On the existence of optimal and $\epsilon-$optimal feedback controls for stochastic second grade fluids

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Abstract

This article deals with a feedback optimal control problem for the stochastic second grade fluids. More precisely, we establish the existence of an optimal feedback control for the two-dimensional stochastic second grade fluids, with Navier-slip boundary conditions. In addition, using the Galerkin approximations, we show that the optimal cost can be approximated by a sequence of finite dimensionl optimal costs, showing the existence of the so-called $\epsilon-$optimal feedback control.

Key words. Feedback optimal control, $\epsilon-$optimal feedback control, Second grade fluids, Stochastic differential equation.

AMS Subject Classification. (35R60, 49K20, 60G15, 60H15, 76D55)

1 Introduction

In this paper, we consider the evolutionary equation for non-Newtonian fluids of the second grade, perturbed by a multiplicative white noise

$$\frac{\partial}{\partial t}(Y-\alpha\Delta Y)=\nu\Delta Y-\mathrm{curl}\,(Y-\alpha\Delta Y)\times Y-\nabla\pi+U(t,Y)+G(t,Y)\,\dot{\mathcal{W}}_t.$$

Let us refer the pioneer papers [14] and [6], where the deterministic second grade equations with the Dirichlet boundary condition were mathematically studied for the first time, and [2] where the deterministic equations were studied with a Navier boundary condition. The well-posedness of this stochastic differential equation, supplemented with Navier-slip boundary condition, and with a distributed mechanical force $U(t)$ independent of the solution has been established in [3] (see also [15], [16], [17]). The corresponding stochastic optimal control problem for the distributed control $U(t)$ and a suitable cost functional has been studied in [4]. In this work, we extend the previous existence and uniqueness results to forces that depend of the solution of the equation. The introduction of these forces is extremely important, because they can be seen as feedback controls acting on the evolution of the physical system.

Our main task consists in the study of the optimal feedback control problem for this class of non-Newtonian fluids. As far as we know, this is the first time that this problem is addressed. In our study, we adopt the methods considered in [1], [11], [12], to the Navier-Stokes equation, first we show the existence of an optimal feedback control for the infinite dimensional evolution system, and after we establish the existence of $\epsilon-$optimal feedback control based on the Galerkin

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finite dimensional approximations, so we will be able to approximate the infinite dimensional control problem by a sequence of finite dimensional ones.

We should mention, that the study of feedback type control problems for fluid mechanics equations is not as easy issue, since the usual approach via the Bellman's *principle of dynamic programming* requires an accurate interpretation and an appropriate analysis of the Hamilton-Jacobi-Bellman equation, which corresponds to a non-linear partial differential equation in infinite dimensional spaces (see [8], [9], [10] concerning Newtonian fluids). Therefore, the existence of $\epsilon-$feedback control, which can be obtained by solving a finite dimensional Hamilton-Jacobi-Bellman equation, can be used numerically to approximate the optimal control.

The remainder of this paper is organized as follows. In Section 2, we define the appropriate functional setting, introduce useful notations and state necessary assumptions on the data. In addition, we formulate the problem, and write the main results of the article. Section 3 establishes the well-posedness for the stochastic state equation (2.1). In Section 4, we present convergence results for suitable sequences of control variables, as well as for the finite dimensional approximations of the solutions. Section 6 is devoted to the proof of the main results, Theorems 2.2 and 2.3.

2 Formulation of the problem and main results

We consider the system for a bounded domain $\mathcal{O}$ of $\mathbb{R}^2$ with sufficiently regular boundary Γ,

$$\begin{cases} dv(Y) = (\nu\Delta Y - \operatorname{curl} v(Y)\times Y - \nabla\pi + U(t,Y))\,dt + G(t,Y)\,d\mathcal{W}_t, & \\ \operatorname{div} Y = 0 & \text{in } (0,T)\times\mathcal{O}, \\ Y\cdot \mathrm{n} = 0, \qquad (\mathrm{n}\cdot DY)\cdot\tau = 0 & \text{on } (0,T)\times\Gamma, \\ Y(0) = Y_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.1}$$

where $Y=(Y_1,Y_2)$ corresponds to the velocity field of the fluid, Δ and ∇ denote the Laplacian and gradient operators, respectively, $\nu>0$ is the viscosity of the fluid and $\alpha>0$ is a constant material modulus. The function π represents the pressure and

$$v(Y) = Y - \alpha\Delta Y.$$

Here, $\mathrm{n}=(n_1,n_2)$ and $\tau=(-n_2,n_1)$ are the unit normal and tangent vectors, respectively, to the boundary Γ, $Dy=\frac{\nabla y+\nabla y^\top}{2}$ corresponds to symmetric part of the velocity gradient. The initial condition Y_0 is a divergence free vector field, and the function U depends on the velocity field and acts as a feedback control of the system. The term
$G(t,\,Y)\,d\mathcal{W}_t$ introduces a stochastic perturbation, which is defined by a standard $\mathbb{R}^m$-valued Wiener process $\mathcal{W}_t=(\mathcal{W}_t^1,\dots,\mathcal{W}_t^m)$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ for $\mathcal{W}_t$. We assume that $\mathcal{F}_0$ contains every P-null subset of Ω.

We introduce some functional spaces. Let X be a real Banach space with norm $\|\cdot\|_X$; we denote by $L^p(0,T;X)$ the space of X-valued p-integrable functions on $[0,T]$, for $p\geq 1$. For $p,r\geq 1$, let $L^p(\Omega,L^r(0,T;X))$ be the space of stochastic processes $Y=Y(\omega,t)$ with values in X, measurable in $\Omega\times[0,T]$, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, and such that

$$\|Y\|_{L^p(\Omega,L^r(0,T;X))} = \left(\mathbb{E}\Big(\int_0^T \|Y\|_X^r\,dt\Big)^{\frac{p}{r}}\right)^{\frac{1}{p}} < \infty, \quad \text{for } 1\leq r<\infty,$$

$$\|Y\|_{L^p(\Omega,L^\infty(0,T;X))} = \left(\mathbb{E}\sup_{t\in[0,T]}\|Y\|_X^p\right)^{\frac{1}{p}} < \infty.$$

In the following, the vector product for 2D vectors $y = (y_1, y_2)$ and $z = (z_1, z_2)$ is calculated as $y \times z = (y_1, y_2, 0) \times (z_1, z_2, 0)$. The curl of the vector y is $\operatorname{curl} y = \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}$, and the vector product of $\operatorname{curl} y$ with vector z is understood as $\operatorname{curl} y \times z = (0, 0, \operatorname{curl} y) \times (z_1, z_2, 0)$.

Given two vectors $y, z \in \mathbb{R}^n$, $y \cdot z = \sum_{i=1}^n y_i z_i$, $|y|^2 = y \cdot y$, and for two matrices A, B of dimensions $n \times n$, $A \cdot B = \sum_{i,j=1}^n A_{ij} B_{ij}$.

Let us introduce the following Hilbert spaces

$$
\begin{aligned}
&H = \left\{ y \in L^2(\mathcal{O}) \mid \operatorname{div} y = 0 \ \text{ in } \mathcal{O} \ \text{ and } \ y \cdot \mathrm{n} = 0 \ \text{ on } \Gamma \right\}, \\
&V = \left\{ y \in H^1(\mathcal{O}) \mid \operatorname{div} y = 0 \ \text{ in } \ \mathcal{O} \text{ and } \ y \cdot \mathrm{n} = 0 \ \text{ on } \ \Gamma \right\}, \\
&W = \left\{ y \in V \cap H^2(\mathcal{O}) \mid (\mathrm{n} \cdot Dy) \cdot \tau = 0 \ \text{ on } \Gamma \right\}, \\
&\widetilde{W} = W \cap H^3(\mathcal{O}).
\end{aligned} \tag{2.2}
$$

We denote by $(\cdot, \cdot)$ the inner product in $L^2(\mathcal{O})$, and by $\|\cdot\|_2$ the associated norm. The norm in the space $H^p(\mathcal{O})$ is denoted by $\|\cdot\|_{H^p}$.

Consider the Helmholtz projector $\mathbb{P} : L^2(\mathcal{O}) \to H$. It is well known that $\mathbb{P}$ is a linear bounded operator being characterized by the equality $\mathbb{P}z = \tilde{z}$, where $\tilde{z}$ is defined by the Helmholtz decomposition

$$
z = \tilde{z} + \nabla \phi, \quad \tilde{z} \in H \quad \text{and} \quad \phi \in H^1(\mathcal{O}).
$$

On the functional spaces V, W and $\widetilde{W}$ defined in (2.2), we introduce the inner products

$$
\begin{aligned}
(y, z)_V &= (y, z) + 2\alpha\, (Dy, Dz)\,, \\
(y, z)_W &= (\mathbb{P}\upsilon(y), \mathbb{P}\upsilon(z)) + (y, z)_V\,, \\
(y, z)_{\widetilde{W}} &= (\operatorname{curl}\upsilon(y), \operatorname{curl}\upsilon(z)) + (y, z)_V
\end{aligned} \tag{2.3}
$$

and denote by $\|\cdot\|_V, \|\cdot\|_W, \|\cdot\|_{\widetilde{W}}$ the corresponding norms. One can prove the equivalence of these norms to the norms $\|\cdot\|_{H^1}, \|\cdot\|_{H^2}, \|\cdot\|_{H^3}$, respectively (see, for instance, [3], [4]).

It is convenient to introduce the following generalized Stokes problem (c.f. [18])

$$
\begin{cases}
h - \alpha \Delta h + \nabla p = f, \qquad \operatorname{div} h = 0 & \text{in } \mathcal{O}, \\
h \cdot \mathrm{n} = 0, \qquad (\mathrm{n} \cdot Dh) \cdot \tau = 0 & \text{on } \Gamma.
\end{cases} \tag{2.4}
$$

Throughout the article, given $f \in L^2(\mathcal{O})$, we denote by $(\widehat{f}, p) \in H^2(\mathcal{O}) \times H^1(\mathcal{O})$ the corresponding solution of the Stokes problem. We recall that

$$
\|\widehat{f}\|_W \leq C \|f\|_2, \quad \text{and consequently} \quad \|\widehat{f}\|_V \leq C \|f\|_V. \tag{2.5}
$$

In addition, we have the relation

$$
(\widehat{f}, z)_V = (f, z), \quad \forall z \in V, \ f \in L^2(\Omega). \tag{2.6}
$$

For any $x = (x_1, x_2, \ldots, x_m) \in (H^1(\mathcal{O}))^m$ and $z \in H^1(\mathcal{O})$, we introduce the following notations

$$
\begin{aligned}
&\|x\|_{H^1}^2 := \sum_{i=1}^m \|x_i\|_{H^1}^2\,, \qquad \|x\|_2^2 := \sum_{i=1}^m \|x_i\|_2^2\,, \\
&(x, z)_{H^1} := ((x_1, z)_{H^1}, \ldots, (x_m, z)_{H_1}), \quad (x, z) := ((x_1, z), \ldots, (x_m, z)).
\end{aligned}
$$

We assume that the diffusion coefficient $G(t,y):[0,T]\times V\to (H^1(\mathcal{O}))^m$ is a $[0,T]\times V$-measurable map, Lipschitz on y and satisfies a linear growth

$$\|G(t,y)-G(t,z)\|_{H^1}^2\leq K\,\|y-z\|_V^2\,,\quad \|G(t,y)\|_{H^1}\leq K\left(1+\|y\|_V\right),\quad \forall y,z\in V,\ t\in[0,T], \tag{2.7}$$

for some positive constant K. According to the previous notations, $\forall y,\ z\in V,\ t\in[0,T]$, we have

$$\|G(t,y)\|_{H^1}^2=\sum_{i=1}^m\|G_i(t,y)\|_{H^1}^2\,,\quad \|G(t,y)\|_2^2=\sum_{i=1}^m\|G_i(t,y)\|_2^2\,,$$

$$(G(t,y),z)_{H^1}=\left((G_1(t,y),z)_{H^1},\dots,(G_m(t,y),z)_{H^1}\right),\quad |(G(t,y),z)_{H^1}|^2=\sum_{i=1}^m(G_i(t,y),z)_{H^1}^2\,,$$

$$(G(t,y),z)=\left((G_1(t,y),z)\,,\dots,(G_m(t,y),z)\right),\quad |(G(t,y),z)|^2=\sum_{i=1}^m(G_i(t,y),z)^2\,.$$

The force term $U(t,Y)$ in equations (2.1) is a function of the solution, and will be considered as a feedback control of the dynamic. We assume that U belongs to an admissible set $\mathcal{A}$, which is defined, for some given $\beta,\mu,\eta>0$, as the set of all $[0,T]\times V$-measurable functions $U:[0,T]\times V\to H^1(\mathcal{O})$, that satisfy

$$\|U(0,0)\|_{H^1}^2\leq\eta\quad\text{and}\quad \|U(t,y)-U(s,z)\|_{H^1}^2\leq\beta|t-s|^2+\mu\|y-z\|_V^2, \tag{2.8}$$

for all $y,z\in V,\ t,s\in[0,T]$.

The solution of the system (2.1) is understood in the variational form.

Definition 2.1. *Let $U\in\mathcal{A}$ and $Y_0\in L^2(\Omega,\widetilde{W})$. A stochastic process $Y\in L^2(\Omega,L^\infty(0,T;\widetilde{W}))$ is a strong solution of* (2.1)*, if for a.e.-P and $\forall t\in[0,T]$, the following equation holds*

$$\begin{aligned}&(\upsilon(Y(t)),\phi)=\int_0^t\left[-2\nu\,(DY(s),D\phi)-(\operatorname{curl}\upsilon(Y(s))\times Y(s),\phi)\right]ds\\&+(\upsilon(Y_0),\phi)+\int_0^t(U(s,Y(s)),\phi)\;ds+\int_0^t(G(s,Y(s)),\phi)\;d\mathcal{W}_s,\quad \textit{for all}\quad \phi\in V.\end{aligned} \tag{2.9}$$

The main aim of this work is to control the dynamic associated to the system (2.1), through the feedback controls U that belong to the admissible set $\mathcal{A}$. The cost functional is given by

$$J(U)=\mathbb{E}\int_0^T L\left(t,U(t,Y_U),Y_U\right)\,dt+\mathbb{E}\,h(Y_U(T)),\qquad U\in\mathcal{A}, \tag{2.10}$$

where Y_U is the solution of the stochastic differential equation (2.1) for the feedback control U. Moreover, $L:[0,T]\times H^1(\mathcal{O})\times V\to[0,+\infty[$ is sequentially weak lower semi-continuous on the variable u and sequentially lower semi-continuous on the variable y, more precisely, for any sequences $(u_n)\subset H^1(\mathcal{O})$, $(y_n)\subset V$

$$u_n\rightharpoonup u\text{ weakly in }H^1(\mathcal{O}),\quad y_n\to y\text{ strongly in }V\Longrightarrow L\left(t,u,y\right)\leq\lim_{n\to\infty}\inf L\left(t,u_n,y_n\right). \tag{2.11}$$

The function $h:V\to[0,+\infty[$ is sequentially lower semicontinuous. (2.12)

Our first purpose is to show the existence of an optimal feedback control for the system (2.1). More precisely, we aim to solve the following problem

$$(\mathcal{P})\begin{cases} \underset{U}{\text{minimize}}\big\{J(U): \ U\in\mathcal{A} \quad \text{and} \\ \qquad\qquad\qquad Y_U \text{ is the solution of system (2.1) for the minimizing } U\in\mathcal{A}\big\}. \end{cases}$$

Let us state the first main results of this article.

Theorem 2.2. *Assume that L and h satisfy the assumptions* (2.11)-(2.12)*, then there exists an optimal feedback control $U^*\in\mathcal{A}$ for the problem $(\mathcal{P})$.*

Our next result establishes the existence of the so-called $\epsilon-$feedback control, namely the infinite dimensional optimal cost can be approximated by a sequence of finite dimensional optimal costs. In order to formulate this result, we introduce a suitable basis and use the *Galerkin's approximation method*. Since the norm $\|\cdot\|_{\widetilde{W}}$ is equivalent to $\|\cdot\|_{H^3}$, the injection operator $I:\widetilde{W}\to V$ is a compact operator, then there exists a basis $\{e_i\}\subset\widetilde{W}$ of eigenfunctions for the problem

$$(y,e_i)_{\widetilde{W}}=\lambda_i\,(y,e_i)_V\,, \qquad \forall y\in\widetilde{W},\ i\in\mathbb{N}, \tag{2.13}$$

being simultaneously an orthonormal basis for V and the corresponding sequence $\{\lambda_i\}$ of eigenvalues verifies $\lambda_i>0$, $\forall i\in\mathbb{N}$ and $\lambda_i\to\infty$ as $i\to\infty$. Let us notice that the ellipticity of equation (2.13) increases the regularity of their solutions. Hence without loss of generality we can consider $\{e_i\}\subset H^4$ (see [2]).

Let us set $W_n=\operatorname{span}\{e_1,\dots,e_n\}$ and denote by $\Pi_n: V\to W_n$ the orthogonal projection onto the space W_n, with respect to the norm $\|\cdot\|_V$. We notice that, if $z\in\widetilde{W}$, we have

$$\Pi_n z=\sum_{j=1}^n (z,e_j)_V e_j=\sum_{j=1}^n (z,\widetilde{e}_j)_{\widetilde{W}}\widetilde{e}_j,$$

where $\widetilde{e}_j=\frac{1}{\sqrt{\lambda_j}}e_j$. This means that the projections of z on W_n, with respect to the norms $\|\cdot\|_V$, $\|\cdot\|_{\widetilde{W}}$, are the same.

In order to introduce the finite dimensional approximations of equation (2.9), we denote $F(t,Y)=\nu\Delta Y(s)-\operatorname{curl}\upsilon(Y(s))\times Y(s)$, and write the equation (2.9) as follows

$$(Y(t),\phi)_V=(Y_0,\phi)_V+\int_0^t(\widehat{F(t,Y)},\phi)_V\,ds+\int_0^t(\widehat{U(t,Y)},\phi)_V\,ds+\int_0^t(\widehat{G(t,Y)},\phi)_V\,d\mathcal{W}_s,$$
$$\forall\phi\in V, \tag{2.14}$$

where $\widehat{F(t,Y)}$, $\widehat{U(t,Y)}$ and $\widehat{G(t,Y)}=(\widehat{G_1(t,Y)},\widehat{G_2(t,Y)},\dots,\widehat{G_m(t,Y)})$ are the solutions of the Stokes problem (2.4), corresponding to $F(t,Y)$, $U(t,Y)$ and $G(t,Y)$, respectively. Therefore, the finite dimensional projection of this equation reads as

$$\begin{aligned}(Y_n(t),\phi)_V=(\Pi_nY_0,\phi)_V&+\int_0^t(\Pi_n\widehat{F(s,Y_n)},\phi)_V\,ds+\int_0^t(\Pi_n\widehat{U(t,Y_n)},\phi)_V\,ds\\ &+\int_0^t(\Pi_n\widehat{G(t,Y_n)},\phi)_V\,d\mathcal{W}_s, \qquad \forall\phi\in W_n.\end{aligned} \tag{2.15}$$

Taking into account the relation (2.6), in the differential form, we have

$$\begin{cases} d\,(\upsilon(Y_n(t)),\phi)=(F(s,Y_n)+\Psi(t,Y_n),\phi)\ dt+(G(t,Y_n),\phi)\,dW_t,\\ Y_n(0)=\Pi_nY_0, \qquad\qquad\qquad\qquad\qquad\qquad \forall\phi\in W_n,\end{cases} \tag{2.16}$$

where $Y_n(t) = \sum_{j=1}^{n} c_j^n(t)e_j,\ c_j^n(t) \in \mathbb{R}$, Ψ belongs to the set

$$\mathcal{A}_n = \{\Psi : \ \Psi(t,y) = \upsilon\left(\Pi_n\widehat{U(t,y)}\right) \ \text{ with } \ U \in \mathcal{A}\},$$

and $\Pi_n Y_0$ is the projection of the divergence free vector field Y_0. The solution of equation (2.16) will be denoted by $Y_{n,\Psi}$.

The corresponding finite dimensional control problem reads

$$(\mathcal{P}_n) \begin{cases} \underset{\Psi}{\text{minimize}}\big\{J_n(\Psi) : \ \Psi \in \mathcal{A}_n \quad \text{and} \\ \qquad\qquad\qquad Y_{n,\Psi} \text{ is the solution of system (2.16) for the minimizing } \Psi \in \mathcal{A}_n\big\} \end{cases}$$

where

$$J_n(\Psi) = \mathbb{E}\int_0^T L\left(t, \Psi(t, Y_{n,\Psi}), Y_{n,\Psi}\right) dt + \mathbb{E}\, h(Y_{n,\Psi}(T)), \qquad \Psi \in \mathcal{A}_n. \tag{2.17}$$

In order to formulate the next result, we impose additional conditions on the Lagrangian L and final condition h. We assume that there exist positive constants $\lambda_L,\ \lambda_h$ such that

$$|L(t,u,y) - L(t,v,x)| \leq \lambda_L\left[\|u-v\|_{H^1}\left(\|u\|_{H^1} + \|v\|_{H^1}\right) + \|x-y\|_V\left(\|y\|_{\widetilde{W}} + \|x\|_{\widetilde{W}}\right)\right], \tag{2.18}$$

$$|h(y) - h(x)| \leq \lambda_h\|y-x\|_V\left(\|y\|_{\widetilde{W}} + \|x\|_{\widetilde{W}}\right), \tag{2.19}$$

for all $t \in [0,T],\ u,v \in H^1(\mathcal{O}),\ x,y \in \widetilde{W}$.

Theorem 2.3. *Assume the hypothesis* (2.18)-(2.19)*. Let $U^* \in \mathcal{A}$ be an optimal feedback control for problem $(\mathcal{P})$ and let $\Psi_n^* \in \mathcal{A}_n$ be a finite dimensional feedback control for problem $(\mathcal{P}_n)$. Then there exists a subsequence $(\Psi_{n_k}^*)$ of (Ψ_n^*) such that for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\forall n_k \geq N_\epsilon$, we have*

$$\left|J_{n_k}\left(\Psi_{n_k}^*\right) - J\left(U^*\right)\right| < \epsilon, \tag{2.20}$$

$$J\left(\Psi_{n_k}^*\right) - J\left(U^*\right) \leq \epsilon. \tag{2.21}$$

3 Well-posedness results and estimates of solutions

In this section, we establish the well-posedness to the stochastic differential equation (2.1), and write the dependence of the solution with respect to the initial data.

Before we go into the results, we state some inequalities involving the non-linear term. Let us introduce the trilinear functional

$$b(\phi,z,y) = (\phi\cdot\nabla z, y), \qquad \forall\phi,z,y \in V.$$

Standard integration by parts shows that

$$(\operatorname{curl}\upsilon(y)\times z, w) = b(w,z,\upsilon(y)) - b(z,w,\upsilon(y)) \quad \text{for any} \quad y,z \in \widetilde{W}, w \in V, \tag{3.22}$$

which implies that

$$(\operatorname{curl}\upsilon(y)\times z, z) = 0 \quad \text{for any} \quad y,z \in \widetilde{W}. \tag{3.23}$$

Lemma 3.1. *The following inequalities hold*

$$|(curl\,\upsilon(y)\times z, w)| \leq C\|y\|_{H^1}\|z\|_{H^3}\|w\|_{H^2} \quad \textit{for any} \quad y,z \in \widetilde{W}, w \in W, \tag{3.24}$$

$$|(curl\,\upsilon(y)\times z, y)| \leq C\|y\|_{H^1}^2\|z\|_{H_3} \quad \textit{for any} \quad y,z \in \widetilde{W}, \tag{3.25}$$

$$|(curl\,\upsilon(y)\times z, w)| \leq C\|y\|_{H^3}\|z\|_{H^1}\|w\|_{H^1} \quad \textit{for any} \quad y \in \widetilde{W}, z,w \in V. \tag{3.26}$$

Proof. The proof of the properties (3.24)-(3.25) can be found in [3], [4]. To verify (3.26), we notice that

$$\begin{aligned}|(\operatorname{curl} \upsilon(y) \times z, w)| &\leq |b(w, z, \upsilon(y))| + |b(z, w, \upsilon(y))| \\ &\leq \|w\|_{L^4} \|\nabla z\|_2 \|\upsilon(y)\|_{L^4} + \|z\|_{L^4} \|\nabla w\|_2 \|\upsilon(y)\|_{L^4},\end{aligned}$$

then the Sobolev's embedding $V \hookrightarrow L^4(\mathcal{O})$ gives the claimed result. □

Now, we are able to generalize the Theorem 4.2 in [3] for forces that depend on the solution, and that will be considered as feedback controls to the stochastic evolution physical system. In addition we state the dependence of the initial data that are fundamental to analyze the control problem.

Theorem 3.2. *Assume that* $U \in \mathcal{A}$ *and* $Y_0 \in L^p(\Omega, V) \cap L^2(\Omega, \widetilde{W})$ *for some* $4 \leq p < \infty$. *Then there exists a unique solution* Y *to equation* (2.9) *which belongs to*

$$L^2(\Omega, L^\infty(0, T; \widetilde{W})) \cap L^p(\Omega, L^\infty(0, T; V)).$$

Moreover, the following estimates hold

$$\mathbb{E} \sup_{s \in [0,t]} \|Y(s)\|_V^2 \leq C(1 + \mathbb{E}\, \|Y_0\|_V^2), \tag{3.27}$$

$$\mathbb{E} \sup_{s \in [0,t]} \|Y(s)\|_V^p \leq C\left(1 + \mathbb{E}\, \|Y_0\|_V^p\right), \tag{3.28}$$

$$\mathbb{E} \sup_{s \in [0,t]} \|Y(s)\|_{\widetilde{W}}^2 \leq C(1 + \mathbb{E}\, \|Y_0\|_{\widetilde{W}}^2). \tag{3.29}$$

Proof. The proof of this theorem is a straightforward adaptation of the proof of the Theorem 4.2 in [4]. The difference is that, here the force U depends on the solution Y, nevertheless the same reasoning can be applied, we just need to introduce minor changes. □

The finite dimension approximation of the equation (2.1) corresponds to the equation (2.16) with $\Psi = \upsilon(\Pi_n \widehat{U})$. Due to the relation (2.6), the finite dimension equation (2.16) can be written in the variational form as follows

$$\begin{aligned}(\upsilon(Y_n(t)), \phi) =& (\upsilon(\Pi_n Y_0), \phi) + \nu \int_0^t (\Delta Y_n(s), \phi) - (\operatorname{curl}(Y_n(s)) \times Y_n(s), \phi)\, ds \\ &+ \int_0^t (U(s, Y_n(s)), \phi)\, ds + \int_0^t (G(s, Y_n(s)), \phi)\, d\mathcal{W}_s, \qquad \forall \phi \in W_n.\end{aligned} \tag{3.30}$$

In order to simplify the notation, given $U \in \mathcal{A}$, we denote by $Y_{n,U}$ the solution of this equation in W_n, namely $Y_{n,U} := Y_{n,\Psi}$, for $\Psi = \upsilon(\Pi_n \widehat{U})$.

4 Convergence results

In order to establish the convergence results, we will need a lemma, which is an adaptation of the Lemma 4.1 in [12]. For the convenience of the reader, we present the proof.

Lemma 4.1. *Let* $(U_n)_{n \in \mathbb{N}}$ *be a sequence in* $\mathcal{A}$. *Then there exists a subsequence* $(U_{n_k})_{k \in \mathbb{N}}$ *of* $(U_n)_{n \in \mathbb{N}}$ *and a mapping* $U \in \mathcal{A}$ *such that for all* $t \in [0, T]$, $x, y \in V$, *we have*

$$\lim_{k \to \infty} (U_{n_k}(t, x), y)_{H^1} = (U(t, x), y)_{H^1}, \tag{4.31}$$

$$\lim_{k \to \infty} \|U_{n_k}(t, x) - U(t, x)\|_2 = 0. \tag{4.32}$$

Proof. Let $\{t_1, t_2, \dots\}$ be a dense subset of $[0,T]$ and $\{h_1, h_2, \dots\}$ be a countable dense subset of V. The sequence $(U_n(t_1,h_1))_{n\in\mathbb{N}}$ is bounded in $H^1(\mathcal{O})$ due to (2.8), then there exists a subsequence $(n_k^{1,1})$ of (n) and $z^{1,1} \in H^1(\mathcal{O})$ such that $U_{n_k^{1,1}}(t_1,h_1) \rightharpoonup z^{1,1}$ in $H^1(\mathcal{O})$.

In particular $(U_{n_k^{1,1}}(t_1,h_1))_{k\in\mathbb{N}}$ is bounded in $H^1(\mathcal{O})$, and by the compact Sobolev's embedding Theorem, there exists a further subsequence $(n_{k'}^{1,1})$ of $(n_k^{1,1})$ and an element $w^{1,1} \in L^2(\mathcal{O})$ such that $U_{n_{k'}^{1,1}}(t_1,h_1) \to w^{1,1}$ strongly in $L^2(\mathcal{O})$. We can show that $z^{1,1} = w^{1,1}$.

Proceeding in the same way for $(U^{1,1}_{n_{k'}}(t_1,h_2))_{n\in\mathbb{N}}$, we find a further subsequence $(n_{k'}^{1,2})$ of $(n_{k'}^{1,1})$ and $z^{1,2} \in H^1(\mathcal{O})$ such that

$$\lim_{k'\to\infty} (U_{n_{k'}^{1,2}}(t_1,h_2), y)_{H^1} = (z^{1,2}, y)_{H^1}, \quad \forall y \in H^1(\mathcal{O}),$$

and

$$\lim_{k'\to\infty} \|U_{n_{k'}^{1,2}}(t_1,h_2) - z^{1,2}\|_2 = 0.$$

We continue this process for $h_3, h_4, \dots$

Considering the diagonal subsequence $(n_{k'}^{1,k'})$, we can verify that

$$\lim_{k'\to\infty} (U_{n_{k'}^{1,k'}}(t_1,h_i), y)_{H^1} = (z^{1,i}, y)_{H^1}, \quad \forall y \in H^1(\mathcal{O}), \quad i = 1, 2, \dots$$

and

$$\lim_{k'\to\infty} \|U_{n_{k'}^{1,k'}}(t_1,h_i) - z^{1,i}\|_2 = 0, \quad i = 1, 2, \dots.$$

Now, we repeat this whole procedure for $(U_{n_{k'}^{1,k'}}(t_2,h_i))_{k'\in\mathbb{N}}$, $i = 1, 2, \dots$, creating further and further subsequences $(n_{k'}^{2,i})$, which by taking the diagonal subsequence $(n_{k'}^{2,k'})$, the above properties hold for t_2 and t_1.

Repeat this whole process for $t_3, t_4, \dots$. We consider the subsequence $(n_{k'}^{k',k'})$, denoted by (n'), which has the properties

$$\lim_{n'\to\infty} (U_{n'}(t_i,h_j), y)_{H^1} = (z^{i,j}, y)_{H^1}, \quad \forall y \in H^1(\mathcal{O}), \quad i, j = 1, 2, \dots$$

and

$$\lim_{n'\to\infty} \|U_{n'}(t_i,h_j) - z^{i,j}\|_2 = 0, \quad i, j = 1, 2, \dots$$

For arbitrary $t \in [0,T]$, $h \in V$, there exist sequences $(t_{i_k})_{k\in\mathbb{N}}$ and $(h_{j_l})_{l\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} t_{i_k} = t \quad (\text{in } \mathbb{R}) \quad \text{and} \quad \lim_{l\to\infty} h_{j_l} = h \quad (\text{in } V).$$

We note that

$$\|z^{i_k,j_k} - z^{i_l,j_l}\|_{H^1}^2 \le \liminf_{n'\to\infty} \|U_{n'}(t_{i_k},h_{j_k}) - U_{n'}(t_{i_l},h_{j_l})\|_{H^1}^2 \le \beta|t_{i_k} - t_{j_l}|^2 + \mu\|h_{i_k} - h_{j_l}\|_V.$$

Thus the sequence $(z^{i_k,j_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. We define

$$U(t,h) := \lim_{k\to\infty} z^{i_k,j_k}$$

(this limit is well defined, since for other sequences $(t_{m_k})_{k\in\mathbb{N}}$, $(h_{p_k})_{k\in\mathbb{N}}$ converging to t and h, z^{m_k,p_k} will converge to the same limit).

Let $t \in [0,T]$, $x, y \in V$. Take sequences $(t_{i_k})_{k\in\mathbb{N}}$ and $(h_{j_l})_{l\in\mathbb{N}}$ converging to t and x as above. Thus for any $\epsilon > 0$ we can take an appropriate $k \in \mathbb{N}$ and $N_k \in \mathbb{N}$ such that for $n' > N_k$, we have

$$|(U(t,x) - U_{n'}(t,x), y)_{H^1}| \le |(U(t,x) - U(t_{i_k}, h_{j_k}), y)_{H^1}| + |(U(t_{i_k}, h_{j_k}) - U_{n'}(t_{i_k}, h_{j_k}), y)_{H^1}|$$
$$+ |(U_{n'}(t_{i_k}, h_{j_k}) - U_{n'}(t,x), y)_{H^1}| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|y\|(\sqrt{\beta}|t_{i_k} - t| + \sqrt{\mu}\|h_{j_k} - x\|_V) \le \epsilon$$

and

$$\|U(t,x) - U_{n'}(t,x)\|_2 \le \|U(t,x) - U(t_{i_k}, h_{j_k})\|_2 + \|U(t_{i_k}, h_{j_k}) - U_{n'}(t_{i_k}, h_{j_k})\|_2$$
$$+ \|U_{n'}(t_{i_k}, h_{j_k}) - U_{n'}(t,x)\|_2 \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + C(\sqrt{\beta}|t_{i_k} - t| + \sqrt{\mu}\|h_{j_k} - x\|_V) \le \epsilon.$$

Therefore, we have

$$\lim_{n'\to\infty}(U_{n'}(t,x), y)_{H^1} = (U(t,x), y)_{H^1} \quad \text{and} \quad \lim_{n'\to\infty}\|U_{n'}(t,x) - U(t,x)\|_2 = 0. \tag{4.33}$$

In particular, using (4.33) for any $s, t \in [0,T]$ and $x, y \in V$, we have

$$\|U(t,x) - U(s,y)\|_{H^1}^2 \le \liminf_{n'\to\infty}\|U_{n'}(t,x) - U_{n'}(s,y)\|_{H^1}^2 \le \beta|t-s|^2 + \mu\|x-y\|_V^2,$$
$$\|U(0,0)\|_{H^1}^2 \le \liminf_{n'\to\infty}\|U_{n'}(0,0)\|_{H^1}^2 \le \eta,$$

then $U \in \mathcal{A}$. □

In order to solve the control problem $(\mathcal{P})$, we take a minimizing sequence $(U_n)_{n\in\mathbb{N}}$, and apply Lemma 4.1 to find a subsequence, still denoted by $(U_n)_{n\in\mathbb{N}}$ such that the convergences in Lemma 4.1 hold, for a certain $U \in \mathcal{A}$. In the next Proposition we will consider this specific subsequence $(U_n)_{n\in\mathbb{N}}$ and function U.

Now, we fix an initial condition $Y_0 \in L^p(\Omega, V) \cap L^2(\Omega, \widetilde{W})$, and given $\Phi \in \mathcal{A}$, we denote by Y_Φ the solution of (2.1) for the feedback control Φ and initial condition Y_0, and by $Y_{n,\Phi}$ the solution of the corresponding finite dimensional problem (3.30) with initial condition $\Pi_n Y_0$.

We have the following result:

Proposition 4.2. *Consider $(U_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $U \in \mathcal{A}$ such that*

$$\lim_{n\to\infty}\|U_n(t,x) - U(t,x)\|_2 = 0, \quad \forall t \in [0,T],\ x \in V. \tag{4.34}$$

Then, we have

$$\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}\|Y_U(s) - Y_{U_n}(s)\|_V^2 = 0, \tag{4.35}$$

$$\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}\|Y_U(s) - Y_{n,U_n}(s)\|_V^2 = 0. \tag{4.36}$$

Proof. We split the proof into two steps.

Step 1. Verification of (4.35). Using the equations for Y_U, Y_{U_n}, and the Itô formula, we have

$$\begin{aligned}
(Y_U(t)-Y_{U_n}(t),e_i)_V^2 =& 2\int_0^t (Y_U(s)-Y_{U_n}(s),e_i)_V(\nu\Delta Y_U(s)-\nu\Delta Y_{U_n},e_i)\,ds\\
&-2\int_0^t (Y_U(s)-Y_{U_n}(s),e_i)_V(\operatorname{curl} v(Y_U(s))\times Y_U(s),e_i)\,ds\\
&+2\int_0^t (Y_U(s)-Y_{U_n}(s),e_i)_V(\operatorname{curl} v(Y_{U_n}(s))\times Y_{U_n}(s),e_i)\,ds\\
&+2\int_0^t (Y_U(s)-Y_{U_n}(s),e_i)_V(U(s,Y_U(s))-U_n(s,Y_{U_n}(s)),e_i)\,ds\\
&+2\int_0^t (Y_U(s)-Y_{U_n}(s),e_i)_V(G(s,Y_U(s))-G(s,Y_{U_n}(s)),e_i)\,d\mathcal{W}_s\\
&+\int_0^t |(G(s,Y_U(s))-G(s,Y_{U_n}(s)),e_i)|^2\,ds,
\end{aligned}$$

so taking the sum over $i=1,2,\dots$, we get

$$\begin{aligned}
\|Y_U(t)-Y_{U_n}(t)\|_V^2 =& 2\nu\int_0^t (\Delta Y_U(s)-\Delta Y_{U_n},Y_U(s)-Y_{U_n}(s))\,ds\\
&-2\int_0^t (\operatorname{curl} v(Y_U(s))\times Y_U(s),Y_U(s)-Y_{U_n}(s))\,ds\\
&+2\int_0^t (\operatorname{curl} v(Y_{U_n}(s))\times Y_{U_n}(s),Y_U(s)-Y_{U_n}(s))\,ds\\
&+2\int_0^t (U(s,Y_U(s))-U_n(s,Y_{U_n}(s)),Y_U(s)-Y_{U_n}(s))\,ds\\
&+2\int_0^t (G(s,Y_U(s))-G(s,Y_{U_n}(s)),Y_U(s)-Y_{U_n}(s))\,d\mathcal{W}_s\\
&+\sum_{i=1}^{\infty}\int_0^t |(G(s,Y_U(s))-G(s,Y_{U_n}(s)),e_i)|^2\,ds. \qquad (4.37)
\end{aligned}$$

For fixed $M>0$, we define the stopping time

$$\mathcal{T}_M=\begin{cases} T, & \text{if } \sup_{s\in[0,T]}\|Y_U(s)\|_{\widetilde{W}}^2<M,\\ \inf\Big\{t\in[0,T]\ :\ \|Y_U(t)\|_{\widetilde{W}}^2\ge M\Big\}, & \text{otherwise.}\end{cases}$$

We'll take the absolute value, the supremum over $s\in[0,t\wedge\mathcal{T}_M]$ and the expectation in equation (4.37), and try to obtain estimates to apply a Gronwall type inequality.

For the first term, integrating by parts first, we have

$$\begin{aligned}
&4\nu\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(D(Y_U(s)-Y_{U_n}(s),D(Y_U(s)-Y_{U_n}(s))|\,ds\\
\le& 4\nu\,C\,\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\mathcal{T}_M]}\|Y_U(r)-Y_{U_n}(r)\|_V^2\,ds.
\end{aligned}$$

The sum of the second and third terms is equal to

$$
\begin{aligned}
&-2\int_0^t (\operatorname{curl} v(Y_U(s)-Y_{U_n}(s))\times Y_U(s), Y_U(s)-Y_{U_n}(s))\,ds\\
&+2\int_0^t (\operatorname{curl} v(Y_{U_n}(s))\times (Y_{U_n}(s)-Y_U(s)), Y_U(s)-Y_{U_n}(s))\,ds.
\end{aligned}
$$

Due to (3.23), the last integral in this expression vanishes. Taking the absolute value, the supremum, the expectation and using (3.25), we deduce

$$
\begin{aligned}
&2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(\operatorname{curl} v(Y_U(s)-Y_{U_n}(s))\times Y_U(s), Y_U(s)-Y_{U_n}(s))|\,ds\\
&\leq 2C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|Y_U(s)\|_{\widetilde{W}}\|Y_U(s)-Y_{U_n}(s)\|_V^2\,ds \leq\ 2CM\,\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\mathcal{T}_M]}\|Y_U(r)-Y_{U_n}(r)\|_V^2\,ds.
\end{aligned}
$$

Let's now look at the fourth term. Using the Lipschitz conditions (2.8), we have

$$
\begin{aligned}
&2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(U(s,Y_U(s))-U_n(s,Y_{U_n}(s)), Y_U(s)-Y_{U_n}(s))|\,ds\\
&\leq 2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|U(s,Y_U(s))-U_n(s,Y_{U_n}(s))\|_2\|Y_U(s)-Y_{U_n}(s)\|_2\,ds\\
&\leq 4\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\,ds + 4\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|U_n(s,Y_U(s))-U_n(s,Y_{U_n}(s))\|_2^2\,ds\\
&\qquad + 2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|Y_U(s)-Y_{U_n}(s)\|_2^2\,ds\\
&\leq 4\,\mathbb{E}\int_0^T \|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\,ds + C\,\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\mathcal{T}_M]}\|Y_U(r)-Y_{U_n}(r)\|_V^2\,ds.
\end{aligned}
$$

For the fifth term, using the Burkholder-Davis-Gundy Inequality, we have

$$
\begin{aligned}
&2\,\mathbb{E}\sup_{r\in[0,t\wedge\mathcal{T}_M]}\Big|\int_0^r (G(s,Y_U(s))-G(s,Y_{U_n}(s)), Y_U(s)-Y_{U_n}(s))\,d\mathcal{W}_s\Big|\\
&\leq C\,\mathbb{E}\Big(\int_0^{t\wedge\mathcal{T}_M} |(G(s,Y_U(s))-G(s,Y_{U_n}(s)), Y_U(s)-Y_{U_n}(s))|^2\,ds\Big)^{\frac{1}{2}}\\
&\leq C\,\mathbb{E}\Big(\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Y_U(s)-Y_{U_n}(s)\|_V^2\int_0^{t\wedge\mathcal{T}_m} \|G(s,Y_U(s))-G(s,Y_{U_n}(s))\|_{H^1}^2\,ds\Big)^{\frac{1}{2}}\\
&\leq C\epsilon\,\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Y_U(s)-Y_{U_n}(s)\|_V^2 + \frac{CK}{\epsilon}\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|Y_U(s)-Y_{U_n}(s)\|_V^2\,ds.
\end{aligned}
\tag{4.38}
$$

Here, we take $\epsilon>0$ such that $C\epsilon=\frac{1}{2}$.

Let's look now at the final term. Taking the solution $\tilde{G}(s)$ to the stokes problem for $G(s,Y_U(s))-G(s,Y_{U_n}(s))$, we write

$$
\mathbb{E}\sup_{r\in[0,t\wedge\mathcal{T}_M]}\Big|\sum_{i=1}^{\infty}\int_0^r |(G(s,Y_U(s))-G(s,Y_{U_n}(s)), e_i)|^2\,ds\Big| = \mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|\tilde{G}(s)\|_V^2\,ds.
$$

Using (2.5) and (2.7), we derive

$$\|\tilde{G}(s)\|_V^2 \le C\|G(s, Y_U(s)) - G(s, Y_{U_n}(s))\|_{H^1}^2 \le CK\|Y_U(s) - Y_{U_n}(s)\|_V^2.$$

Therefore

$$\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\tilde{G}(s)\|_V^2\, ds \le C\,\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\mathcal{T}_M]} \|Y_U(r) - Y_{U_n}(r)\|_V^2\, ds.$$

Now, we use all the previous inequalities to obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Y_U(s) - Y_{U_n}(s)\|_V^2 &\le C\,\mathbb{E}\int_0^T \|U(s, Y_U(s)) - U_n(s, Y_U(s))\|_2^2\, ds \\ &\quad + C\,\mathbb{E}\int_0^t \sup_{r\in[0,s\wedge\mathcal{T}_M]} \|Y_U(r) - Y_{U_n}(r)\|_V^2\, ds.\end{aligned}$$

Using a Gronwall type inequality, we deduce

$$\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Y_U(s) - Y_{U_n}(s)\|_V^2 \le C\,\mathbb{E}\int_0^T \|U(s, Y_U(s)) - U_n(s, Y_U(s))\|_2^2\, ds. \tag{4.39}$$

Due to (2.8), we have $\|U(s, Y_U(s)) - U_n(s, Y_U(s))\|_2^2 \le C(1 + \sup_{s\in[0,T]} \|Y_U(s)\|_V^2)$.

Since the right hand side is integrable, we can apply the Lebesgue's Theorem and Lemma 4.1 to obtain

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \|U(s, Y_U(s)) - U_n(s, Y_U(s))\|_2^2\, ds = 0.$$

Using (4.39), we conclude that

$$\lim_{n\to\infty} \mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Y_U(s) - Y_{U_n}(s)\|_V^2 = 0.$$

Taking $t = T$, we have

$$\lim_{n\to\infty} \mathbb{E}\sup_{s\in[0,\mathcal{T}_M]} \|Y_U(s) - Y_{U_n}(s)\|_V^2 = 0.$$

Using the estimate (3.29), we can verify that $\mathcal{T}_M \nearrow T$ in probability. Taking into account [1], Proposition B.3, in Appendix B, we introduce the stochastic process

$$Q_n(t) = \sup_{s\in[0,t]} \|Y_U(s) - Y_{U_n}(s)\|_V^2.$$

For all $M > 0$, we have $\mathbb{E}|Q_n(\mathcal{T}_M)| \to 0$, as $n \to \infty$. Given $\delta, \varepsilon > 0$, there exists $M_0 > 0$, such that $P(\mathcal{T}_{M_0} < T) \le \frac{\varepsilon}{2}$, and there exists $N \in \mathbb{N}$, such that $\forall n > N$,

$$\mathbb{E}|Q_n(\mathcal{T}_{M_0})| \le \frac{\delta\,\varepsilon}{2}.$$

Therefore

$$P\left(|Q_n(T)| \ge \delta\right) \le P(\mathcal{T}_{M_0} < T) + P\left(\mathcal{T}_{M_0} = T \ \wedge\ |Q_n(T)| \ge \delta\right) \le \frac{\varepsilon}{2} + \frac{1}{\delta}\mathbb{E}|Q_n(\mathcal{T}_{M_0})| = \varepsilon,$$

which gives $|Q_n(T)| \to 0$ in probability, as $n \to \infty$. The estimate (3.28) gives

$$\mathbb{E}\left(|Q_n(T)|^2\right) = \mathbb{E} \sup_{s\in[0,T]} \|Y_U(s) - Y_{U_n}(s)\|_V^4 \leq C,$$

for some $C > 0$ independent of n, then the sequence $(|Q_n(T)|)$ is uniformly integrable. Since the uniform integrability and convergence in probability imply convergence in $L^1(\Omega)$, we deduce

$$\lim_{n\to\infty} \mathbb{E} \sup_{s\in[0,T]} \|Y_U(s) - Y_{U_n}(s)\|_V^2 = 0.$$

Step 2. Verification of (4.36). Consider now $Y_{n,U_n}(t)$, the solution to the finite dimensional equation (3.30) with control U_n, and consider also $\Pi_n Y_U(t)$. We may use Itô's formula as before, to obtain

$$\begin{aligned}
\|\Pi_n Y_U(t) - Y_{n,U_n}(t)\|_V^2 = {} & 2\nu \int_0^t (\Delta Y_U(s) - \Delta Y_{n,U_n}(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& - 2\int_0^t (\operatorname{curl} v(Y_U(s)) \times Y_U(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& + 2\int_0^t (\operatorname{curl} v(Y_{n,U_n}(s)) \times Y_{n,U_n}(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& + 2\int_0^t (U(s, Y_U(s)) - U_n(s, Y_{n,U_n}(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& + 2\int_0^t (G(s, Y_U(s)) - G(s, Y_{n,U_n}(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, d\mathcal{W}_s \\
& + \sum_{i=1}^n \int_0^t |(G(s, Y_U(s)) - G(s, Y_{n,U_n}(s)), e_i)|^2\, ds. \qquad (4.40)
\end{aligned}$$

We'll consider the same stopping time, $\mathcal{T}_M$, for a fixed $M > 0$. As before, we'll take the absolute value, the supremum over $s \in [0, t \wedge \mathcal{T}_M]$, and the expectation in equation (4.40) and try to obtain estimates to apply a Gronwall type inequality.

For the first term, integrating by parts first, we have

$$\begin{aligned}
& 4\nu\, \mathbb{E} \int_0^{t\wedge\mathcal{T}_M} |(DY_U(s) - DY_{n,U_n}(s), D\Pi_n Y_U(s) - DY_{n,U_n}(s))|\, ds \\
& \leq C\, \mathbb{E} \int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds + C\, \mathbb{E} \int_0^T \|Y_U(s) - \Pi_n Y_U(s)\|_V^2\, ds.
\end{aligned}$$

The sum of the second and third terms can be written as

$$\begin{aligned}
& 2\int_0^t (\operatorname{curl} v(\Pi_n Y_U(s) - Y_U(s)) \times Y_U(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& + 2\int_0^t (\operatorname{curl} v(\Pi_n Y_U(s)) \times (\Pi_n Y_U(s) - Y_U(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds \\
& - 2\int_0^t (\operatorname{curl} v(\Pi_n Y_U(s) - Y_{n,U_n}(s)) \times \Pi_n Y_U(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, ds.
\end{aligned}$$

Now, we take the absolute value, the expectation and estimate. Using (3.26) and (3.25), we deduce

$$
\begin{aligned}
&2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(\operatorname{curl} v(\Pi_n Y_U(s) - Y_U(s)) \times Y_U(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))|\, ds \\
&\le 2\,C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_U(s)\|_{\widetilde{W}} \|Y_U(s)\|_V \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V\, ds \\
&\le\; CM\,\mathbb{E}\int_0^{T} \|\Pi_n Y_U(s) - Y_U(s)\|_{\widetilde{W}}^2\, ds \;+\; CM\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds,
\end{aligned}
$$

$$
\begin{aligned}
&2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(\operatorname{curl} v(\Pi_n Y_U(s)) \times (\Pi_n Y_U(s) - Y_U(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))|\, ds \\
&\le 2\,C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s)\|_{\widetilde{W}} \|\Pi_n Y_U(s) - Y_U(s)\|_V \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V\, ds \\
&\le\; CM\,\mathbb{E}\int_0^{T} \|\Pi_n Y_U(s) - Y_U(s)\|_{\widetilde{W}}^2\, ds \;+\; CM\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(\operatorname{curl} v(\Pi_n Y_U(s) - Y_{n,U_n}(s)) \times \Pi_n Y_U(s), \Pi_n Y_U(s) - Y_{n,U_n}(s))|\, ds \\
&\le C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2 \|\Pi_n Y_U(s)\|_{\widetilde{W}}\, ds \le\; CM\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds.
\end{aligned}
$$

Let's now look at the fourth term. We have

$$
\begin{aligned}
&2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} |(U(s, Y_U(s)) - U_n(s, Y_{n,U_n}(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))|\, ds \\
&\le 2\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|U(s, Y_U(s)) - U_n(s, Y_{n,U_n})\|_2 \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_2\, ds \\
&\le 4\mathbb{E}\int_0^{T} \|U(s, Y_U(s)) - U_n(s, Y_U(s))\|_2^2\, ds + C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|Y_U(s) - \Pi_n Y_U(s)\|_V^2\, ds \\
&\qquad + C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds.
\end{aligned}
$$

For the stochastic term, using the Burkholder-Davis-Gundy Inequality as in (4.38), and (2.7), we have

$$
\begin{aligned}
&2\,\mathbb{E}\sup_{r\in[0,t\wedge\mathcal{T}_M]} \left| \int_0^r (G(s, Y_U(s)) - G(s, Y_{n,U_n}(s)), \Pi_n Y_U(s) - Y_{n,U_n}(s))\, d\mathcal{W}_s \right| \\
&\le C\epsilon\,\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2 + 2\frac{CK}{\epsilon}\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|Y_U(s) - \Pi_n Y_U(s)\|_V^2\, ds \\
&\qquad + 2\frac{CK}{\epsilon}\mathbb{E}\int_0^{t\wedge\mathcal{T}_M} \|\Pi_n Y_U(s) - Y_{n,U_n}(s)\|_V^2\, ds.
\end{aligned}
$$

We take $\epsilon > 0$ such that $C\epsilon = \frac{1}{2}$.

To deal with the final term, we first take $\overline{G}(s)$ defined as the solution to the stokes problem for $G(s,Y_U(s))-G(s,Y_{n,U_n}(s))$. Then, the last term becomes

$$\mathbb{E}\sup_{r\in[0,t\wedge\mathcal{T}_M]}\Big|\sum_{i=1}^{n}\int_0^r|(G(s,Y_U(s))-G(s,Y_{n,U_n}(s)),e_i)|^2\,ds\Big|\le\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|\overline{G}(s)\|_V^2\,ds$$

By the property (2.5), we have

$$\begin{aligned}\|\overline{G}(s)\|_V^2&\le C\|G(s,Y_U(s))-G(s,Y_{n,U_n}(s))\|_{H^1}^2\le CK\|Y_U(s)-Y_{n,U_n}(s)\|_V^2\\&\le C\|Y_U(s)-\Pi_nY_U(s)\|_V^2+C\|\Pi_nY_U(s)-Y_{n,U_n}(s)\|_V^2.\end{aligned}$$

We conclude that

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|\overline{G}(s)\|_V^2\,ds\le C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|\Pi_nY_U(r)-Y_{nU_n}(r)\|_V^2\,ds\\+C\,\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|Y_U(s)-\Pi_nY_U(r)\|_V^2\,ds.\end{aligned}$$

Collecting all the previous inequalities, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|\Pi_nY_U(s)-Y_{n,U_n}(s)\|_V^2\le\ C\mathbb{E}\int_0^t\sup_{r\in[0,s\wedge\mathcal{T}_M]}\|\Pi_nY_U(r)-Y_{n,U_n}(r)\|_V^2\,ds\\&+C\mathbb{E}\int_0^T\|Y_U(s)-\Pi_nY_U(s)\|_{\widetilde{W}}^2ds+4\mathbb{E}\int_0^T\|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\,ds.\end{aligned}$$

The Gronwall type inequality yields

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|\Pi_nY_U(s)-Y_{n,U_n}(s)\|_V^2\le C\mathbb{E}\int_0^T\|Y_U(s)-\Pi_nY_U(s)\|_{\widetilde{W}}^2\\+4\mathbb{E}\int_0^T\|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\,ds\end{aligned}$$

Since

$$\begin{aligned}&\|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\le C(1+\sup_{s\in[0,T]}\|Y_U(s)\|_V^2),\\&\|Y_U(s)-\Pi_nY_U(s)\|_{\widetilde{W}}^2\le C(1+\sup_{s\in[0,T]}\|Y_U(s)\|_{\widetilde{W}}^2),\end{aligned}$$

and the right hand sides of these inequalities are integrable, we can use the Lebesgue's Theorem and Lemma 4.1 to conclude that

$$\lim_{n\to\infty}C\mathbb{E}\int_0^T\|Y_U(s)-\Pi_nY_U(s)\|_{\widetilde{W}}^2+4\mathbb{E}\int_0^T\|U(s,Y_U(s))-U_n(s,Y_U(s))\|_2^2\,ds=0,$$

and consequently

$$\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|\Pi_nY_U(s)-Y_{n,U_n}(s)\|_V^2=0.$$

Arguing as in the final part of *Step 1*, we obtain

$$\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}\|\Pi_nY_U(s)-Y_{n,U_n}(s)\|_V^2=0. \tag{4.41}$$

Now, we prove that

$$\lim_{n\to\infty} \mathbb{E} \sup_{s\in[0,T]} \|Y_U(s) - \Pi_n Y_U(s)\|_V^2 = 0.$$

The Itô formula gives

$$\begin{aligned}
&\|Y_U(t) - \Pi_n Y_U(t)\|_V^2 = 2\nu \int_0^t (\Delta Y_U(s), Y_U(s) - \Pi_n Y_U(s))\, ds \\
&- 2\int_0^t (\mathrm{curl}(Y_U(s)) \times Y_U(s), Y_U(s) - \Pi_n Y_U(s))\, ds + \|Y_0 - \Pi_n Y_0\|_V^2 \\
&+ 2\int_0^t (U(s, Y_U(s)), Y_U(s) - \Pi_n Y_U(s))\, ds + 2\int_0^t (G(s, Y_U(s)), Y_U(s) - \Pi_n Y_U(s))\, d\mathcal{W}_s \\
&+ 2\sum_{i=n+1}^{\infty} \int_0^t |(G(s, Y_U(s)), e_i)|^2\, ds.
\end{aligned}$$

Next, we take the absolute value, the supremum over $[0,T]$, the expected value, and prove that each term converges to zero.

For the first term, we have

$$4\nu\mathbb{E}\int_0^T |(DY_U(s), D(Y_U(s) - \Pi_n Y_U(s)))| ds \leq 4\nu\mathbb{E}\int_0^T \|Y_U(s)\|_V \|Y_U(s) - \Pi_n Y_U(s)\|_V\, ds,$$

which converges to zero, by the Lebesgue's Theorem.

For the second term, we have

$$\begin{aligned}
&2\mathbb{E}\int_0^T |(\mathrm{curl}(Y_U(s)) \times Y_U(s), Y_U(s) - \Pi_n Y_U(s))|\, ds \\
&\leq 2\mathbb{E}\int_0^T \|Y_U(s)\|_{\widetilde{W}} \|Y_U(s)\|_V \|Y_U(s) - \Pi_n Y_U(s)\|_V\, ds \\
&\leq 2\Big(\mathbb{E}\int_0^T \|Y_U(s)\|_{\widetilde{W}}^2\, ds\Big)^{\frac{1}{2}} \Big(\mathbb{E}\int_0^T \|Y_U(s)\|_V^4\, ds\Big)^{\frac{1}{4}} \Big(\mathbb{E}\int_0^T \|Y_U(s) - \Pi_n Y_U(s)\|_V^4\, ds\Big)^{\frac{1}{4}},
\end{aligned}$$

which converges to zero, by taking into account previous estimates on Y_U.

It is clear that $\mathbb{E}\|Y_0 - \Pi_n Y_0\|_V^2$ converges to zero. Let's look at the force term. We have

$$2\mathbb{E}\int_0^T |(U(s, Y_U(s)), Y_U(s) - \Pi_n Y_U(s))|\, ds \leq 2C\mathbb{E}\int_0^T \|U(s, Y_U(s))\|_{H^1} \|Y_U(s) - \Pi_n Y_U(s)\|_V,$$

which converges to zero, because $\|U(s, Y_U(s))\|_{H^1} \|Y_U(s) - \Pi_n Y_U(s)\|_V \leq C(1 + \|Y_U(s)\|_V^2)$, and we can apply the Lebesgue's Theorem.

For the stochastic term, the Burkholder-Davis-Gundy inequality gives

$$\begin{aligned}
&2\mathbb{E}\sup_{t\in[0,T]} \Big|\int_0^t (G(s, Y_U(s)), Y_U(s) - \Pi_n Y_U(s))\, d\mathcal{W}_s\Big| \\
&\leq C\Big(\mathbb{E}\int_0^T \|G(s, Y_U(s))\|_{H^1}^2 \|Y_U(s) - \Pi_n Y_U(s)\|_V^2\, ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $\|G(s, Y_U(s))\|_{H^1}^2 \|Y_U(s) - \Pi_n Y_U(s)\|^2 \leq C(1 + \|Y_U(s)\|_V^2)\|Y_U(s)\|_V^2$, using the Lebesgue's Theorem, we show that this term converges to zero.

For the last term, we consider the solution $\tilde{G}$ of the stokes problem for $G(s, Y_U(s))$, and notice that

$$2\,\mathbb{E}\sup_{t\in[0,T]}\sum_{i=n+1}^{\infty}\int_0^t |(G(s,Y_U(s)),e_i)|^2\,ds = 2\mathbb{E}\int_0^T \|\tilde{G}-\Pi_n\tilde{G}\|_V^2\,ds. \tag{4.42}$$

Using (2.5), we know that $\|\tilde{G}\|_V^2 \le C\|G(s,Y_U(s))\|_{H^1}^2 \le C(1+\|Y_U(s)\|_V^2)$. Again, applying the Lebesgue's Theorem, we can show that the right hand side in (4.42) converges to zero.

We have proved that

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T]}\|Y_U(t)-\Pi_n Y_U(t)\|_V^2 = 0. \tag{4.43}$$

Finally, (4.41) and (4.43) yield (4.36). □

Corollary 4.3. *Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence such that $\Psi_n \in \mathcal{A}_n$, $\forall n\in\mathbb{N}$, and consider Y_{Ψ_n}, Y_{n,Ψ_n} the solutions of equations* (2.1) *and* (2.16)*, respectively, corresponding to the feedback control Ψ_n. Then, there exists a subsequence $(\Psi_{n_k})_{k\in\mathbb{N}}$ such that the following convergence holds*

$$\lim_{k\to\infty}\mathbb{E}\sup_{s\in[0,T]}\|Y_{\Psi_{n_k}}(s)-Y_{n_k,\Psi_{n_k}}(s)\|_V^2 ds = 0. \tag{4.44}$$

Proof. According to Lemma 4.1, there exists a subsequence $(\Psi_{n_k})_{k\in\mathbb{N}}$ of (Ψ_n), and $\Psi\in\mathcal{A}$ such that

$$\begin{aligned}
&\Psi_{n_k}(t,x) \rightharpoonup \Psi(t,x) \quad \text{weakly in } H^1(\mathcal{O}),\\
&\Psi_{n_k}(t,x) \to \Psi(t,x) \quad \text{strongly in } H.
\end{aligned}$$

Applying the result of Proposition 4.2 to $(\Psi_{n_k})_{k\in\mathbb{N}}$ and Ψ, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\|Y_{\Psi_{n_k}}(s)-Y_{n_k,\Psi_{n_k}}(s)\|_V^2\\
&\le \mathbb{E}\sup_{s\in[0,T]}\|Y_{\Psi_{n_k}}(s)-Y_{\Psi}(s)\|_V^2 + \mathbb{E}\sup_{s\in[0,T]}\|Y_{\Psi}(s)-Y_{n_k,\Psi_{n_k}}(s)\|_V^2 \to 0, \quad \text{as} \quad k\to\infty.
\end{aligned}$$

□

5 Proof of Theorems 2.2 and 2.3

Proof of Theorem 2.2. Let us consider a minimizing sequence (U_n) in $\mathcal{A}$. Then we have

$$\inf_{U\in\mathcal{A}} J(U) = \lim_{n\to\infty} J(U_n). \tag{5.45}$$

Due to Lemma 4.1, there exist a subsequence (U_{n_k}) of (U_n) and an admissible control U^*, such that for all $t\in[0,T]$ and $x,y\in V$, we have

$$\lim_{k\to\infty}(U_{n_k}(t,x),y)_{H^1} = (U^*(t,x),y)_{H^1}, \tag{5.46}$$

and

$$\lim_{k\to\infty}\|U_{n_k}(t,x)-U^*(t,x)\|_2 = 0. \tag{5.47}$$

According to (4.35), there exists a subsequence of $(Y_{U_{n_k}})$ still denoted by $(Y_{U_{n_k}})$ such that for all $t \in [0,T]$ and a.e.$-P$, we have

$$Y_{U_{n_k}} \to Y_{U^*} \quad \text{strongly in } V. \tag{5.48}$$

The relations (5.46), (5.48), give that for all $t \in [0,T]$ and a.e.$-P$

$$U_{n_k}(t, Y_{U_{n_k}}) \rightharpoonup U^*(t, Y_{U^*}) \quad \text{weakly in } H^1(\mathcal{O}). \tag{5.49}$$

Taking into account the lower semicontinuity assumptions (2.11)-(2.12) and the Fatou's Lemma, we deduce

$$\begin{aligned}\mathbb{E}\int_0^T L(t, U^*(t, Y_{U^*}), Y_{U^*})dt &\le \mathbb{E}\int_0^T \lim_{k\to\infty}\inf L(t, U_{n_k}(t, Y_{U_{n_k}}), Y_{U_{n_k}})dt \\ &\le \lim_{k\to\infty}\inf \mathbb{E}\int_0^T L(t, U_{n_k}(t, Y_{U_{n_k}}), Y_{U_{n_k}})dt\end{aligned}$$

and

$$\mathbb{E}h(Y_{U^*(T)}) \le \mathbb{E}\lim_{k\to\infty}\inf h(Y_{U_{n_k}}(T)) \le \lim_{k\to\infty}\inf \mathbb{E}h(Y_{U_{n_k}}(T)),$$

which imply $J(U^*) = \inf_{U\in\mathcal{A}} J(U)$.

Proof of Theorem 2.3. Let us initiate the proof by establishing a useful technical lemma.

Lemma 5.1. *Assume $\Phi \in \mathcal{A}$, $\varphi_n \in \mathcal{A}_n$, and let Y_Φ, Y_{n,φ_n} be the solutions of the equations* (2.1), (2.16) *associated with the feedback controls Φ, φ_n, and initial conditions Y_0, $\Pi_n Y_0$, respectively. Then there exists a positive constant M independent on n such that*

$$\begin{aligned}|J_n(\varphi_n) - J(\Phi)| \le & M\Bigg\{\left(\mathbb{E}\sup_{t\in[0,T]}\|Y_{n,\varphi_n}(t) - Y_\Phi(t)\|_V^2\right)^{\frac12} \\ & + \left(\mathbb{E}\int_0^T \|\varphi_n(t, Y_\Phi(t)) - \Phi(t, Y_\Phi(t))\|_{H^1}^2\right)^{\frac12}\Bigg\}.\end{aligned} \tag{5.50}$$

Proof. Using (2.8), (2.18)-(2.19), (3.27)-(3.29) and the Hölder inequality, we deduce

$$\begin{aligned}&|J_n(\varphi_n) - J(\Phi)| \le \mathbb{E}\int_0^T |L(t, \varphi_n(t, Y_{n,\varphi_n}), Y_{n,\varphi_n}) - L(t, \Phi(t, Y_\Phi), Y_\Phi)|\,dt \\ &+ \mathbb{E}\,|h(Y_{n,\varphi_n}(T)) - h(Y_\Phi(T))| \\ &\le \lambda_L\sqrt{\mu T}\left(\mathbb{E}\sup_{t\in[0,T]}\|Y_{n,\varphi_n} - Y_\Phi\|_V^2\right)^{\frac12}\left(\mathbb{E}\int_0^T \left(\|\varphi_n(t, Y_{n,\varphi_n})\|_{H^1} + \|\Phi(t, Y_\Phi)\|_{H^1}\right)^2\right)^{\frac12} \\ &+ \lambda_L\left(\mathbb{E}\int_0^T \|\varphi_n(t, Y_\Phi) - \Phi(t, Y_\Phi)\|_{H^1}^2\right)^{\frac12}\left(\mathbb{E}\int_0^T \left(\|\varphi_n(t, Y_{n,\varphi_n})\|_{H^1} + \|\Phi(t, Y_\Phi)\|_{H^1}\right)^2\right)^{\frac12} \\ &+ \sqrt{T}\lambda_L\left(\mathbb{E}\sup_{t\in[0,T]}\|Y_{n,\varphi_n} - Y_\Phi\|_V^2\right)^{\frac12}\left(\mathbb{E}\int_0^T \left(\|Y_{n,\varphi_n}\|_{\widetilde{W}} + \|Y_\Phi\|_{\widetilde{W}}\right)^2\right)^{\frac12} \\ &+ \lambda_h\left(\mathbb{E}\|Y_{n,\varphi_n}(T) - Y_\Phi(T)\|_V^2\right)^{\frac12}\left(\mathbb{E}\left(\|Y_{n,\varphi_n}(T)\|_{\widetilde{W}} + \|Y_\Phi(T)\|_{\widetilde{W}}\right)^2\right)^{\frac12} \\ &\le M\Bigg\{\left(\mathbb{E}\sup_{t\in[0,T]}\|Y_{n,\varphi_n} - Y_\Phi\|_V^2\right)^{\frac12} + \left(\mathbb{E}\int_0^T \|\varphi_n(t, Y_\Phi) - \Phi(t, Y_\Phi)\|_{H^1}^2\right)^{\frac12}\Bigg\},\end{aligned}$$

where M is a constant independent of n. □

Let us take $\epsilon > 0$. For each $n \in \mathbb{N}$, let Ψ_n^* be the solution of the $n-$dimensional control problem. By the Corollary 4.3, there exists a subsequence $(\Psi_{n_k}^*)$ of (Ψ_n^*) such that

$$\lim_{k\to\infty} \mathbb{E} \sup_{s\in[0,T]} \|Y_{\Psi_{n_k}^*}(s) - Y_{n_k,\Psi_{n_k}^*}(s)\|_V^2 ds = 0.$$

We will show that the sequence $(\Psi_{n_k}^*)$ satisfy the claim of the theorem. There exists $N_\epsilon^1 \in \mathbb{N}$, such that $\forall k > N_\epsilon^1$

$$\mathbb{E} \sup_{s\in[0,T]} \|Y_{\Psi_{n_k}^*}(s) - Y_{n_k,\Psi_{n_k}^*}(s)\|_V^2 < \frac{\epsilon^2}{4M^2}.$$

Let us fix $k \in \mathbb{N}$. If we assume that $J_{n_k}(\Psi_{n_k}^*) \leq J(U^*)$, then due to (5.50), we have

$$\begin{aligned}|J_{n_k}(\Psi_{n_k}^*) - J(U^*)| &= J(U^*) - J_{n_k}(\Psi_{n_k}^*) \leq J(\Psi_{n_k}^*) - J_{n_k}(\Psi_{n_k}^*)\\ &\leq M\left(\mathbb{E} \sup_{t\in[0,T]} \|Y_{n_k,\Psi_{n_k}^*}(t) - Y_{\Psi_{n_k}^*}(t)\|_V^2\right)^{\frac{1}{2}} \leq \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

In the case $J_{n_k}(\Psi_{n_k}^*) \geq J(U^*)$, we define $U_{n_k} = \Pi_{n_k} U^*$ and notice that $U_{n_k}(t,x) \to U^*(t,x)$ strongly in $H^1(\mathcal{O})$, $\forall t \in [0,T],\ x \in V$. Using the Lebesgue's Theorem, we can verify that

$$\mathbb{E} \int_0^T \|U_{n_k}(t, Y_{U^*}) - U^*(t, Y_{U^*})\|_{H^1}^2 \to 0, \quad \text{as} \quad k \to \infty.$$

Using the same reasoning as in the proof of (4.36), in the Proposition 4.2, we can show that

$$\mathbb{E} \sup_{t\in[0,T]} \|Y_{n_k,U_{n_k}} - Y_{U^*}\|_V^2 \to 0, \quad \text{as} \quad k \to \infty.$$

Then there exists $N_\epsilon^2 \in \mathbb{N}$, such that $\forall k > N_\epsilon^2$

$$\mathbb{E} \int_0^T \|U_{n_k}(t, Y_{U^*}) - U^*(t, Y_{U^*})\|_{H^1}^2 < \frac{\epsilon^2}{16M^2} \quad \text{and} \quad \mathbb{E} \sup_{t\in[0,T]} \|Y_{n_k,U_{n_k}} - Y_{U^*}\|_V^2 < \frac{\epsilon^2}{16M^2}.$$

Therefore, using (5.50), we deduce

$$\begin{aligned}&|J_{n_k}(\Psi_{n_k}^*) - J(U^*)| = J_{n_k}(\Psi_{n_k}^*) - J(U^*) \leq J_{n_k}(U_{n_k}) - J(U^*)\\ &\leq M\left\{\left(\mathbb{E} \sup_{t\in[0,T]} \|Y_{n_k,U_{n_k}} - Y_{U^*}\|_V^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\int_0^T \|U_{n_k}(t, Y_{U^*}) - U^*(t, Y_{U^*})\|_{H^1}^2\right)^{\frac{1}{2}}\right\} \leq \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Taking $N_\epsilon = \max\{N_\epsilon^1, N_\epsilon^2\}$, the assertion of the theorem holds.

Finally, to show (2.21), we just notice that

$$J\left(\Psi_{n_k}^*\right) - J\left(U^*\right) \leq |J\left(\Psi_{n_k}^*\right) - J_{n_k}\left(\Psi_{n_k}^*\right)| + |J_{n_k}\left(\Psi_{n_k}^*\right) - J\left(U^*\right)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

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