\right)^{\gamma}\|A_{S_t}^* g\|_{\mathcal{E}} \le \frac{1}{2}\|A_{S_t}^* g\|_{\mathcal{E}}. \tag{6.13}$$

Combining inequalities (6.6), (6.9), and (6.13), we arrive at

$$\|A_{S_t}^* g\|_{\mathcal{E}} \le \nu\|M^{\mathcal{D}}g\|_{\mathcal{E}} + \frac{1}{2}\|A_{S_t}^* g\|_{\mathcal{E}}.$$

It follows from this inequality, Theorem 4, and the boundedness of the Hardy-Littlewood maximal operator on $\mathcal{E}(X,d,\mu)$ that for all finite sparse families $S_t\subset S$ and all $g\in\mathcal{E}(X,d,\mu)$, one has

$$\|A_{S_t}^* g\|_{\mathcal{E}} \le 2\nu\|M^{\mathcal{D}}g\|_{\mathcal{E}} \le 2\nu C_{HK}(X)\|Mg\|_{\mathcal{E}} \le 2\nu C_{HK}(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|g\|_{\mathcal{E}}. \tag{6.14}$$

Combining (6.4) and (6.14) with [3, Chap. 1, Lemma 2.8], we see that

$$\|A_{S_t}f\|_{\mathcal{E}'} = \sup\left\{\left|\int_X (A_{S_t}f)(x)g(x)\,d\mu(x)\right| \ :\ g\in\mathcal{E}(X,d,\mu),\ \|g\|_{\mathcal{E}}\le 1\right\} \le 2\nu C_{HK}(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|f\|_{\mathcal{E}'}$$

for all $t\in\mathbb{N}$. Passing in this inequality to the limit as $t\to\infty$ and taking into account (6.2), we get

$$\|A_S f\|_{\mathcal{E}'} \le 2\nu C_{HK}(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|f\|_{\mathcal{E}'}.$$

It follows from this inequality and inequality (6.1) that

$$\|M^{\mathcal{D}}f\|_{\mathcal{E}'} \le 2a\nu C_{HK}(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|f\|_{\mathcal{E}'}. \tag{6.15}$$

for every dyadic grid $\mathcal{D}\in\bigcup_{t=1}^K\mathcal{D}^t$. In turn, inequality (6.15) and Theorem 4 imply that

$$\|Mf\|_{\mathcal{E}'} \le 2a\nu K C_{HK}^2(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|f\|_{\mathcal{E}'} \tag{6.16}$$

for every nonnegative function $f\in L^1(X,d,\mu)\cap\mathcal{E}'(X,d,\mu)$.

Now let $f\in\mathcal{E}'(X,d,\mu)$ be an arbitrary complex-valued function. Since X is σ-finite, there are measurable sets $\{A_n\}_{n\in\mathbb{N}}$ such that $\mu(A_n)<\infty$ for all $n\in\mathbb{N}$, $A_i\subset A_j$ for $i<j$, and $\bigcup_{n\in\mathbb{N}}A_n = X$. Let $f_n = |f|\chi_{A_n}$ for $n\in\mathbb{N}$. Then $f_n\in L^1(X,d,\mu)\cap\mathcal{E}'(X,d,\mu)$ for all $n\in\mathbb{N}$ in view of axiom (A5) in the definition of a Banach function space. By (6.16), for all $n\in\mathbb{N}$,

$$\|Mf_n\|_{\mathcal{E}'} \le 2a\nu K C_{HK}^2(X)\|M\|_{\mathcal{B}(\mathcal{E})}\|f_n\|_{\mathcal{E}'}. \tag{6.17}$$

Since $f_n\uparrow|f|$ a.e., we have $Mf_n\uparrow Mf$ a.e. (cf. [6, Lemma 2.2]). Passing to the limit in inequality (6.17), we conclude from the Fatou property that inequality (6.16) holds for all $f\in\mathcal{E}'(X,d,\mu)$. Thus, the Hardy-Littlewood maximal operator M is bounded on the space $\mathcal{E}'(X,d,\mu)$ whenever it is bounded on the space $\mathcal{E}(X,d,\mu)$ and the latter space satisfies the condition $\mathcal{A}_\infty$. □

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