Growths of endomorphisms of finitely generated semigroups

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ABSTRACT

This paper studies the growths of endomorphisms of finitely generated semigroups. This is a certain dynamical characteristic describing how iterations of the endomorphism 'stretch' balls in the Cayley graph of the semigroup. We make a detailed study of the relation of the growth of an endomorphism of a finitely generated semigroup and the growth of the restrictions of the endomorphism to finitely generated invariant subsemigroups. We also study the possible values endomorphism growths can attain. We show the role of linear algebra in calculating the growths of endomorphisms of homogeneous semigroups. Proofs are a mixture of syntactic algebraic rewriting techniques and analytical tricks. We state various problems and suggestions for future research.

Keywords: growth; finitely generated semigroups; endomorphism; Cayley graph.

1 INTRODUCTION

The important connections between the geometry of Cayley graphs of groups and their intrinsic algebraic properties are well-known, the best examples perhaps being Gromov's original proof that word-hyperbolic groups have linear Dehn function [Gro87], and Muller & Schupp's proof that groups with context-free word problem are precisely the finitely generated virtually free groups [MS83], which relies heavily on the notion of ends of Cayley graphs.

When one generalizes from groups to semigroups, there is *some* geometry on Cayley graphs: for instance, there are several possible definitions of hyperbolicity for semigroups [Cai13, CM12, DG04]; one can also define ends of finitely generated semigroups and prove results about them similar to those about ends of groups [JK09, KMC15]. For semigroups of finite geometric type, the Cayley graphs behave quite nicely (see, for example, [CM, § 11] or [SS04], and there have even been attempts to generalize to semigroups such crucial results as Švarc–Milnor Lemma and its consequences [GK11, GK13]. However, all these results, though natural and beautiful, are proved by methods that indicate that semigroups are not very geometric objects.

In this paper we take a different approach: to study not the geometry of semigroups themselves, but certain a geometric feature of their endomorphisms, namely *growth*. Informally, growth characterizes the extent to which balls in the Cayley graph of a finitely generated semigroup are 'stretched' by iterations of the endomorphism. (See Section 2 for the formal definition.) There has been some study of the growths of endomorphisms of finitely generated groups, but the literature seems to be limited to the seminal paper of Bowen [Bow78], some studies of growths of endomorphisms of free groups [BFH00, DV93, LL00], and some general results proved in [FFK11]. To read about other dynamical characteristics of endomorphisms, we refer the reader to [MS06] and references therein, and to the recent preprint [DGB13]. In the broader setting of semigroups, endomorphisms can be much more 'exotic' and unexpected results often arise (see, for instance, [MR12]).

Let us outline the structure of the paper. Section 2 contains the necessary definitions and facts we will use throughout. Section 3 shows that every real number $r \geqslant 1$ arises as the growth of an endomorphism of some finitely generated semigroup. Section 4 shows how the growth of an endomorphism of a finitely generated semigroup is connected to the growth of the restriction of this endomorphism to various types of invariant finitely generated subsemigroups. Section 5 studies the interaction of growths and two fundamental semigroup constructions, namely direct products and free products. Finally, Section 6 examines growths of endomorphisms of semigroups of special classes, namely homogeneous, group-embeddable, and free inverse semigroups.

2 DEFINITIONS

2.1 *Growth*

Our definitions basically follow those for group theory [FFK11, MS06], but we use slightly more precise notation.

Let S be a finitely generated semigroup and let A be a finite generating set for S. For any $w \in S$, the *length* of w over A is the length of the shortest product of elements of A that equals w; the length of w over A is denoted by $|w|_A$ or simply by $|w|$. Denote by $B_{n,A}$ the standard ball of radius n in the Cayley graph of S with respect to A; that is, $B_{n,A} = \{ w \in S : |w|_A \leqslant n \}$.

Let $\varphi : S \to S$ be an endomorphism of S. For convenience here and throughout the paper, define, for any subset X of S,

$$K(\varphi, X, A) = \max_{x \in X} |x\varphi|_A. \tag{2.1}$$

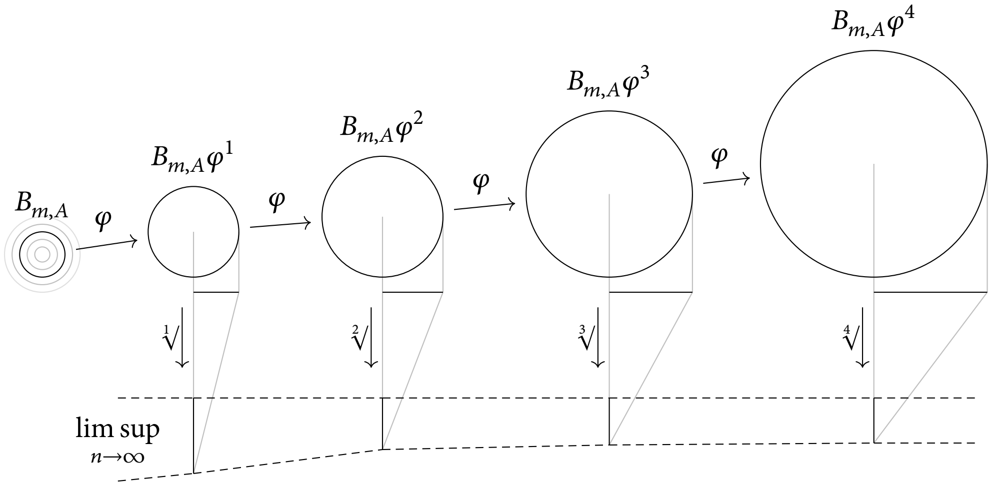


FIGURE 1
The definition of $\Gamma(\varphi)$: each iteration of φ has a 'multiplicative' effect on the size of the ball $B_{m,A}$ (not in terms of the number of elements, but only on their lengths). Taking n-th roots 'scales' the size of $B_{m,A}\varphi^n$ to a size comparable to $B_{m,A}\mathrm{id}^n = B_{m,A}$, and then taking lim sup gives the asymptotic effect of iterations of φ on the size of $B_{m,A}$. Finally, we take the supremum over all possible balls $B_{m,A}$.

We will usually set $X = B_{m,A}$ for some $m \in \mathbb{N}$ or $X = B_{1,A} = A$. Note that $K(\varphi, X, A) \geqslant 1$ because we deal with *semigroup* generating sets. The single real number that describes how balls $B_{m,A}$ are stretched by φ is the *growth* of φ and is defined by

$$\Gamma(\varphi) = \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} \sqrt[n]{K(\varphi^n, B_{m,A}, A)}.$$

(This definition is originally due to Bowen [Bow78].) We will see (in Proposition 2.4) that the definition of $\Gamma(\varphi)$ does not depend on the choice of the generating set A; this justifies omitting it on the left-hand side of this definition. Figure 1 gives an intuitive illustration of the definition.

LEMMA 2.1. *Let A be finite generating sets for a semigroup S and let $\varphi : S \to S$ be an endomorphism. Then:*

a) *For all $m \in \mathbb{N}$, the inequality $K(\varphi, B_{m,A}, A) \leqslant mK(\varphi, A, A)$ holds.*
b) *If X and Y are subsets of S with $X \subseteq Y$, then $K(\varphi, X, A) \leqslant K(\varphi, Y, A)$.*
c) *If A' is also a finite generating set for S and $A \subseteq B_{m,A'}$ for some $m \in \mathbb{N}$, then $K(\varphi, X, A') \leqslant mK(\varphi, X, A)$.*
d) *If $\psi : S \to S$ is also an endomorphism of S, then $K(\varphi\psi, A, A) \leqslant K(\varphi, A, A)K(\psi, A, A)$.*

Proof of 2.1. a) Let $x \in B_{m,A}$. Then $x = a_1 \cdots a_\ell$ for some $\ell \leqslant m$ and $a_i \in A$. Therefore

$$|x\varphi|_A = |(a_1 \cdots a_\ell)\varphi|_A \leqslant |a_1\varphi|_A + \cdots + |a_\ell\varphi|_A \leqslant mK(\varphi, A, A),$$

where the last inequality holds since $\ell \leqslant m$ and $|a_i\varphi|_A \leqslant K(\varphi, A, A)$ for all i. Since $x \in B_{m,A}$ was arbitrary, $K(\varphi, B_{m,A}, A) \leqslant \ell K(\varphi, A, A)$.

b) By the definition, we have $K(\varphi, X, A) = \max_{x\in X}|x\varphi|_A \leqslant \max_{x\in Y}|x\varphi|_A = K(\varphi, Y, A)$.

c) Let $x \in X$. Then $|x\varphi|_A = p \leqslant K(\varphi, X, A)$. Thus $x\varphi = a_1 \cdots a_p$ for some $a_i \in A$. Since $A \subseteq B_{m,A'}$, we have $|a_i|_{A'} \leqslant m$, and so $|x\varphi|_{A'} \leqslant mp \leqslant mK(\varphi, X, A)$. Since $x \in X$ was arbitrary, it follows that $K(\varphi, X, A') \leqslant mK(\varphi, X, A)$.

d) Let $a \in A$. Then $a\varphi = a_1 \cdots a_p$ for some $a_i \in A$ and $p \leqslant K(\varphi, A, A)$. Thus

$$\begin{aligned} |a\varphi\psi|_A &= \left|(a_1 \cdots a_p)\psi\right|_A \\ &\leqslant |a_1\psi|_A + \cdots + |a_p\psi|_A \\ &\leqslant pK(\psi, A, A) \\ &\leqslant K(\varphi, A, A)K(\psi, A, A). \end{aligned}$$

Since $a \in A$ was arbitrary, we have $K(\varphi\psi, A, A) \leqslant K(\varphi, A, A)K(\psi, A, A)$.

2.1

The following proposition gives some elementary properties of growth.

PROPOSITION 2.2. *Let A be a finite generating set for a semigroup S and let $\varphi : S \to S$ and $\psi : S \to S$ be endomorphisms. Then:*

a) $\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} = \inf\{ \sqrt[n]{K(\varphi^n, A, A)} : n \in \mathbb{N} \}$;

b) $\Gamma(\varphi) \leqslant K(\varphi, A, A) = \max_{a\in A}|a\varphi|_A$.

c) $\Gamma(\varphi^k) = \Gamma(\varphi)^k$ *for all* $k \in \mathbb{N}$.

The proofs of these properties follow closely the analogous result for groups [FFK11, Theorem 2.1]. We include proofs for completeness and because certain technicalities are not emphasized in the group-theoretical proofs.

Proof of 2.2. First we must prove a technical lemma about the limits of certain kinds of sequences:

LEMMA 2.3. a) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $a_{i+j} \leqslant a_i + a_j$ for all $i, j \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n/n$ exists and equals* $\inf\{ a_n/n : n \in \mathbb{N} \}$.

b) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than or equal to 1 satisfying $a_{i+j} \leqslant a_i a_j$ for all $i, j \in \mathbb{N}$. Then $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and equals* $\inf\{ \sqrt[n]{a_n} : n \in \mathbb{N} \}$.

Proof of 2.3. a) Since all the a_n are positive, $\{ a_n/n : n \in \mathbb{N} \}$ is bounded below by 0 and so has an infimum ℓ. The aim is to prove that $a_n/n \to \ell$ as $n \to \infty$. Let $\epsilon > 0$. Let $m \in \mathbb{N}$ be such that $a_m/m < \ell + \epsilon/2$; such an m must exist since ℓ is the infimum of $\{ a_n/n : n \in \mathbb{N} \}$. Choose $N \in \mathbb{N}$ large enough such that $a_i/N < \epsilon/2$ for $i = 1, \ldots, m-1$. Let $n \geqslant N$; we aim to prove that $a_n/n \leqslant \ell + \epsilon$. There exists $q \in \mathbb{N} \cup \{0\}$ and $r \in \{0, \ldots, m-1\}$ such that $n = qm + r$. Note that $a_n = a_{qm+r} \leqslant qa_m + a_r$ by the hypothesis about the sequence $(a_n)_{n\in\mathbb{N}}$, where we formally take $a_0 = 0$ if $r = 0$. Thus

$$
\begin{aligned}
& a_n/n \\
\leqslant\ & (qa_m + a_r)/n \\
=\ & qa_m/n + a_r/n \\
=\ & qa_m/(qm + r) + a_r/n \\
\leqslant\ & qa_m/qm + a_r/n \\
<\ & a_m/m + \epsilon/2 && \text{(since } n \geqslant N \text{ and by the choice of } N\text{)} \\
<\ & \ell + \epsilon/2 + \epsilon/2 && \text{(by the choice of } m\text{)} \\
=\ & \ell + \epsilon.
\end{aligned}
$$

Hence $\lim_{n\to\infty} a_n/n$ exists and equals $\ell = \inf\{\, a_n/n : n \in \mathbb{N}\,\}$.

b) Let $b_n = \log a_n$ for all $n \in \mathbb{N}$. Then $(b_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers, and $b_{i+j} = \log a_{i+j} \leqslant \log a_i a_j = \log a_i + \log a_j = b_i + b_j$. So, by part 1, $\lim_{n\to\infty} b_n/n$ exists and equals $\inf\{\, b_n/n : n \in \mathbb{N}\,\}$. Thus

$$\lim_{n\to\infty} (1/n) \log a_n = \inf\{\, (1/n) \log a_n : n \in \mathbb{N}\,\}. \tag{2.2}$$

Hence

$$
\begin{aligned}
& \inf\{\, \sqrt[n]{a_n} : n \in \mathbb{N}\,\} \\
=\ & \exp \log \inf\{\, \sqrt[n]{a_n} : n \in \mathbb{N}\,\} \\
=\ & \exp \inf\{\, \log \sqrt[n]{a_n} : n \in \mathbb{N}\,\} && \text{(since log preserves } \leqslant\text{)} \\
=\ & \exp \inf\{\, (1/n) \log a_n : n \in \mathbb{N}\,\} \\
=\ & \exp \lim_{n\to\infty} (1/n) \log a_n && \text{(by (2.2))} \\
=\ & \exp \lim_{n\to\infty} \log \sqrt[n]{a_n} \\
=\ & \lim_{n\to\infty} \exp \log \sqrt[n]{a_n} && \text{(since exp is continuous)} \\
=\ & \lim_{n\to\infty} \sqrt[n]{a_n}
\end{aligned}
$$

2.3

By Lemma 2.1(4), $K(\varphi^{i+j}, A, A) \leqslant K(\varphi^i, A, A) K(\varphi^j, A, A)$, and therefore, by Lemma 2.3(2),

$$\lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} = \inf\{\, \sqrt[n]{K(\varphi^n, A, A)} : n \in \mathbb{N}\,\} \tag{2.3}$$

Thus:

$$
\begin{aligned}
\Gamma(\varphi) &= \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} \sqrt[n]{K(\varphi^n, B_{m,A}, A)} && \text{(by definition)} \\
&\leqslant \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} \sqrt[n]{mK(\varphi^n, A, A)} && \text{(by Lemma 2.1(1))} \\
&= \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} && \text{(since } \sqrt[n]{m} \to 1\text{)} \\
&= \limsup_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} && \text{(since } m \text{ is not present)} \\
&= \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} && \text{(since the limit exists by (2.3))} \\
&= \inf\{\, \sqrt[n]{K(\varphi^n, A, A)} : n \in \mathbb{N}\,\} && \text{(by (2.3))}
\end{aligned}
$$

In particular,

$$\Gamma(\varphi) = \inf\{ \sqrt[n]{K(\varphi^n, A, A)} : n \to \infty \} \leqslant \sqrt[1]{K(\varphi^1, A, A)} = K(\varphi, A, A),$$

which is part 2.

Next, let $k \in \mathbb{N}$. Then

$$\begin{aligned}\Gamma(\varphi) &= \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} \\ &= \lim_{n\to\infty} \sqrt[nk]{K(\varphi^{kn}, A, A)} \\ &= \lim_{n\to\infty} \left(\sqrt[n]{K((\varphi^k)^n, A, A)} \right)^{1/k} \\ &= \left(\lim_{n\to\infty} \sqrt[n]{K((\varphi^k)^n, A, A)} \right)^{1/k} \\ &= (\Gamma(\varphi^k))^{1/k};\end{aligned}$$

this proves part 3. 2.2

PROPOSITION 2.4. *Let S be a semigroup and let $\varphi : S \to S$ be an endomorphism of S. Then $\Gamma(\varphi)$ is not dependent on the choice of finite generating set for S.*

Proof of 2.4. Let A and A' be finite generating sets for S. Choose $m, p \in \mathbb{N}$ such that $A \subseteq B_{m,A'}$ and $A' \subseteq B_{p,A}$. Then

$$\begin{aligned}K(\varphi^n, A', A') &\leqslant mK(\varphi^n, A', A) && \text{(by Lemma 2.1(3))} \\ &\leqslant mK(\varphi^n, B_{p,A}, A) && \text{(by Lemma 2.1(2))} \\ &\leqslant mpK(\varphi^n, A, A) && \text{(by Lemma 2.1(1))}\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A', A')} \\ \leqslant\ &\lim_{n\to\infty} \sqrt[n]{mpK(\varphi^n, A, A)} \\ \leqslant\ &\lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)}; && \text{(since } \sqrt[n]{mp} \to 1 \text{ as } n \to \infty)\end{aligned}$$

Repeating the same reasoning with A and A' interchanged shows the opposite inequality. Hence

$$\lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} = \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A', A')},$$

and thus $\Gamma(\varphi)$ is independent the choice of generating set. 2.4

2.2 *Rewriting systems*

We now recall the terminology of rewriting systems, which we will use heavily throughout the paper; see [BO93] or [BN98] for further background reading. Let A be a finite alphabet. By a *rewriting system* we will mean a subset of $A^* \times A^*$, where A^* denotes the free monoid over A. Every element (u, v) of a system Σ is called a *rule* and normally denoted by $u \to_\Sigma v$ or simply $u \to v$. The relation $\to$ is then extended to a relation on A^* by letting $w_1 \to w_2$ if and only if w_1 and w_2 admit decompositions $w_1 = puq$ and $w_2 = pvq$ for some rule $u \to_\Sigma v$ and $p, q \in A^*$. The reflexive and transitive closure of $\to$ is denoted by $\to^*$. A rewriting system Σ is

- *length-reducing* if $|u| > |v|$ for all rules $u \to v$;
- *terminating* if there is no infinite chain $u_0 \to u_1 \to u_2 \to \cdots$;
- *locally confluent* if for all $u, v, w \in A^*$ such that $w \to u$ and $w \to v$, there exists $x \in A^*$ with $u \to^* x$ and $v \to^* x$.
- *confluent* if for all $u, v, w \in A^*$ such that $w \to^* u$ and $w \to^* v$, there exists $x \in A^*$ with $u \to^* x$ and $v \to^* x$.

Note that a length-reducing rewriting system is necessrily terminating. Any rewriting system that is locally confluent and terminating is confluent. Rewriting systems which are terminating and confluent are called *complete*. Complete systems are computationally pleasant in the following sense: if a semigroup S is defined by a presentation $\mathrm{Sg}\langle A \mid u_i = v_i \quad (i \in I)\rangle$ such that the rewriting system $\{\,(u_i, v_i) : i \in I\,\}$ is complete, then S is in one-to-one correspondence with the non-empty *normal forms* of this rewriting system: that is, the words from A^+ that do not contains subwords from $\{\,u_i : i \in I\,\}$ and thus cannot be rewritten further. This allows us to work with such monoids S in a very convenient syntatic way.

3 VALUES FOR GROWTH

THEOREM 3.1. *Let $r \in \mathbb{R}$ with $r \geqslant 1$. Then there is a finitely generated semigroup S and an endomorphism $\varphi : S \to S$ such that $\Gamma(\varphi) = r$.*

Proof of 3.1. Obviously the growth of the identity endomorphism on any semgroup is 1, so assume without loss of generality that $r > 1$.

Define $p_n = \lceil r^{n+1} \rceil + n$ for $n \in \mathbb{N} \cup \{0\}$. Let $A = \{a, b\}$ and let Σ consist of the following rewriting rules over A:

$$a^{p_j}(a^{p_i}b^{p_i}ab)^{p_j}a(a^{p_i}b^{p_i}ab) \to a^{p_{i+j+1}}b^{p_{i+j+1}}ab \qquad \text{for } i, j \in \mathbb{N} \cup \{0\}.$$

Note that $2 \leqslant p_0 < p_1 < p_2 < \cdots$. Therefore there cannot be any non-trivial overlaps between any left-hand sides of these rewriting rules, and so this rewriting system is confluent. This system is also terminating, since it is length-reducing, because

$$\begin{aligned}
|a^{p_j}(a^{p_i}b^{p_i}ab)^{p_j}a(a^{p_i}b^{p_i}ab)| &= p_j + (2p_i+2)p_j + 2p_i + 3 \\
&> 2p_ip_j + 8 \\
&= 2(\lceil r^{i+1}\rceil + i)(\lceil r^{j+1}\rceil + j) + 8 \\
&\geqslant 2r^{i+j+2} + 2i + 2j + 2ij + 8 \\
&\geqslant 2(r^{i+j+2} + 1 + i + j + 2) + 2 \\
&\geqslant 2p_{i+j+1} + 2 \\
&= |a^{p_{i+j+1}}b^{p_{i+j+1}}ab|.
\end{aligned}$$

Thus the rewriting system Σ is complete.

Furthermore, since the rewriting system is length-reducing, the length of an element is the length of its unique normal form word.

Let $S = \mathrm{Sg}\langle A \mid \Sigma\rangle$. Define an endomorphism $\varphi : S \to S$ by $a \mapsto a$ and $b \mapsto a^{p_0}b^{p_0}ab$. To check that φ is well-defined, note that it maps the two sides of each rewriting rule to words that are equal in the semigroup (for clarity, underlines indicate where rewriting is applied):

$$\begin{aligned}
&\big(a^{p_j}(a^{p_i}b^{p_i}ab)^{p_j}a(a^{p_i}b^{p_i}ab)\big)\varphi \\
&= a^{p_j}\big(\underline{a^{p_i}(a^{p_0}b^{p_0}ab)^{p_i}a(a^{p_0}b^{p_0}ab)}\big)^{p_j}a\big(\underline{a^{p_i}(a^{p_0}b^{p_0}ab)^{p_i}a(a^{p_0}b^{p_0}ab)}\big) \\
&\to \underline{a^{p_j}(a^{p_{i+1}}b^{p_{i+1}}ab)^{p_j}a(a^{p_{i+1}}b^{p_{i+1}}ab)} \\
&\to a^{p_{i+j+2}}b^{p_{i+j+2}}ab
\end{aligned}$$

and

$$\begin{aligned}
(a^{p_{i+j+1}}b^{p_{i+j+1}}ab)\varphi &= \underline{a^{p_{i+j+1}}(a^{p_0}b^{p_0}ab)^{p_{i+j+1}}a(a^{p_0}b^{p_0}ab)} \\
&\to a^{p_{i+j+2}}b^{p_{i+j+2}}ab.
\end{aligned}$$

Since φ fixes a, we have that $|a\varphi^n| = 1$ for all n. Note that

$$(a^{p_i}b^{p_i}ab)\varphi = a^{p_i}(a^{p_0}b^{p_0}ab)^{p_i}a(a^{p_0}b^{p_0}ab) \to a^{p_{i+1}}b^{p_{i+1}}ab,$$

and this, together with $b\varphi = a^{p_0}b^{p_0}ab$, shows that $b\varphi^n = a^{p_{n-1}}b^{p_{n-1}}ab$ for all $n \in \mathbb{N}$. Since words $a^{p_{n-1}}b^{p_{n-1}}ab$ are in normal form, this shows that

$$|b\varphi^n| = |a^{p_{n-1}}b^{p_{n-1}}ab| = 2(\lceil r^n\rceil + n - 1) + 2 = 2\lceil r^n\rceil + 2n.$$

Hence $K(\varphi^n, A, A) = 2\lceil r^n\rceil + 2n$ and so

$$\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{2\lceil r^n\rceil + 2n} = r. \qquad \boxed{3.1}$$

REMARK 3.2. Using the same general technique as in the proof of Theorem 3.1, we could have constructed a *surjective* endomorphism φ with the same growth r: to the alphabet A we add two letters c and d, and to the previous set of rewriting rules Σ we add the following ones:

$$\begin{aligned}
&ca^{p_0}b^{p_0}abd \to b \\
&ca^{p_n}b^{p_n}abd \to a^{p_{n-1}}b^{p_{n-1}}ab \qquad \text{for } n \in \mathbb{N}.
\end{aligned}$$

Then the resulting rewriting system is still complete and length-reducing. The endomorphism φ given by $a \mapsto a$, $b \mapsto a^{p_0}b^{p_0}ab$, $c \mapsto c$ and $d \mapsto d$ is again well-defined, and since $(cbd)\varphi = ca^{p_0}b^{p_0}abd \to b$ and φ fixes a, c, and d, it follows that φ is surjective. As previously, we still have $\Gamma(\varphi) = r$. Thus every real number greater than or equal to 1 also arises as the growth of a surjective endomorphism of a finitely generated semigroups.

Two natural questions arising from this discussion are

QUESTION 3.3. What are the growths of endomorphisms of finitely *presented* semigroups? Are they always computable?

QUESTION 3.4. What are the growths of endomorphisms of semigroups presented by finite complete rewriting systems?

4 ENDOMORPHISM GROWTH IN RELATION TO INVARIANT SUBSEMIGROUPS

Consider the following situation: let $\varphi : S \to S$ be an endomorphism of a finitely generated semigroup S, and let T be a finitely generated subsemigroup of S such that $T\varphi \subseteq T$. The following natural question arises: how are the growths of $\Gamma(\varphi)$ and $\Gamma(\varphi|_T)$ related? In this section, we will study this question, and we will also apply some of the results from this section in Section 6.

4.1 *General case: no relationship*

We might initially hope that the growths of the endomorphism and its restriction to the subsemigroup are related by an inequality like $\Gamma(\varphi) \leqslant \Gamma(\varphi|_T)$ or $\Gamma(\varphi) \geqslant \Gamma(\varphi|_T)$. In this subsection, we give examples to show that neither of these inequalities holds.

EXAMPLE 4.1. Let $S = (\{a\}^+)^0$ (that is, S is the free semigroup of rank 1 with a zero adjoined), $T = \{0\}$ and define $\varphi : S \to S$ by $a \mapsto a^2$, and $0 \mapsto 0$. Note that $T\varphi = T$.

Since $|0\varphi^n| = |0| = 1$ and $|a\varphi^n| = |a^{2^n}| = 2^n$, we have

$$\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, \{a, 0\}, \{a, 0\})} = \lim_{n\to\infty} \sqrt[n]{2^n} = 2,$$
$$\Gamma(\varphi|_T) = \lim_{n\to\infty} \sqrt[n]{K(\varphi^n, \{0\}, \{0\})} = \lim_{n\to\infty} \sqrt[n]{1} = 1$$

by Proposition 2.21. Therefore in this case we have $\Gamma(\varphi) \nleqslant \Gamma(\varphi|_T)$.

EXAMPLE 4.2. Let A be the alphabet $\{a, b, c, d\}$. Let φ be the endomorphism of the free semigroup A^+ defined by $a \mapsto ab$, $b \mapsto ba$, $c \mapsto c$ and $d \mapsto d$. Let S

be the semigroup defined by the following infinite rewriting system

$$\begin{aligned} a^n c^n a^n d &\to a\varphi^n && \text{for } n \in \mathbb{N}; \\ b^n c^n b^n d &\to b\varphi^n && \text{for } n \in \mathbb{N}; \\ (a\varphi^k)^n c^n (a\varphi^k)^n d &\to a\varphi^{k+n} && \text{for } k, n \in \mathbb{N}; \\ (b\varphi^k)^n c^n (b\varphi^k)^n d &\to b\varphi^{k+n} && \text{for } k, n \in \mathbb{N}. \end{aligned}$$

Since every application of a rule reduces the number of symbols c, it follows immediately that this system is terminating. The system is also confluent, since if two left-hand sides of rules overlap, the exponents n must coincide and it is easy to see that if $(x\varphi^k)^n = (y\varphi^\ell)^n$ for some $x, y \in \{a, b\}$ and $k, \ell, n \in \mathbb{N}$, then $k = \ell$ and $x = y$. Thus the rewriting system is complete.

It is straightforward to check that the endomorphism $\varphi : A^+ \to A^+$ maps the two sides of every rule to words which rewrite to the same normal form. Therefore the endomorphism φ of the free semigroup A^+ factors to give an endomorphism of S, which we also denote by φ. It also follows that $\{a, b\}$ forms a free basis for $T = \langle a, b\rangle$. Note that $T\varphi \subseteq T$. It is immediate that $K(\varphi^n, \{a, b\}, \{a, b\}) = 2^n$ and so $\Gamma(\varphi|_T) = 2$. But from the presentation for S it follows that

$$\begin{aligned} |a\varphi^n| &= |a^n c^n a^n d| \leqslant 3n + 1, \\ |b\varphi^n| &= |b^n c^n b^n d| \leqslant 3n + 1, \\ |c\varphi^n| &= |c| = 1, \\ |d\varphi^n| &= |d| = 1; \end{aligned}$$

thus $K(\varphi^n, A, A) \leqslant 3n + 1$, so $\Gamma(\varphi) = 1$.

Thus, in this case, $\Gamma(\varphi) \ngeqslant \Gamma(\varphi|_T)$.

4.2 *Mapping into a subsemigroup: growths coincide*

In the restricted situation where the endomorphism φ maps the semigroup S into the subsemigroup T, we have a positive result:

PROPOSITION 4.3. *Let T be a finitely generated subsemigroup of a finitely generated semigroup S. Let φ be an endomorphism of S such that $S\varphi \subseteq T$. Then $\Gamma(\varphi) = \Gamma(\varphi|_T)$.*

Proof of 4.3. Let B be a finite generating set for T and extend it to a finite generating set A for S. Let $m = K(\varphi, A, B)$. (Note that $a\varphi \in T$ for all $a \in A$ and so $K(\varphi, A, B) = \max_{a\in A} |a\varphi|_B$ is defined.)

Let $a \in A$. Then $a\varphi \in T$ and so $a\varphi = b_1 \cdots b_p$ for some $p \leqslant m$.

$$\begin{aligned} |a\varphi^{n+1}|_A &\leqslant |a\varphi^{n+1}|_B \\ &= \big|(a\varphi)\varphi^n\big|_B \\ &= \big|(b_1 \cdots b_p)\varphi|_T^n|_B \\ &\leqslant \big|b_1\varphi|_T^n\big|_B + \ldots + \big|b_p\varphi|_T^n\big|_B \\ &= pK(\varphi|_T^n, B, B) \\ &\leqslant mK(\varphi|_T^n, B, B). \end{aligned}$$

Thus $K(\varphi^{n+1}, A, A) \leqslant mK(\varphi|_T^n, B, B)$ and so

$$\begin{aligned}
&\Gamma(\varphi) \\
&= \lim_{n\to\infty} \sqrt[n+1]{K(\varphi^{n+1}, A, A)} \\
&\leqslant \lim_{n\to\infty} \sqrt[n+1]{mK(\varphi|_T^n, B, B)} \\
&= \lim_{n\to\infty} \sqrt[n+1]{K(\varphi|_T^n, B, B)} \qquad \text{(since } \sqrt[n+1]{m} \to 1) \\
&\leqslant \lim_{n\to\infty} \sqrt[n]{K(\varphi|_T^n, B, B)} \\
&= \Gamma(\varphi|_T).
\end{aligned} \tag{4.1}$$

Now let $b \in B$. Let $q = |b\varphi|_T^n|_A$ so that $b\varphi|_T^n = a_1 \cdots a_q$ for some $a_i \in A$. Note that $q \leqslant K(\varphi^n, B, A)$. Then

$$\begin{aligned}
|b\varphi|_T^{n+1}|_B &= |(a_1\varphi) \cdots (a_q\varphi)|_B \\
&\leqslant |a_1\varphi|_B + \ldots + |a_q\varphi|_B \\
&\leqslant mq \\
&\leqslant mK(\varphi^n, B, A) \\
&\leqslant mK(\varphi^n, A, A). \qquad \text{(by Lemma 2.1(2))}
\end{aligned}$$

Since b was arbitrary, this shows that $K(\varphi|_T^{n+1}, B, B) \leqslant mK(\varphi^n, A, A)$. By reasoning similar to (4.1), $\Gamma(\varphi|_T) \leqslant \Gamma(\varphi)$.

Therefore $\Gamma(\varphi) = \Gamma(\varphi|_T)$. 4.3

4.3 *Finite number of cosets: only one direction of inequality*

When a finite-index subgroup of a group is preserved by an endomorphism, the growth of the endomorphism and the growth of the restriction to the subgroup are equal [FFK11, Theorem 3.1]. For semigroups, using an analogy of the notion of coset, an inequality holds in one direction. The proof partly follows the group-theoretic result, but some extra care is needed because in the semigroup case an element may lie in more than one 'coset'.

PROPOSITION 4.4. *Let T be a finitely generated subsemigroup of a finitely generated semigroup S such that there exists a finite subset $R \subseteq S$ with $S = RT$. Let $\varphi : S \to S$ be an endomorphism of S such that $T\varphi \subseteq T$. Then $\Gamma(\varphi) \leqslant \Gamma(\varphi|_T)$.*

Proof of 4.4. Let A be a finite generating set for T. Obviously $A \cup R$ is a (finite) generating set for S. First, for every $a \in A$ we have

$$|a\varphi^n|_{A\cup R} \leqslant |a\varphi|_T^n|_A \leqslant K(\varphi_T^n, A, A) \tag{4.2}$$

Now, take any $r \in R$. The aim is to calculate an upper bound for $|r\varphi^n|_{A\cup R}$. To begin, for every $r \in R$ fix a canonical decomposition $r\varphi = r'w$ where $r' \in R$ and $w \in T$, and let

$$C = \max\{\, |w|_A : r\varphi \text{ has canonical decomposition } r'w \text{ for some } r \in R \,\}.$$

Now, $r\varphi$ decomposes as $r_1 w_1$, with $r_1 \in R$ and $|w_1|_A \leqslant C$. Then $r\varphi^2 = (r_1\varphi)(w_1\varphi) = r_2 w_2 (w_1\varphi)$, where $r_1\varphi$ decomposes as $r_2 w_2$, with $r_2 \in R$ and $|w_2|_A \leqslant C$. Proceeding by induction we obtain the expansion

$$r\varphi^n = r_n w_n (w_{n-1}\varphi)(w_{n-2}\varphi^2)\cdots(w_1\varphi^{n-1}),$$

where $r_n \in R$ and $|w_i|_A \leqslant C$ for all i. Thus

$$|r\varphi^n|_{A\cup R} \leqslant 1 + C + CK(\varphi, A, A) + CK(\varphi^2, A, A) + \cdots + CK(\varphi^{n-1}, A, A).$$

Let $\gamma = \Gamma(\varphi|_T)$. Since $\sqrt[n]{K(\varphi^n, A, A)} \to \gamma$ as $n \to \infty$, for every $\epsilon > 0$ there exists $M > 1$ such that $K(\varphi^n, A, A) \leqslant M(\gamma+\epsilon)^n$ for all $n \geqslant 1$. Then

$$\begin{aligned} |r\varphi^n|_{A\cup R} &\leqslant 1 + C + CM(\gamma+\epsilon) + CM(\gamma+\epsilon)^2 + \cdots + CM(\gamma+\epsilon)^{n-1} \\ &\leqslant \frac{CM(\gamma+\epsilon)^n}{1 - 1/(\gamma+\epsilon)}. \end{aligned}$$

Combining this with (4.2) gives

$$\begin{aligned} K(\varphi^n, A\cup R, A\cup R) &= \max_{a\in A\cup R} |a\varphi^n| \\ &\leqslant \max\left\{K(\varphi|_n, A, A), \frac{CM(\gamma+\epsilon)^n}{1-1/(\gamma+\epsilon)}\right\}. \end{aligned}$$

Taking n-th roots on both sides and then the limit as $n \to \infty$, and recalling that $\gamma = \Gamma(\varphi|_T)$ shows that $\Gamma(\varphi) \leqslant \Gamma(\varphi|_T)$. 4.4

The following example shows that the inequality in Proposition 4.4 can be strict.

EXAMPLE 4.5. Let $L = \{a, b, c\}$ and let

$$L = \{\, ab^{n2^k} c^n : k \geqslant 0, n \text{ is positive and odd}\,\}.$$

We are going to construct a rewriting system Σ over A and so define a monoid $S = \mathrm{Mon}\langle A \mid \Sigma\rangle$. The rewriting system Σ will have the following properties:

a) Σ is complete;
b) the left-hand-sides of the rules of Σ form exactly the set L;
c) every word from $A^+ - A^*LA^*$ appears on the right-hand-side of some rule in Σ;
d) there is a well-defined endomorphism $\varphi : S \to S$ defined by $a \mapsto a$, $b \mapsto b^2$ and $c \mapsto c$;
e) $\Gamma(\varphi) = 1$.

Once we have constructed Σ, we reason as follows: first of all, by (1) and (2) the language $A^+ - A^*LA^*$ is a set of normal forms of S. Therefore by (2) and (3), for any normal form word w, there is some word in L (beginning with a and with all other letters from $\{b, c\}$) that rewrites to w, and so $S = \{1, a\}T$, where $T = \mathrm{Mon}\langle b, c\rangle$. By (4), φ is an endomorphism of S. Notice further that $T\varphi \subseteq T$. Furthermore, since every rule on Σ has a letter a on the left-hand side by 2), it follows that Σ is free on $\{b, c\}$ and so clearly $\Gamma(\varphi|_T) = 2$. Hence, by (5), we have $\Gamma(\varphi) < \Gamma(\varphi|_T)$.

We now have to construct Σ with the required properties. We will define Σ in stages by iteratively defining $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ with $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq$ and then letting Σ be the union of all the Σ_i. Define the first set of rules Σ_0 as follows: for all $n \in \mathbb{N}$, let p_n be the n-th odd prime number. Then Σ_0 consists of the following rules:

$$ab^{p_n 2^k} c^{p_n} \to b^{2^{k+n}} \qquad \text{for } n \in \mathbb{N} \text{ and } k \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

For the next stages, enumerate all the words from $A^* - A^*LA^*$ in some order: $u_1, u_2, u_3, \ldots$ and set $n_0 = 1$. Now iterate the following procedure. The i-th step of the procedure, for $i \in \mathbb{N}$, is to take the first word u_j from the list which does not appear as a right-hand-side a rule in Σ_{i-1}. Take also any odd composite number $n_i > n_{i-1}$ such that $n_i > |u_j|$. Define Σ_i to be the rules of Σ_{i-1} together with

$$ab^{n_i 2^k} c^{n_i} \to u_j \varphi^k \qquad \text{for } k \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Note that the left-hand sides of the newly added rules do not appear as left-hand sides in Σ_{i-1}, because if we had $n_i 2^k = n_{i'} 2^{k'}$ for some $i' < i$, then since the n_i are chosen to be odd, we would have $n_i = n_{i'}$ by the fundamental theorem of arithmetic, contradicting the fact that the n_i form a strictly increasing sequence. Note also that since $n_i > |u_j|$ that each rule in Σ_i strictly decreases the total number of symbols a and c.

Let $\Sigma' = \bigcup_{i \in \mathbb{N} \cup \{0\}} \Sigma_i$. Note that by construction of Σ', for every odd n one of two cases holds: either every element of the set $\{ ab^{n2^k} c^n : k \geqslant 0 \}$ appears as a left-hand side in Σ', or no element of this set does. Now let Σ be Σ' together with the rules

$$ab^{n2^k} c^n \to a \quad \text{where } k \geqslant 0 \text{ and } ab^n c^n \text{ is not a left-hand side in } \Sigma'. \tag{4.5}$$

It is clear that every word in L is the left-hand side of exactly one rule in Σ, so (2) holds. Similarly, by construction of the Σ_i, every word in $A^* - A^*LA^*$ appears on the right-hand side at least one rule in Σ, so (3) is satisfied.

Each application of a rule of Σ strictly decreases the total number of symbols a and c, and so Σ is terminating. Since left-hand sides of rules have no non-trivial overlaps, Σ is locally confluent and thus confluent. So Σ is complete, and so (1) is satisfied.

Now we have to check that the endomorphism φ is well-defined, which means checking that φ maps the two sides of each rule to words that are equal in S. First consider a rule $ab^{p_n 2^k} c^{p_n} \to b^{2^{k+n}}$ of the form (4.3). Then $(ab^{p_n 2^k} c^{p_n})\varphi = ab^{p_n 2^{k+1}} c^{p_n}$ and $b^{2^{k+n}} \varphi = b^{2^{k+n+1}}$, and $ab^{p_n 2^{k+1}} c^{p_n} \to b^{2^{k+n+1}}$ is also a rule in Σ_0. Now consider a rule $ab^{n_i 2^k} c^{n_i} \to u_j \varphi^k$ of the form (4.4). Then $(ab^{n_i 2^k} c^{n_i})\varphi = ab^{n_i 2^{k+1}} c^{n_i}$ and $(u_j \varphi^k)\varphi = u_j \varphi^{k+1}$, and $ab^{n_i 2^{k+1}} c^{n_i} \to u_j \varphi^{k+1}$ is also a rule in Σ_i. Finally, consider a rule $ab^{n2^k} c^n \to a$ of the form (4.5). Then $(ab^{n2^k} x^n)\varphi = ab^{n2^{k+1}} c^n$ and $a\varphi = a$, and $ab^{n2^{k+1}} c^n \to a$ is also a rule of the form (4.5). So φ is a well-defined endomorphism, which is (4).

Note that $|a\varphi^n|_A = |a|_A = 1$ and $|c\varphi^n|_A = |c|_A = 1$. Furthermore, $|b\varphi^n|_A = |b^{2^n}|_A = |ab^{p_n}c^{p_n}|_A \leqslant 2p_n + 1$. Hence $K(\varphi^n, A, A) \leqslant 2p_n + 1$, and so $\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{2p_n + 1}$. Since $p_{n-1} \leqslant n(\ln n + \ln\ln n)$ for all $n \geqslant 6$ (see [RS62, Corollary to Theorem 3]), it follows that $\Gamma(\varphi) = 1$, which is condition (5).

4.4 *Finite Green index subsemigroups: growths coincide*

With the notion of 'finitely many cosets' used in Proposition 4.4, we have an inequality, possibly strict by Example 4.5, showing that the growth of the endomorphism of the semigroup is bounded above by the growth of the restriction to a subsemigroup. In this subsection, we show that the Green index serves as a better analogy of the group index [GR08] and gives us equality, directly generalizing the result for groups. The Green index, which was introduced by [GR08], has proven to be a very useful generalization of both the group-theoretic notion of index and the more established Rees index for semigroups, and has yielded many Reidemeister–Schreier-type theorems about the inheritance of various finiteness properties by subsemigroups or extensions of finite index; see, for example, [CGR12, CM14, GR08, GMMR, KMC15, MMR09]. We recall the definition here: Let T be a subsemigroup of a semigroup S. For $x, y \in S$, let

$$x \mathcal{R}^T y \Leftrightarrow xT \cup \{x\} = yT \cup \{y\}$$
$$x \mathcal{L}^T y \Leftrightarrow Tx \cup \{x\} = Ty \cup \{y\},$$

and let $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$. Then $\mathcal{R}^T$, $\mathcal{L}^T$, and $\mathcal{H}^T$ are equivalence relations on S that respect T. The *Green index* of T in S is $1 + |(S \setminus T)/\mathcal{H}^T|$.

PROPOSITION 4.6. *Let T be a finite Green index subsemigroup of a finitely generated semigroup S, and let $\varphi : S \to S$ be an endomorphism of S such that $T\varphi \subseteq T$. Then $\Gamma(\varphi) = \Gamma(\varphi|_T)$.*

Proof of 4.6. Let A be a finite generating set for S. The proof of [CGR12, Theorem 4.3] constructs a finite generating set B for T such that for every $w \in T$, $|w|_B \leqslant |w|_A$. In particular, for every $b \in B$, we have $|b\varphi|_T^n|_B = |b\varphi|_T^n|_A$. Hence $K(\varphi|_T^n, B, B) = K(\varphi|_T^n, B, A) \leqslant K(\varphi^n, A, A)$, where the second inequality follows from Lemma 2.1(2). Thus $\Gamma(\varphi|_T) \leqslant \Gamma(\varphi)$. We observed that T is finitely generated, and so Proposition 4.4 applies to show that $\Gamma(\varphi) \leqslant \Gamma(\varphi|_T)$. Therefore we have $\Gamma(\varphi) = \Gamma(\varphi|_T)$. 4.6

4.5 *Ideals: exact formula via factor semigroups*

In the setting of semigroups, the counterpart of a 'normal subgroup' is a notion borrowed from ring theory: a subset I of a semigroup S is called an *ideal* if $IS \cup SI \subseteq I$. To every ideal I in S, one associates the *Rees congruence* $\rho_I = \mathrm{id}_{S\setminus I} \cup (I \times I)$ (see [How95, § 1.7]). The corresponding factor semigroup is called the Rees factor and is denoted by S/I.

Let ρ be a congruence on a semigroup S and let $\varphi : S \to S$ be an endomorphism that respects ρ, in the sense that $x \,\rho\, y \Rightarrow x\varphi \,\rho\, y\varphi$ for all $x, y \in S$. Then

ρ factors to give a well-defined endomorphism φ/ρ of the factor semigroup S/ρ, defined by $[x]_\rho\varphi = [x\varphi]_\rho$.

Before stating the result on Rees factor semigroups, we note the following immediate observation, which is worth stating separately.

LEMMA 4.7. *Let $\varphi : S \to S$ be an endomorphism of a finitely generated semigroup S and let ρ be a congruence on S such that φ respects ρ. Then $\Gamma(\varphi/\rho) \leqslant \Gamma(\varphi)$.*

Proof of 4.7. Let A be a finite generating set for S. Let $A/\rho = \{\,[a]_\rho : a \in A\}$; notice that A/rho generates S/rho. Let $x \in S$ and let $p = |x|_A$. Then $x = a_1 \cdots a_p$ for some $a_i \in A$. Thus $[x]_\rho = [a_1 \cdots a_p]_\rho = [a_1]_\rho \cdots [a_p]_\rho$, and so $\big|[x]_\rho\big|_{A/\rho} \leqslant |x|_A$. Consequently, $K(\varphi^n, A/\rho, A/\rho) \leqslant K(\varphi^n, A, A)$ for all $n \in \mathbb{N}$ and so $\Gamma(\varphi/\rho) \leqslant \Gamma(\varphi)$. 4.7

PROPOSITION 4.8. *Let I be a finitely generated ideal of a finitely generated semigroup S, and let $\varphi : S \to S$ be an endomorphism of a semigroup S such that $I\varphi \subseteq I$. Then $\Gamma(\varphi) = \max\{\Gamma(\varphi|_I), \Gamma(\varphi/\rho_I)\}$.*

Proof of 4.8. Let B be a finite generating set for I and extend B to a finite generating set A for S.

Part 1: $\geqslant$. First, Lemma 4.7 gives $\Gamma(\varphi/\rho_I) \leqslant \Gamma(\varphi)$, so it remains to show that $\Gamma(\varphi|_I) \leqslant \Gamma(\varphi)$. Our first aim is to prove that there exists a constant $m \in \mathbb{N}$ such that for every $w \in S$ and $b \in B$, we have $|bw|_B \leqslant m|w|_A$. So let $b \in B$ and $w = a_1 \cdots a_p$ where $a_i \in A$ and $p = |w|_A$. Put $C = \max\{|ba|_B : a \in A, b \in B\}$. Then

$$bw = ba_1 \cdots a_p = w_1 a_2 \cdots a_p,$$

where w_1 is a word over B with $|w_1|_B \leqslant m$. Take the last letter b' from w_1 and repeat the process for the subword $b' a_2 \cdots a_p$. Proceeding in this way, we eventually obtain the an expression of bw as a product $w_1 \cdots w_p$ of elements w_i of B, with $|w_i|_B \leqslant m$ for all i; thus $|bw|_B \leqslant mp = m|w|_A$.

Now let $b \in B$ be arbitrary. Consider a shortest expression of $b\varphi^n$ as a product of elements of B: we have $b\varphi^n = b_1 \cdots b_p$ with $b_i \in B$ and $p \leqslant K(\varphi^n, B, B)$. Then in

$$b\varphi^{2n} = (b_1 b_2 \cdots b_p)\varphi^n = (b_1\varphi^n)(b_2\varphi^n) \cdots (b_p\varphi^n)$$

we take a shortest expression for $b_1\varphi^n = ub'$ as a product of elements of B, and shortest expressions for $b_2\varphi^n, \ldots, b_p\varphi^n$ as products of elements of A. Then

$$\begin{aligned}
&|b\varphi^{2n}|_B \\
&\leqslant |b_1\varphi^n|_B - 1 + |b'(b_2\varphi^n) \cdots (b_p\varphi^n)|_B \\
&\leqslant |b_1\varphi^n|_B - 1 + m|(b_2\varphi^n) \cdots (b_p\varphi^n)|_A \\
&\leqslant |b_1\varphi^n|_B - 1 + m(|b_2\varphi^n|_A + \cdots + |b_p\varphi^n|_A) \\
&\leqslant |b_1\varphi^n|_B + mpK(\varphi^n, B, A) \\
&\leqslant |b_1\varphi^n|_B + mpK(\varphi^n, A, A) && \text{(by Lemma 2.1(2))} \\
&\leqslant mK(\varphi^n, B, B) + mK(\varphi^n, B, B)K(\varphi^n, A, A) \\
&= mK(\varphi^n, B, B)\big(1 + K(\varphi^n, A, A)\big).
\end{aligned}$$

Since $n \in B$ was arbitrary, this shows that $K(\varphi^2 n, B, B) \leqslant mK(\varphi^n, B, B)(1 + K(\varphi^n, A, A))$, and so taking the limit as $n \to \infty$ in

$$\sqrt[2n]{K(\varphi^{2n}, B, B)} \leqslant \sqrt[2n]{m}\sqrt[2n]{K(\varphi^n, B, B)}\sqrt[2n]{1 + K(\varphi^n, A, A)},$$

we obtain $\Gamma(\varphi|_I) \leqslant \Gamma(\varphi|_I)\Gamma(\varphi)$. Thus we also have $\Gamma(\varphi|_I) \leqslant \Gamma(\varphi)$, as required.

Part 2: $\leqslant$. For each $a \in A$, there are two possibilities: either $a\varphi^n \in I$ for some $n \in \mathbb{N}$, or $a\varphi^n \in S \setminus I$ for all $n \in \mathbb{N}$. Let $A' = \{a \in A : (\exists n \in \mathbb{N})(a\varphi^n \in I)\}$. Let k be such that $a'\varphi^k \in I$ for all $a' \in A'$, and let $m = \max_{a' \in A'} |a'\varphi^k|_B$. Let $a' \in A'$ and $n \geqslant k$. Then $|a'\varphi^n|_A \leqslant |a'\varphi^n|_B \leqslant |(a'\varphi^k)\varphi^{n-k}|_B \leqslant mK(\varphi^{n-k}, B, B)$.

On the other hand, let $a \in A - A'$, so that $a\varphi^n \in S \setminus I$ for all $n \geqslant 1$. Then it follows that $|a\varphi^n|_A = |[a\varphi^n]_{\rho_I}|_{A/\rho_I}$ and so $|a\varphi^n|_A \leqslant K(\varphi^n, A/\rho_I, A/\rho_I)$ for all $n \geqslant 1$.

So $K(\varphi^n, A, A) \leqslant \max\{mK(\varphi^{n-k}, B, B), K(\varphi^{n-k}, A/\rho_I, A/\rho_I)\}$. This proves that $\Gamma(\varphi) \leqslant \max\{\Gamma(\varphi|_I), \Gamma(\varphi/\rho_I)\}$. 4.8

REMARK 4.9. As Example 4.1 shows, the inequality $\Gamma(\varphi|_I) \leqslant \Gamma_S(\varphi)$ can be strict, and thus the term $\Gamma_{S/I}(\varphi/I)$ cannot be eliminated from the formula in Proposition 4.8.

5 CONSTRUCTIONS

In this section we consider the interaction of endomorphism growth with two fundamental semigroup constructions, namely free and direct products. The first result is about free products is straightforward to prove:

PROPOSITION 5.1. *Let φ and ψ be endomorphisms of a finitely generated semigroups S and T respectively. Let $\varphi \cup \psi$ be the lift of these endomorphisms to an endomorphism of the free product $S * T$. Then $\Gamma(\varphi \cup \psi) = \max\{\Gamma(\varphi), \Gamma(\psi)\}$.*

Proof of 5.1. Let A and B be finite generating sets for S and T respectively. Then $A \cup B$ is a finite generating set for $S * T$. Since $S * T$ is a free product, $|x|_A = |x|_{A\cup B}$ for any element $x \in S$ and $|y|_B = |y|_{A\cup B}$ for any element $y \in T$. Hence, since $a(\varphi \cup \psi)^n = a\varphi^n \in S$ for all $a \in A$ and $b(\varphi \cup \psi)^n = b\psi^n \in T$ for all $b \in B$, we have

$$\begin{aligned}
&K((\varphi \cup \psi)^n, A \cup B, A \cup B) \\
&= \max\{K((\varphi \cup \psi)^n, A, A \cup B), K((\varphi \cup \psi)^n, B, A \cup B)\} \\
&= \max\{K(\varphi^n, A, A \cup B), K(\psi^n, B, A \cup B)\} \\
&= \max\{K(\varphi^n, A, A), K(\psi^n, B, B)\},
\end{aligned}$$

and the result follows. 5.1

The situation with direct products of semigroups has some special features that do not arise for groups, because a direct product of finitely generated semigroups is not necessarily itself finitely generated. Robertson et al. [RRW98] characterized direct products of semigroups are finitely generated: $S \times T$ is finitely generated if and only if both S and T are finitely generated and

- if S and T are both infinite, then $S^2 = S$ and $T^2 = T$;
- if S is finite and T is infinite, then $S^2 = S$;
- if S is infinite and T is finite, then $T^2 = T$.

PROPOSITION 5.2. *Let φ and ψ be endomorphisms of finitely generated semigroup S and T respectively. Suppose $S \times T$ is finitely generated. Let $\varphi \oplus \psi$ be the endomorphism of $S \times T$ with $(s,t) \mapsto (s\psi, t\psi)$. Then $\Gamma(\varphi \oplus \psi) = \max\{\Gamma(\varphi), \Gamma(\psi)\}$.*

Proof of 5.2. Interchanging S and T if necessary, it is sufficient to consider the following two cases:

a) S is finite and $S^2 = S$. Let A be a finite generating set for T. Then $S \times A$ is a finite generating set for $S \times T$. Let $(s,a) \in S \times A$ be arbitrary. Let $|a\varphi^n|_A = p \leqslant K(\psi^n, A, A)$. Then $a\psi^n = a_1 \cdots a_p$ for some $a_i \in A$. Let also $s\varphi^n = s_1 \cdots s_p$ be any decomposition of $s\varphi^n \in S$ into a product of p elements of S. (This decomposition exists since $S^2 = S$). Then $|(s,a)(\varphi \oplus \psi)^n|_{S\times A} = |(s\varphi^n, a\psi^n)|_{S\times A} = |(s_1, a_1)\cdots(s_p, a_p)|_{S\times A} \leqslant p \leqslant K(\psi^n, A, A)$. Thus $K((\varphi \oplus \psi)^n, S \times A, S \times A) \leqslant K(\psi^n, A, A)$ and so $\Gamma(\varphi \oplus \psi) \leqslant \Gamma(\psi)$.
b) Both S and T are infinite and $S^2 = S$ and $T^2 = T$. As was proved in [RRW98], S and T admit finite generating sets A and B satisfying the additional conditions that $A \subseteq A^2$, $B \subseteq B^2$ and $A \times B$ is a finite generating set for $S \times T$. Let $(a,b) \in A \times B$. Let $a\varphi^n = a_1 \cdots a_p$ and $b\psi^n = b_1 \cdots b_q$ where $p = |a\varphi^n|_A$ and $q = |b\psi^n|_B$. By the conditions $A \subseteq A^2$ and $B \subseteq B^2$, we may find alternative decompositions $a\varphi^n = a'_1 \cdots a'_r$ and $b\psi^n = b'_1 \cdots b'_r$ where $r = \max\{p,q\}$. This implies that $|(a,b)(\varphi \oplus \psi)^n|_{A\times B} \leqslant r \leqslant \max\{K(\varphi^n, A, A), K(\psi^n, B, B)\}$.

Thus $\Gamma(\varphi \oplus \psi) \leqslant \max\{\Gamma(\varphi), \Gamma(\psi)\}$. By Lemma 4.7, $\max\{\Gamma(\varphi), \Gamma(\psi)\} \leqslant \Gamma(\varphi \oplus \psi)$ and so the result holds. 5.2

6 SPECIAL CLASSES OF SEMIGROUPS

6.1 *Homogeneous semigroups*

Let S be a semigroup admitting a homogeneous presentation over a generating set $A = \{a_1, \ldots, a_k\}$: that is, a presentation such that in every defining relation the length of the left-hand side equals the length of the right-hand side. Therefore if two products of generators from A are equal in S, they must have the same length. Let $\varphi : S \to S$ be an endomorphism. The map φ is determined by its effect on the generators: $a_1 \mapsto w_1, \ldots, a_k \mapsto w_k$. Denote by $x_{ij}^{(n)}$ the number of letters a_i in $a_j\varphi^n$ for all $1 \leqslant i, j \leqslant k$ and $n \in \mathbb{N}$. Note that each $x_{ij}^{(n)}$ is a non-negative integer.

Now, $x_{ij}^{(n+1)}$ is the number of a_i in $a_j\varphi^{n+1}$. For each h, there are $x_{hj}^{(n)}$ symbols a_h in $a_j\varphi^n$, and the image of each of these symbols under φ contributes $x_{ih}^{(1)}$ symbols a_i to the total $x_{ij}^{(n+1)}$. That is,

$$x_{ij}^{(n+1)} = \sum_{h=1}^{k} x_{ih}^{(1)} x_{hj}^{(n)}.$$

Therefore,

$$\begin{bmatrix} x_{1j}^{(n+1)} \\ \vdots \\ x_{kj}^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_{11}^{(1)} & \cdots & x_{1k}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{k1}^{(1)} & \cdots & x_{kk}^{(1)} \end{bmatrix} \begin{bmatrix} x_{1j}^{(n)} \\ \vdots \\ x_{kj}^{(n)} \end{bmatrix} = P \begin{bmatrix} x_{1j}^{(n)} \\ \vdots \\ x_{kj}^{(n)} \end{bmatrix}$$

where P is the matrix whose i, j-th entry is $x_{ij}^{(1)}$. Then, since S is homogeneous,

$$|a_j\varphi^n| = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} P^{n-1} \begin{bmatrix} x_{1j}^{(1)} \\ \vdots \\ x_{kj}^{(1)} \end{bmatrix}.$$

Since

$$K(\varphi^n, A, A) = \max_{a \in A} |a\varphi^n|_A \leqslant \sum_{a \in A} |a\varphi^n|_A \leqslant k \max_{a \in A} |a\varphi^n|_A \leqslant kK(\varphi^n, A, A),$$

and since $\lim_{n\to\infty} \sqrt[n]{K(\varphi^n, A, A)} = \lim_{n\to\infty} \sqrt[n]{kK(\varphi^n, A, A)}$, it follows that

$$\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{\sum_{a\in A} |a\varphi^n|}$$

and so we have

$$\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} P^{n-1} \begin{bmatrix} x_{11}^{(1)} + \ldots + x_{1k}^{(1)} \\ \vdots \\ x_{k1}^{(1)} + \ldots + x_{kk}^{(1)} \end{bmatrix}}.$$

If $x_{i1}^{(1)} + \cdots + x_{ik}^{(1)} > 0$ for all i, then it follows that

$$\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{\|P^n\|},$$

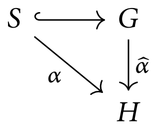
where $\|X\|$ is the sum of the absolute values of all entries of the matrix X. If $x_{i1}^{(1)} + \ldots + x_{ik}^{(1)} = 0$ for some i, then φ maps S to the subsemigroup $T = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \rangle$, which is obviously also a homogeneous semigroup, and so by Proposition 4.3 we reduce the calculation of $\Gamma(\varphi)$ to calculation of the growth of the endomorphism $\varphi|_T$ on the subsemigroup T, which has a smaller generating set than S.

Therefore there is a correspondence between endomorphisms of S and non-negative integer $k \times k$ matrices. In the case when S is free, any such matrix corresponds to an endomorphism. Thus we reduce the problem of describing the growths of endomorphisms of homogeneous semigroups to studying the asymptotics of the powers of such matrices. It remains to notice that by Gelfand's formula, we immediately obtain that $\Gamma(\varphi) = \lim_{n\to\infty} \sqrt[n]{\|P^n\|} = \rho(P)$ (the spectral radius of P), and so $\Gamma(\varphi)$ is the largest eigenvalue of a non-negative integer matrix. In particular, we have the following result:

THEOREM 6.1. *The growth of an endomorphism of a homogeneous semigroup is an algebraic number.*

6.2 *Group-embeddable semigroups*

For every group-embeddable semigroup S, there exists a *universal group* G, containing S and generated by S as a group, such that for every group H and homomorphism $\alpha : S \to H$ with $\mathrm{Gr}\langle S\alpha\rangle = H$, there exists a homomorphism $\widehat{\alpha} : G \to H$ such that the following diagram commutes (see [Cai05] and [CP67, Chapter 12]):



Let S be a semigroup generated by a finite set A and $\varphi : S \to S$ an endomorphism. We may treat φ as a homomorphism from S to the subgroup $\mathrm{Gr}\langle S\varphi\rangle$ of G and so φ extends to an endomorphism $\widehat{\varphi} : G \to G$ of the group G. Obviously for every generator $a \in A$

$$|a^{-1}\widehat{\varphi}^n|_{A\cup A^{-1}} = |a\widehat{\varphi}^n|_{A\cup A^{-1}} \leqslant |a\varphi^n|_A,$$

and so $\Gamma(\widehat{\varphi}) \leqslant \Gamma(\varphi)$. However, this inequality may be strict, as the following example shows:

EXAMPLE 6.2. Let $A = \{a, b\}$ and let S_k be the semigroup defined by $\mathrm{Sg}\langle A \mid ab = ba^k\rangle$. The semigroup S_k is one of the Baumslag–Solitar semigroups, which are well-known to be group-embeddable. The universal group of S_k is $G_k = \mathrm{Gp}\langle A \mid ab = ba^k\rangle$. Define an endomorphism $\varphi : S_k \to S_k$ by $a \mapsto a^k$ and $b \mapsto b$. It is easy to check that φ is well-defined. Note that $a\varphi^n = a^{k^n}$, and that no other word over A equals a^{k^n} since the defining relation cannot be applied to a word that does not contain symbols b. Hence $|a\varphi^n| = k^n$ and so, since b is fixed by φ, we have $K(\varphi^n, A, A) = k^n$ and so $\Gamma(\varphi) = k$.

However, $a^k =_{G_k} b^{-1}ab$ and so $a\widehat{\varphi}^n = a^{k^n} = b^{-n}ab^n$. Thus $K(\widehat{\varphi}, A \cup A^{-1}, A \cup A^{-1}) \leqslant 2n + 1$ and so $\Gamma(\widehat{\varphi}) = 1$.

Note that the Baumslag–Solitar semigroups belong to a special class of group-embeddable semigroups: left-reversible semigroups, or equivalently those semigroups which admit groups of right quotients; see [CP67, § 1.10]. This suggests that in the general case there is little hope for an exact formula relating $\Gamma(\varphi)$ and $\Gamma(\widehat{\varphi})$.

However, we conjecture that the equality $\Gamma(\widehat{\varphi}) = \Gamma(\varphi)$ holds for the class of finitely generated subsemigroups of free semigroups (perhaps surprisingly, this class has a rich theory; see for example [CRR06, Lal79]).

QUESTION 6.3. Is it true that $\Gamma(\widehat{\varphi}) = \Gamma(\varphi)$ for every endomorphism φ of a finitely generated subsemigroup of a free semigroup?

6.3 *Free inverse semigroups*

We close by briefly examining endomorphisms of free inverse semigroups, which we believe will be an important area for further research.

We assume familiarity with the use of *Munn trees* to represent the elements of a free inverse semigroup FIS(A) over a basis A (see [Law98, Chapter 6] for details). Let φ be an endomorphism of FIS(A). Recall the relation $\equiv$ on FIS(A) defined by $u \equiv v$ if and only if $\mathrm{red}(u) = \mathrm{red}(v)$, where $\mathrm{red}(w)$ stands for the reduced word in the free group FG(A) of the word $w \in$ FIS(A). This relation $\equiv$ is the minimal group congruence of FIS(A) and the factor monoid FIS(A)$/\equiv$ is isomorphic to FG(A). Let $\widehat{\varphi}$ be the induced endomorphism on FG(A). Then of course $\Gamma(\widehat{\varphi}) \leqslant \Gamma(\varphi)$ by Lemma 4.7.

When φ is an endomorpism of a free *monogenic* inverse semigroups we actually have $\Gamma(\widehat{\varphi}) = \Gamma(\varphi)$:

PROPOSITION 6.4. *Let φ be an endomorphism of* FIS(a)*, the free inverse semigroup of rank* 1*. Then*

a) *if $a\varphi$ is an idempotent (equivalently,* $\mathrm{red}(a\varphi) = \varepsilon$*) then $\Gamma(\varphi) = \Gamma(\widehat{\varphi}) = 1$;*
b) *otherwise, $\Gamma(\varphi) = \Gamma(\widehat{\varphi}) = |\mathrm{red}(a\varphi)|_{\{a,a^{-1}\}}$.*

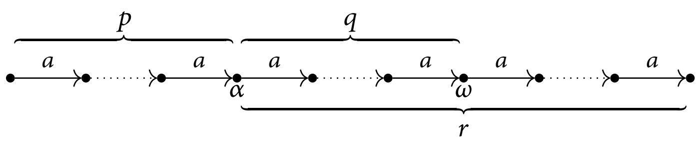
Proof of 6.4. Recall that FIS(a) can be viewed as the set

$$\{\,(p,q,r) : p,q,r \in \mathbb{Z}, p \leqslant 0, r \geqslant 0, p \leqslant q \leqslant r\,\}$$

with multiplication

$$(p,q,r)(p',q',r') = (\min\{p, p' + q\}, q + q', \max\{r, q + r'\}).$$

A tuple (p,q,r) corresponds to the following Munn tree, where p, q, and r record the 'x-coordinates' of, respectively, the left-most vertex, the final vertex ω, and the right-most vertex, with the 'origin' at the initial vertex α:



The generator a is $(0,1,1)$. An element (p,q,r) has inverse $(-r,-q,-p)$. The element (p,q,r) is equal to the product $a^p a^{-p} a^r a^{-r} a^q$ and so $|(p,q,r)| \leqslant 2|p| + |q| + 2|r|$. The image of (p,q,r) in FG(a) is a^q. Idempotents are elements of the form $(p,0,r)$.

a) Suppose $a\varphi$ is an idempotent. Then, $a\varphi$ is of the form $(p,0,r)$. Thus $(a\varphi)^{-1}$ is $(-r,0,-p)$. For $n \geqslant 1$, the element $a\varphi^n$ is a product of $a\varphi$ and $a\varphi^{-1}$. An easy induction shows that $a\varphi^n$ and $a^{-1}\varphi^n$ are triples $(x,0,y)$, where $x \in \{p,-r\}$ and $y \in \{r,-p\}$, and so $a\varphi^n$ and $a^{-1}\varphi^n$ have bounded length over $A \cup A^{-1}$. Hence $\Gamma(\varphi) = 1$. Since $1 \leqslant \Gamma(\widehat{\varphi}) \leqslant \Gamma(\varphi) = 1$, the result follows.
b) Suppose $a\varphi$ is not an idempotent. Then $a\varphi = (p,q,r)$ for some $q \neq 0$. Suppose $q > 0$; the other case is similar. It is easy to see that $(x,y,z)\varphi = (xq+p, yq, zq+p)$; thus, by induction, $a\varphi^n = (q^n p + \ldots + qp + p, q^{n+1}, q^n r +$

$\ldots + qr + r)$. Hence

$$
\begin{aligned}
&K(\varphi^n, \{a, a^{-1}\}, \{a, a^{-1}\}) \\
&= |a\varphi^n| \\
&\leqslant 2|q^n p + \ldots + qp + p| + |q^{n+1}| + 2|q^n r + \ldots + qr + r| \\
&\leqslant 2|q^{n+1} p| + |q^{n+1}| + 2|q^{n+1} r| \\
&\leqslant Cq^n \qquad \text{for a constant } C.
\end{aligned}
$$

Hence $\Gamma(\varphi) \leqslant \lim_{n\to\infty} \sqrt[n]{q^n} = q$. On the other hand,

$$K(\widehat{\varphi}^n, \{a, a^{-1}\}, \{a, a^{-1}\}) = a\widehat{\varphi}^n = a^{q^n},$$

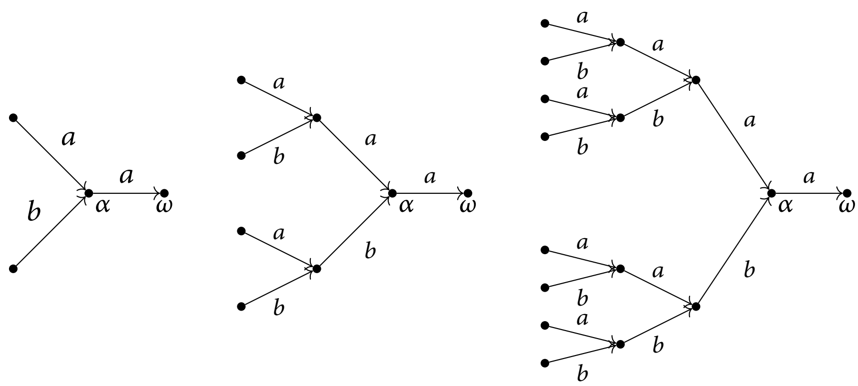
so $\Gamma(\widehat{\varphi}) = q$. Hence $q = \Gamma(\widehat{\varphi}) \leqslant \Gamma(\varphi) \leqslant q$ and so

$$\Gamma(\widehat{\varphi}) = \Gamma(\varphi) = q = |a^q| = |\mathrm{red}(a\varphi)|_{\{a,a^{-1}\}}. \qquad \boxed{6.4}$$

However, in the general case the inequality may be strict and $\Gamma(\varphi)$ may depend strongly on the overlaps between the Munn trees of the elements to which φ maps the generators in A. We provide an example to illustrate: let $A = \{a, b\}$ and define φ by

$$
\begin{aligned}
a &\mapsto a^{-1}ab^{-1}ba \\
b &\mapsto a^{-1}ab^{-1}bb.
\end{aligned}
$$

Then $\widehat{\varphi}$ is the identity map on $FG(a, b)$ and so $\Gamma(\widehat{\varphi}) = 1$. To calculate $\Gamma(\varphi)$, by symmetry it suffices to consider only the iterations of a. The Munn trees of $a\varphi^n$ look like rooted trees: the Munn trees of $a\varphi$, $a\varphi^2$, and $a\varphi^3$ are, respectively:



where α and ω indicate the initial and final vertices of the Munn trees.

For every element $w \in FIS(A)$, let $e(w)$ denote the number of edges in the Munn tree of w. In general, $e(w) \leqslant |w|_{A\cup A^{-1}} \leqslant 2e(w)$ because at least $e(w)$ edges are traversed in a path visiting all vertices of the tree, and at most $2e(w)$ edge-traversals are required to start from α, visit every vertex, and finish at ω.

Clearly, $e(a\varphi^n) = e(b\varphi^n) = 2^{n+1} - 1$. Together with the observations in the previous paragraph, this shows that $2^{n+1} - 1 \leqslant K(\varphi^n, A, A) \leqslant 2^{n+2} - 2$, and so $\Gamma(\varphi) = 2$.

QUESTION 6.5. Is there any formula to calculate the growth of an endomorphism of a free inverse semigroup relative to the growth of the corresponding endomorphism of the free group? Is this growth always an algebraic number?

QUESTION 6.6. Are there connections between growths of endomorphisms of free inverse semigroups and Lindenmayer systems?

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