PHILOSOPHICAL TRANSACTIONS A

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Cite this article: Cain AJ. 2019 Visual thinking and simplicity of proof. *Phil. Trans. R. Soc. A* **377**: 20180032.
http://dx.doi.org/10.1098/rsta.2018.0032

Accepted: 11 September 2018

One contribution of 11 to a theme issue 'The notion of 'simple proof' - Hilbert's 24th problem'.

Subject Areas:
mathematical logic

Keywords:
visual thinking, spatial thinking, diagrams, simplicity of proof

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Visual thinking and simplicity of proof

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This paper studies how spatial thinking interacts with simplicity in [informal] proof, by analysing a set of example proofs mainly concerned with Ferrers diagrams (visual representations of partitions of integers) and comparing them to proofs that do not use spatial thinking. The analysis shows that using diagrams and spatial thinking can contribute to simplicity by (for example) avoiding technical calculations, division into cases, and induction, and creating a more surveyable and explanatory proof (both of which are connected to simplicity). In response to one part of Hilbert's 24th problem, the area between two proofs is explored in one example, showing that between a proof that uses spatial reasoning and one that does not, there is a proof that is less simple yet more impure than either. This has implications for the supposed simplicity of impure proofs.

This article is part of the theme issue 'The notion of 'simple proof' - Hilbert's 24th problem'.

1. Introduction

The use of visual thinking in mathematics—broadly encompassing visual perception and visual imagination—has received substantial attention from philosophers of mathematics. (See, for example, [1–3].) This is unsurprising, given the importance that mathematicians themselves place on visual thinking: famously, Jacques Hadamard wrote of those he surveyed: 'Practically all of them avoid not only the use of mental words but also, just as I do, the mental use of algebraic or any other precise signs; also as in my case, they use vague images' [4, p. 84]. (Notable exceptions are G.H. Hardy, who wrote: 'I think almost entirely in words and formulae... I find thought almost impossible if my hands are cold and I cannot write in comfort' [5, p. 114], and Bertrand Russell,

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who noted, in the context of William James' distinction between the visual mind and auditory mind: 'I never think except in words which I imagine spoken' [6, p. 57].)

Within the broad area of visual thinking, one can distinguish *spatial thinking*, meaning specifically thinking that involves visual images (perceived or imagined) of objects in particular geometric configurations. One could perhaps call it 'geometrical thinking', but this suggests thinking that requires some level of formal geometrical knowledge, and some of the thinking examined here depends only on a 'pre-mathematical' conception of space. Spatial thinking, as the term is used here, does not include the kind of visual thinking involved in symbol manipulation (see, for example [7, ch. 10]) or in visualizing structure such as ordered sets, where the geometric configuration can be varied without affecting the reasoning (see [8]). These kinds of visual thinking, of course, deserve attention, but for reasons of length, this essay considers only spatial thinking.

The epistemological role of spatial thinking and visual thinking generally has been a particular focus of philosophers' attention. In particular, there is the question of how a particular diagram (which may only illustrate a special case) yields knowledge about a general case. (For a discussion, see [9]; note also the aphorism 'Geometry is the art of correct reasoning on incorrect figures' [10, p. 208].) If a proof requires visual thinking, is one justified in accepting the theorem? Visual thinking can be misleading, particularly (but not only) with regard to limits of infinite processes (see, for example [7, pp. 3ff.]).

This paper leaves aside the epistemological controversy and instead focuses on how spatial thinking interacts with simplicity in proof. The motivation includes two aspects of Hilbert's 24th problem (see [11]): 'criteria of simplicity' and 'the area lying between the two routes' (the 'routes' here being proofs, thought of as ways of reaching a 'destination' theorem). Does spatial thinking contribute to simplicity? Many mathematicians seem to think so; witness the famous 'proofs without words' features in *Mathematics Magazine* and *The College Mathematics Journal* and collected in [12,13]. Gardner said of these that 'in many cases a dull proof can be supplemented by a geometric analogue so *simple* and beautiful that the truth of a theorem is almost seen at a single glance' [14, ch. 16; emphasis added].

This essay considers three questions. Two of them arise directly from Hilbert's 24th problem: (1) What criteria of simplicity are satisfied by proofs using spatial thinking but not by other proofs of a result? (Put another way: In what ways does spatial thinking contribute to simplicity?) (2) What can be said about the 'area lying between' two proofs that use and that do not use spatial thinking? The third question arises from the idea, discussed by Arana [15], that impure proofs (that is, proofs that draw on resources that are not close to or intrinsic to the result) are simpler: (3) Is the gain in simplicity that comes from spatial reasoning compatible with the idea that impure proofs are simpler?

The approach is to examine carefully particular proofs that use spatial thinking (for brevity, henceforth 'spatial proofs') to see how spatial thinking contributes to simplicity. All the proofs considered in this paper are informal proofs. Although diagrammatic formal systems have been developed (see, for example [16]), most proofs that employ spatial thinking are informal. Furthermore, simplicity is a property that is more readily identified in informal proofs; as Iemhoff has noted, it seems difficult to determine the extent to which simplicity is preserved under formalization [17, p. 147]. The main (but not sole) focus is what can broadly be called 'pebble diagrams': representing natural numbers using dots, or, as the Pythagoreans did, pebbles. (See [18, pp. 100ff.] or [19, p. 33] for historical background and sources.) Such diagrams are still used in modern mathematics, for instance, in the guise of Ferrers diagrams of partitions (for the definition, see §4 below). There are two reasons for this choice. First, with one exception, the *theorems* discussed do not require any spatial thinking to understand. Second, considering only proofs about discrete quantities may at least mitigate some of the concerns about whether the proofs are legitimate, as these avoid situations where limits of infinite processes play a part. Proofs that do not use spatial thinking (for brevity, 'non-spatial proofs') are sometimes outlined insofar as necessary to compare them with the spatial proofs. (It is perhaps worth noting that Sylvester, author of the paper in which Ferrers diagrams were first published, declared himself perfectly

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satisfied by proofs that use them; he described such a proof as 'so simple and instructive, that I am sure every logician will be delighted to meet with it here or elsewhere' [20, p. 597]. Ferrers diagrams were suggested by Sylvester's correspondent Ferrers; see [21] for a discussion of the origin of the name.) Third, the use of spatial thinking in proving theorems that are not themselves spatial is, prima facie, an impurity.

Detlefsen [22, p. 87] distinguished *verificational* simplicity and *inventional* simplicity of a proof: the former is the simplicity found by a reader who sets out to work through and check the proof; the latter is the simplicity involved in the discovery of the proof in the first place. To this, one can add *explanatorial* simplicity: that is, the simplicity found by the reader who sets out to understand how the proof explains the theorem. The notion of explanation in mathematics and how it compares with an explanation in the natural sciences, is debated (see, for example, [23,24]). In particular, there is disagreement over whether proofs that use induction are explanatory [25,26].

This essay does not consider how proofs are discovered but only examines established proofs from the literature. It is thus concerned only with verificational and explanatorial simplicity. From this perspective, a proof has greater simplicity if it (for instance) avoids technical calculations, complicated notation, division into many cases, 'deus-ex-machina auxiliary functions' [27, pp. 239–240] or (given the debate over explanatoriness mentioned in the last paragraph) induction. Length has also been suggested as a criterion of simplicity; for instance by Thiele & Wos [28] and Arara [15] (but see the countervailing views of Iemhoff [17]), but consideration of length is avoided in what follows, because of the difficulties of comparing lengths of proof (or proof steps) that use diagrams with those that do not. In the course of examining proofs, other contributions to simplicity arise from spatial thinking: simpler auxiliary definitions; simpler inference of properties from those definitions; greater unity; better surveyability; easier exhaustion of cases. This is not an attempt at a systematic taxonomy of criteria of simplicity (or even just of those exhibited by spatial proofs), for a more extensive examination of theorems would likely yield other criteria of simplicity. Rather it is a first foray into the interaction of simplicity and spatial thinking in proof. Simplicity seems to be a multi-faceted condition, and this essay makes no attempt to define it. All the identified criteria are comparative: they concern how a spatial proof is simpler than a non-spatial one.

The main body of the essay is the examination of proofs in §§3–6. In particular, §5 studies proofs of a result connecting a class of inversions and a class of partitions, making a detailed comparison of a spatial and a non-spatial proof. Furthermore, this section also considers Hilbert's remark, in the above-mentioned notes he made on his 24th problem, that, given two routes (that is, proofs), 'it is necessary to investigate the area lying between the two routes' [11, 2]. Between the examined spatial and non-spatial proofs in this section, there lies a proof that is less simple than either (in terms of criteria of simplicity identified elsewhere in the essay). This is interesting in itself, but it also has implications for the relationship between impurity and simplicity as regards spatial and non-spatial proofs, which is the focus of the final §7.

2. Diagrams and spatial thinking

Before proceeding, it is important to note that there is a distinction between spatial thinking and using diagrams. Of course, diagrams are often used in proofs that employ spatial thinking, to illustrate or exemplify that thinking. But it is quite possible for a proof to use spatial thinking and yet involve no actual diagram. Euler's polyhedron formula asserts that for any polyhedron with V vertices, E edges and F faces, the equality $V + E - F = 2$ holds. A standard proof of this begins by transforming the polyhedron into a planar graph by removing one of the faces and deforming the polyhedron by pulling apart the vertices around the missing face. The vertices, edges and faces of the polyhedron become vertices, edges and regions of the planar graph, with the missing face replaced by the unbounded region of the plane around the graph. There are then various ways to proceed to show that the formula holds for finite connected planar graphs. The proof step of deforming the polyhedron to obtain a planar graph appeals to spatial thinking, but no diagram is used. (The validity of proofs beginning thus is part of Lakatos's famous study *Proofs and Refutations* [29]; recall, however, that this essay leaves aside questions of validity.)

In contrast, a proof can use diagrams and not call upon spatial thinking. Standard examples would be 'diagram chasing' proofs of results using commutative diagrams (see, for instance, [30, Lemma VIII.4.1, pp. 202–203]). In such proofs, an element is 'chased' around a diagram of arrows between objects. These arrows represent morphisms, and alternative paths of arrows between different objects correspond to equal compositions of morphisms. Here, the diagram serves in a book-keeping role: it stores in a compact and accessible way information about the compositions of morphisms and allows easy manipulation of this information. In this way, diagrams can certainly contribute to the verificational and possibly the inventional simplicity of proofs. But at no point is spatial thinking actually used: the actual configuration of the diagram in space is unimportant: it could be redrawn in a messy form (say, with arrows unnecessarily criss-crossing and objects scattered arbitrarily), and yet precisely the same sequence of deductions could be drawn from it, albeit in a way that seems less simple in verificational terms. Indeed, the same deductions could be made from a list of equalities of compositions of morphisms, in a still less simple way.

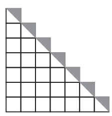
Diagrams of arrows, or diagrams generally, can of course contribute to simplifying proofs. (Indeed, this is the very starting point of category theory, according to MacLane [30, p. 1]). However, further analysis of how diagrams can contribute to simplicity, independently of spatial thinking, is beyond the scope of this essay.

3. Spatial thinking and induction

One of the canonical examples of a 'proof without words' [12, p. 60] and of the use of spatial thinking in proving a numerical result, is of the formula for the sum of the first n integers:

Theorem. $\sum_{i=1}^{n} i = n^2/2 + n/2$.

Spatial proof.



(3.1)

■

Non-spatial proof, abbreviated. Proceed by induction on n:

Base of induction: $$\sum_{i=1}^{1} i = 1 = \frac{1^2}{2} + \frac{1}{2}.$$

Inductive hypothesis: $$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$$

Inductive step: $$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1) = \frac{n^2}{2} + \frac{n}{2} + \frac{2n}{2} + \frac{2}{2} = \frac{(n+1)^2}{2} + \frac{n+1}{2}.$$ ■

How does the spatial proof 'work'? Brown [31, pp. 40–41] notes that mathematicians and philosophers are divided into two roughly equal groups. One group holds that the diagram *encodes* an induction and is thus a legitimate proof. The other group holds that the diagram is a heuristic device that *suggests* an induction and that it is this suggested induction that is a legitimate proof. Brown himself holds that there is no induction present, arguing that 'Some "pictures" ... are windows to Plato's heaven' [31, p. 40], or, less poetically but perhaps more accurately, that 'As telescopes help the unaided eye, so some diagrams are instruments ... which help the unaided mind's eye.' [31, p. 40].

Do these opposing views—encoding or suggesting an induction versus aiding the mind's eye—have implications for simplicity in proof? Let us accept that the picture supplies a valid proof in one of these ways. *With this proviso*, I think that few would disagree that the spatial proof is simpler than the inductive proof.

Let us consider first the possibility that the diagram encodes or suggests an induction. Chihara [32, pp. 302–303] describes the thought-process that the diagram triggers. In abbreviated form, it is as follows: When one examines the diagram, one sees that to go from the nth to the $(n+1)$th row, one has to duplicate the nth row and add another square, and so there are $n+1$ squares in the $(n+1)$th row. Thus, 'an intuitive version of mathematical induction' allows one to see that for any n, the number of squares in the nth row will be n. Therefore, calculating $\sum_{i=1}^{n} i$ requires calculating the total number of squares in the diagram, which is $n^2/2 + n/2$, by the formula for the area of a triangle plus the number of squares supplied by the shaded half-squares.

Note that Chihara's suggested induction does not lead directly to the theorem: rather, it leads to the result that $\sum_{i=1}^{n} i$ is the number of squares in the diagram with n rows. Put another way, Chihara's induction shows that the diagram (3.1) can be extended to any number of rows n without changing the fact that its area is $\sum_{i=1}^{n} i$. It is from this statement that the result is then deduced. By contrast, the non-spatial induction leads directly to the result. Thus, it cannot be the case that greater simplicity of the spatial proof is just because the diagram serves to guide the reader through the induction in a way that is simpler than the non-spatial proof, for the two inductions have different conclusions.

It does not seem relevant whether Chihara's account of the thought-process is seen as extracting an induction encoded in the diagram, or as constructing an induction suggested by it: the point is that *if* the proof is through an induction triggered by the diagram, it must be simpler than the induction in the non-spatial proof. Of course, the spatial proof does not require any notation and is simpler in that respect, but it seems difficult to identify other ways in which it is simpler.

Now return to the possibility suggested by Brown that the diagram aids the mind's eye, without any inductive process. In this case, I suggest that the thought-process is that one immediately perceives that the partially shaded squares form a diagonal, and that two conclusions follow from this immediately: the diagram is n squares wide and n squares high, the particular number n in the actual diagram never being noted. Therefore, the lengths of the rows are $1, 2, \ldots, n$. Calculating $\sum_{i=1}^{n} i$ thus requires calculating the total number of squares in the diagram, which is $n^2/2 + n/2$, by the formula for the area of a triangle plus the number of squares supplied by the shaded half-squares, as in Chihara's description above. As the particular number n in the diagram remained unnoted, this holds for all n.

The simplicity here seems to arise from the fact that the perception of the diagonal is immediate and the conclusions from it are almost built into spatial perception. That is, it is the absence of induction and its replacement with spatial thinking (or, if one prefers, the view of Plato's heaven) that makes the picture proof simpler.

In summary, this discussion has suggested several ways in which spatial reasoning leads to gains in simplicity: first, *decreased complexity* (*or elimination*) *of notation*; second, depending on the position one takes on the triggering of an induction, either an *easier-to-understand induction* or an *avoidance of induction*. However, it could be argued that in the 'non-inductive' thought-process I described above there could be a hidden induction that a mathematically trained reader can do 'without thinking' and that an untrained reader could intuit. A more convincing example of avoidance of induction will be considered in §5 below.

4. Definitions and inferences

This section and the following ones consider spatial proofs involving partitions of numbers, using Ferrers diagrams. Only the background necessary for understanding the proofs below is recalled here; see [33] for further reading.

A *partition* of a natural number (that is, positive integer) n is a finite non-increasing sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ that sums to n; the λ_i are the *parts* of the partition. For example, a partition of 24 is $(8, 6, 6, 3, 1)$. A *Ferrers diagram* or *Ferrers graph* is a visual representation of such a partition as an array of dots, left-aligned, with the ith row (counting from the top) containing λ_i dots. For example, the following Ferrers diagram represents the partition $(8, 6, 6, 3, 1)$:

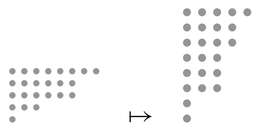
As a partition is a non-increasing sequence, the rows of a Ferrers diagram are non-increasing in length from top to bottom, and a left-aligned array of dots forms a Ferrers diagram (that is, represents a partition) precisely when this condition is satisfied.

Finite non-increasing sequences of natural numbers and Ferrers diagrams are thus alternative representations of partitions. As Starikova [34] has pointed out, alternative representations of mathematical objects can certainly ease the doing of mathematics, not just in proof, but in the discovery and in the formulation of new concepts.

A fundamental result in partition theory, due to Euler, is the following:

Theorem ([33, Theorem 1.4]). *The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.*

Spatial proof. Reflect the Ferrers diagram of a partition along its main diagonal:


(4.1)

This gives a bijection between the two classes of partitions, because the non-increasing lengths of rows from top to bottom require that the columns have non-decreasing lengths from left to right. ■

Note that the spatial proof uses the only reflection: this is a notion that is not dependent on having any geometrical knowledge but is rather an appeal to basic ideas about space.

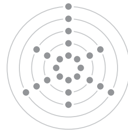
This bijection could be defined non-spatially as

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \mapsto (\mu_1, \mu_2, \ldots, \mu_p), \quad \begin{cases} \text{where } \mu_i \text{ is the number of} \\ \lambda_j \text{ that are greater than } i. \end{cases} \tag{4.2}$$

However, this definition is less simple. First, the reader grasps the spatial definition 'reflecting along the diagonal' immediately, whereas one has to pause and think to understand (4.2). Second, there are two inferences that are immediate from the spatial definition, but require some thought to deduce from (4.2). The first inference is that the map is well-defined, in the sense that the image is a partition, and indeed a partition *of the same natural number n*. This is not *difficult* to prove from (4.2), but is trivial from the spatial definition, as reflection preserves both the non-increasing lengths of lines and the total number of dots. The second inference is that the map is a bijection (and indeed an involution): trivial from the spatial definition, but requiring some work to prove from (4.2).

This is another gain in simplicity from spatial reasoning: *simpler definitions of transformations using visual representations*, and *simpler inference of properties of those transformations* (such as bijectivity). Note that it is the particular choice of visual representation using Ferrers diagrams that enables the definition of the bijection using reflection in (4.1). Different visual representations are possible, such as representing the partition (8, 6, 6, 3, 1) as


or

or

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but none of these representations gives rise to such a simple definition of the bijection as (4.1), nor to such simple inference of its properties.

5. Ferrers diagrams in a more spatial proof

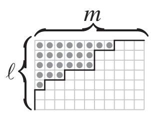
Let $\ell, m, n \in \mathbb{N}$. Let $\xi_1\xi_2\cdots\xi_{\ell+m}$ be a permutation of $1^\ell 2^m$. An *inversion* in this permutation is a pair (i, j) such that $i < j$ and $\xi_i = 2$ and $\xi_j = 1$. (That is, a symbol 1 after a symbol 2.) Let $\mathrm{Inv}(m, \ell; n)$ denote the set of permutations $\xi_1\xi_2 \ldots x_{\ell+m}$ of $1^m 2^\ell$ that have exactly n inversions. Let $P(m, \ell; n)$ denote the set of partitions of n into at most ℓ parts, each at most m.

Theorem ([33, Theorem 3.5]). $|P(\ell, m; n)| = |\mathrm{Inv}(\ell, m; n)|$.

The following proof is a modified version of [33, Proof of Theorem 3.5].

Spatial proof. The aim is to define an explicit bijection between the set of partitions $P(m, \ell; n)$ and the set of permutations $\mathrm{Inv}(m, \ell; n)$.

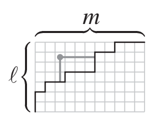
Consider a partition of n with at most ℓ parts, each at most m. Place an $m \times \ell$ grid around its Ferrers diagram and 'wrap' the diagram with a path following the grid, as follows:



(5.1)

Trace the steps made by the path through the grid starting at the upper-right and moving leftwards and downwards. If the path makes a downwards step, write 2. If it makes a leftward step, write 1. The resulting sequence contains m symbols 1 and ℓ symbols 2, and so is a permutation of $1^m 2^\ell$.

Note that there is a one-to-one correspondence between dots of the Ferrers diagram and symbols 1 (leftward steps) after symbols 2 (downward steps), as these leftward steps and downward steps, respectively, determine a column and a row and thus a unique dot in the Ferrers diagram, and vice versa:



(5.2)

So there are exactly n inversions in this permutation.

As there is a one-to-one correspondence between partitions and paths, this defines a bijection between the permutations of $1^m 2^\ell$ with exactly n inversions and the number of partitions of n with at most ℓ parts, each of size at most m. This completes the proof. ∎

(a) Comparison of simplicity relative to a non-spatial proof

Spatial thinking is used at the following steps of the proof:

(S1) The definition of the path that wraps a Ferrers diagram of the partition. (Note that the definition is incomplete without the diagram. In particular, the reader is expected to understand the meaning of 'wrap' directly from the diagram.)

(S2) The definition of the sequence determined by the path, which uses the spatial idea of travelling along the path, and of making leftward or downward steps.

(S3) Deduction that this sequence is a permutation of 1^m2^ℓ, which is immediate from the correspondence of symbols 1 and 2 to (respectively) leftward and downward steps, and the horizontal and vertical distance that the path must cover.

(S4) Deduction that there is a one-to-one correspondence between inversions in the permutation and points of n, which follows from the spatial arrangement of the dots in the Ferrers diagram.

(S5) Deduction that there is a one-to-one correspondence between partitions and paths, which is immediate because the Ferrers diagram is seen to precisely fill the path, so that a path determines the corresponding Ferrers diagram and vice versa.

A non-spatial syntactic proof could proceed (in outline) as follows:

(N1) Definition of a map ϕ taking partitions to sequences, given by

$$\phi(\lambda) = 1^{m-\lambda_1}21^{\lambda_1-\lambda_2}21^{\lambda_2-\lambda_3}2\cdots 1^{\lambda_{k-1}-\lambda_k}21^{\lambda_k}2^{\ell-k}. \tag{5.3}$$

(N2) Deduction that $\phi(\lambda)$ is a permutation of 1^m2^ℓ, which requires an explicit calculation of the total number of symbols 1 and 2 in the sequence defined by (5.3).

(N3) Deduction that $\phi(\lambda)$ contains exactly n inversions, which again requires an explicit calculation using the definition (5.3).

(N4) Deduction that there is a one-to-one correspondence between $P(m,\ell;n)$ and $\mathrm{Inv}(m,\ell;n)$:

- (N4a) Proof of injectivity: given $\phi(\lambda)$, the tail of symbols 2 at the end determines k; the string of symbols 1 immediately to the right of the kth symbol 2 determines λ_k; by induction, knowing λ_{i+1}, the string of symbols 1 immediately to the right of the ith symbol 2 determines λ_i.
- (N4b) Proving that the image of $P(m,k;n)$ is $\mathrm{Inv}(m,k;n)$: a notationally messy argument producing a partition mapping onto a given permuation of 1^m2^ℓ with n inversions.

Note that there is a broad correspondence in overall structure between the given spatial proof and the non-spatial proof outlined above:

$$\left.\begin{matrix}\text{(S1)}\\ \text{(S2)}\end{matrix}\right\} \longleftrightarrow \text{(N1)},\quad \text{(S3)} \longleftrightarrow \text{(N2)},\quad \text{(S4)} \longleftrightarrow \text{(N3)},\quad \text{(S5)} \longleftrightarrow \text{(N4)}\left\{\begin{matrix}\text{(N4a)}\\ \text{(N4b)}\end{matrix}\right. .$$

Considered on this level of overall structure, therefore, there is little difference in the simplicity of the two proofs. However, within corresponding steps, there are major differences in simplicity. The definition (N1) is notationally less simple than (S1)–(S2), and in particular, in (N1) it has been necessary to introduce notation for a partition and its parts. Steps (S3) and (S4) seem simpler than (N2) and (N3): the former are immediate on making certain spatial observations, while the latter require explicit (though admittedly not difficult) calculations. Most importantly, step (S5) is simpler, again being immediate, but (N4) in particular uses an induction.

There seem to be two gains in simplicity from spatial thinking here: *simpler definitions of maps using spatial representations*, and *simpler inference of properties of those maps*. This is to be distinguished from the gains in simplicity related to transformations discussed in §4. Here, there is greater simplicity even though the spatial representation is being used to define, not another spatial representation of [a mathematical object of] the same species, but a sequence.

The contrast between the use of induction in step (N4) but nowhere in the spatial proof requires further attention as, as discussed above in §3, there is disagreement over whether the diagram (3.1) triggers an induction (whether by encoding or suggestion). It is thus necessary to consider the possibility that, at least for some readers, (5.1) triggers an induction.

There is an obvious contrast with (3.1), because the increasing length of rows in (3.1) corresponds to the statement one wishes to deduce. That is, the spatial perception (of how additional rows increasing in length) parallels the induction (on the number of rows). There seems no such obvious correspondence in (5.1). At this point, the proof considers the Ferrers diagram of a partition of n with at most ℓ parts, each at most m. An induction on ℓ and/or m corresponds to extending the grid to the right or downwards. Certainly, the reader is called upon to visualize an $m \times \ell$ grid for arbitrary ℓ and m, but no conclusion is drawn directly from this in the way that the conclusion about lengths of rows is drawn in (5.1). One could consider induction on n, or the number of parts, or on the lengths of the parts. But, this would involve assuming the existence of a unique path wrapping around the Ferrers diagram and satisfying certain properties, then adding one or more dots and deducing the existence of a unique path wrapping around the diagram and satisfying certain other properties. (One hesitates to try to write down such an induction.) However, this seems very far from the spatial thinking of wrapping the Ferrers diagram. So, this seems to be a rather more certain instance of spatial thinking leading to a gain in simplicity by *avoidance of induction* than the proof discussed in §3.

Returning to the relative simplicity of the two proofs: the spatial proof is also simpler because it has *greater unity*: the definition of the path and all but one deduction step appeal to spatial thinking by calling on the same diagram. The remaining deductive step appeals to spatial thinking about a second diagram that is very closely related to the first. One could even combine the diagrams, but this requires using a diagram for (S1)–(S3) that contains information that is not relevant until (S4). This would arguably be a decrease in simplicity; it certainly seems to be a decrease in elegance. This increased unity is perhaps linked with *better surveyability* of the proof: the unity increases the ease with which '[t]he mathematician *surveys* the proof in its entirety and thereby comes to *know* the conclusion' [35, p. 59; emphases in original].

(b) Between the proofs

In (S1), the path is defined by example and by appealing to the spatial idea of wrapping around the Ferrers diagram. One could avoid this and use an explicit definition:

(T1) For the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ the path is the map $\pi_\lambda : [0,1] \to \mathbb{R}^2$ defined by…,

giving explicit coordinates for the images of each point $\pi_\lambda(t)$. This still defines the path as an object in the Euclidean plane, but without requiring spatial thinking of the reader. (Although many readers would doubtless try to visualize the definition.) Is (T1) more or less simple than (S1)? On the one hand, (T1) has introduced a new concept, in the form of an explicit coordinate system; on the other hand, it has eliminated a concept, in the form of the Ferrers diagram.

There could conceivably be many proofs beginning with (T1): here only one is considered. Suppose the next steps are as follows:

(T2) Definition of a sequence from the path π_λ.
(T3) Deduction of what the sequence is in terms of λ.

The conclusion of (T3) would be (5.3); the proof could then proceed via (N2)–(N4). Now, (T1) was an attempt to explicitly define the path used in (S1) without using spatial thinking. Thus the proof (T1)–(T3), (N2)–(N4) can be thought of as between the spatial proof (S1)–(S5) and the non-spatial proof (N1)–(N4). However, it is certainly less simple than either the spatial or the non-spatial proofs. It shares the explicit calculations and induction of the non-spatial proof, and yet it uses the path π_λ to do what (N1) does by definition.

Thus, a naïve attempt to 'de-spatialize' the original spatial proof may result in a decrease in simplicity, even if there is another non-spatial proof that is of greater simplicity in than the resulting 'de-spatialized' one. Of course, in this particular case, a hypothetical mathematician who had formulated (T1)–(T3) would doubtless note that all mention of the paths π_λ could be eliminated and would obtain the non-spatial proof (N1)–(N4). The point is that there is a third proof (T1)–(T3), (N2)–(N4) that lies between the spatial and non-spatial proofs and is less simple than either. Section 7 considers the implications of this for the suggested connection between simplicity and impurity in proofs.

6. Spatial thinking and exhaustion of cases

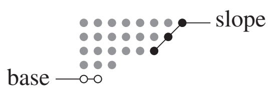
Euler's pentagonal number theorem is the following identity:

Theorem.

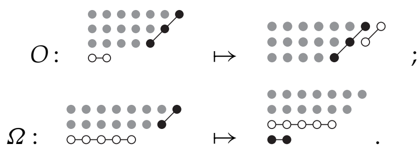
$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=0}^{\infty} c_n x^n, \quad \textit{where } c_n = \begin{cases} 0 & \textit{if } n \neq \dfrac{k(3k \pm 1)}{2}, \\ (-1)^k & \textit{if } n = \dfrac{k(3k \pm 1)}{2}. \end{cases}$$

It is related to partitions because it can be shown that $c_n = p_e(n) - p_o(n)$, where $p_e(n)$ and $p_o(n)$ are, respectively, the numbers of partitions of n into an even number and into an odd number of unequal parts. The combinatorial proof by Franklin [36], as explained by Hardy [37, §6.2] and Hardy & Wright [38, §19.11], uses transformations of Ferrers diagrams to establish a near-correspondence between partitions of n into an even number and into an odd number of unequal parts, with exact correspondence or an excess of one partition of each type giving rise to the cases where $p_e(n) - p_o(n)$ is 0, 1, or −1.

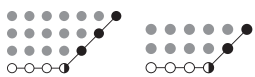
Spatial proof, key steps only. Define the *base* to be the bottommost row of the diagram, and define the *slope* to be the longest line of dots starting at the top-rightmost and moving towards the bottom-left, as in the following diagram:



Define partial transformations O and Ω by (respectively) moving the base to lie parallel to the slope, provided that the result is a Ferrers diagram with unequal parts, and moving the slope to form a new base, provided that the result is a Ferrers diagram with unequal parts:



Clearly, O and Ω are mutually inverse and so are partial bijections. The key to the proof is establishing that one of O or Ω is defined except when the base and the slope meet and when the lengths of the base and slope are equal or when the length of the base exceeds the length of the slope by 1:



(6.1)

as it is precisely in these cases that for both O and Ω, attempting the transformations yields a configuration of dots that does not form a Ferrers diagram with unequal parts.

The proof concludes by showing that $n = k(3k \pm 1)/2$, with an excess of one partition of odd or even type depending on the parity of k. ■

Here, the key role played by spatial thinking is in realizing that O or Ω is defined except in the specified cases. It is unnecessary to introduce notation or explicitly enumerate the cases for lengths of the base and slope. That is, spatial reasoning allows *easier exhaustion of cases*. Furthermore, it is spatial thinking that makes it possible to use the condition 'provided that the result is a Ferrers diagram with unequal parts'. Giving a non-spatial definition of the partial transformations O and Ω would require first defining the length of the base and the slope of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with unequal parts: the former is just λ_k, but the length of the slope is the maximum i such that $i \leq k$ and $\lambda_1, \ldots, \lambda_i$ is a sequence of consecutive decreasing natural numbers. Then the map O would be defined when λ_k is less than this i. That is, preliminary technical definitions would be required before the map O can be defined. Thus, we have an even stronger instance of how spatial thinking can allow a simpler definition of a [partial] transformation, as discussed in §4 above. Here, there is a greater contrast between the simplicity of the spatial and non-spatial definitions, even though the transformation itself is more complicated than (4.1).

(In his original proof, Franklin [36, pp. 448–449] does not attempt a general definition, but only describes in text the transformation O when the base contains 1, 2 or 3 elements, and remarks that this can be extended step by step to all partitions, excepting those 'in which the indicated process is not applicable' [36, p. 449; my translation]. As Franklin does not give the conditions under which the transformation O is defined, even without proof, I suspect that his text is actually describing a process that he was visualizing using Ferrers diagrams or some similar representation. Ferrers diagrams were first published by Sylvester in 1853 [20]. Franklin's proof was published in 1881, so it is likely he was acquainted with Ferrers diagrams. Furthermore, Sylvester [20, p. 597] speculates that Euler used Ferrers diagrams, so the idea may have been 'in the air' for much longer. However, this is no more than a suspicion.)

7. Spatial thinking, purity and simplicity

Arana [15, pp. 209–211] has pointed out that, in many instances, impure proofs are held to have greater simplicity. Impure proofs are those that establish a theorem by calling upon a different area of mathematics: for example, Dirichlet's theorem that for any coprime natural numbers a and d there are infinitely many primes of the form $a + nd$, a result whose statement is part of elementary number theory, was first proved by analysis. (For an overview of the notion of purity in proof, see [39].)

In the case studies concerning partitions above, the use of Ferrers diagrams seems to introduce impurity. Ferrers diagrams are more than simply another form of notation like sequences $(\lambda_1, \ldots, \lambda_k)$. They are more than just a book-keeping device. They are a vehicle that enables spatial thinking about partitions, and it is this spatial thinking that forms the impurity and permits the simpler proofs. This is perhaps especially evident in the proof in §5: this is a statement about the equality of the numbers of elements in a particular set of partitions and a particular set of permutations. The proof (S1)–(S5) uses the Ferrers diagram spatial representation of partitions, a correspondence between paths and permutations, and the spatial notion of wrapping the latter around the former, and the determination of rows and columns by steps of the path. These spatial notions are introduced to prove a purely combinatorial result; the proof is thus impure. But, the non-spatial proof (N1)–(N4) is pure.

However, there is a difference in the kind of impurity in, say, the proof of Dirichlet's theorem when compared with the spatial proofs examined above. The proof of Dirichlet's theorem is impure because, in Arana's terminology [15, pp. 207–208], it is both *elementally distant* (that is, it draws on resources that are more advanced and complicated than the theorem itself) and *topically distant* (that is, it draws on concepts that are not part of the content of the theorem). The spatial proofs discussed in this essay are certainly topically distant, given that they use spatial thinking to prove results about number theory, combinatorics or algebra. But they are *not* elementally distant. The spatial thinking that is used in them is neither advanced nor complicated. In particular, with one exception, none of the spatial thinking used is geometrical, in the sense of being imported from geometry. The one exception is the formula for the area of the triangle used in §3. The proofs involving Ferrers diagrams use reflection, 'wrapping' paths and moving parts of the diagram around: these are informal notions from pre-mathematical spatial thinking rather than from geometry as a systematized field.

But the discussion of the third proof (T1)–(T3), (N2)–(N4) in §5 complicates the posited relationship between impurity and simplicity. This third proof is not topically closer to the theorem than the spatial proof (S1)–(S5), as it involves the concepts of the plane and paths in the plane. Unlike the spatial proof, the third proof (T1)–(T3), (N2)–(N4) seems to be elementally further from the theorem than the spatial proof (S1)–(S5) or the non-spatial proof (N1)–(N4), because it draws on the formal definition of a path as a map from an interval to the plane, and the formal identification of the plane with the set of pairs of real numbers $\mathbb{R}^2$. Thus, the third proof (T1)–(T3), (N2)–(N4) is less simple than the spatial proof (S1)–(S5) or the non-spatial proof (N1)–(N4), but is also more impure, which goes against the idea that impure proofs have greater simplicity.

Of course, the proof (T1)–(T3), (N2)–(N4) is in some sense an 'unnatural' proof that was found by naïve despatialization of the proof (S1)–(S5) as part of the exploration of 'the area lying between the two routes', as Hilbert proposed. (This is not the place to analyse what a 'natural' proof is; let us take it to mean a proof that practising mathematicians would be likely to discover, publish and read.) It is perhaps to be expected that exploring the area between two natural proofs would often throw up unnatural proofs. But the argument above has shown that, at least when making a Hilbertian exploration like this, the connection between impurity and greater simplicity cannot be assumed to hold.

Data accessibility. This article has no additional data.
Competing interests. I declare I have no competing interests.
Funding. The author was supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through an Investigador FCT research fellowship (IF/01622/2013/CP1161/CT0001). This work was partially supported by the Fundação para a Ciência e a Tecnologia through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações) and the projects PTDC/MHC-FIL/2583/2014 and PTDC/MAT-PUR/31174/2017.
Acknowledgements. The author thanks the two anonymous referees for their careful reading of the paper and many constructive criticisms and suggestions.

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