A C^*-algebra of Singular Integral Operators with Shifts and Piecewise Quasicontinuous Coefficients

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Abstract. The C^*-algebra $\mathfrak{B}$ of bounded linear operators on the space $L^2(\mathbb{T})$, which is generated by all multiplication operators by piecewise quasicontinuous functions, by the Cauchy singular integral operator and by the range of a unitary representation of a group G of orientation-preserving diffeomorphisms of $\mathbb{T}$ onto itself that have the same finite set of fixed points for all $g \in G \setminus \{e\}$, is studied. A Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ and a Fredholm criterion for the operators $B \in \mathfrak{B}$ are established by using spectral measures and the local-trajectory method for studying C^*-algebras associated with C^*-dynamical systems.

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1. Introduction

Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$, by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators in $\mathcal{B}(\mathcal{H})$, and by $I \in \mathcal{B}(\mathcal{H})$ the identity operator on $\mathcal{H}$. If $S, T \in \mathcal{B}(\mathcal{H})$ and $S - T \in \mathcal{K}(\mathcal{H})$, we say that the operators S and T are equivalent and write $S \simeq T$. For an operator $A \in \mathcal{B}(\mathcal{H})$, we denote by $A^\pi := A + \mathcal{K}(\mathcal{H})$ the coset of A in

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the Calkin algebra $\mathcal{B}^\pi(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and by $|A|$ the essential norm of A, that is,

$$|A| = \|A^\pi\| = \inf\{\|A+K\| : K \in \mathcal{K}(\mathcal{H})\}.$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is Fredholm on a Hilbert space $\mathcal{H}$ if and only if the coset A^π is invertible in the C^*-algebra $\mathcal{B}^\pi(\mathcal{H})$. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \cong \mathcal{B}$ if these C^*-algebras are *-isomorphic and hence isometric.

Let $\mathcal{K} := \mathcal{K}(L^2(\mathbb{T}))$ be the ideal of compact operators acting on the Lebesgue space $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ with the length measure and anticlockwise orientation, and let $PQC(\mathbb{T})$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the quasicontinuous functions, $QC(\mathbb{T})$, and by the piecewise continuous functions, $PC(\mathbb{T})$, on $\mathbb{T}$. Consider the C^*-algebra

$$\mathfrak{A} := \mathrm{alg}(PQC(\mathbb{T}), S_\mathbb{T}) \subset \mathcal{B}(L^2(\mathbb{T})) \tag{1.1}$$

generated by all the multiplication operators aI with $a \in PQC(\mathbb{T})$ and by the Cauchy singular integral operator $S_\mathbb{T}$ defined by

$$(S_\mathbb{T}\varphi)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int_{\mathbb{T}\setminus\mathbb{T}(t,\varepsilon)} \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

where $\mathbb{T}(t,\varepsilon) = \{\tau \in \mathbb{T} : |\tau - t| < \varepsilon\}, \quad t \in \mathbb{T}$.

Let G be a discrete group of orientation-preserving diffeomorphisms of $\mathbb{T}$ onto itself that have the same finite set $\Lambda \subset \mathbb{T}$ of fixed points for all $g \in G \setminus \{e\}$, where e is the unit of G and the group operation is given by $(gh)(t) = h(g(t))$ for all $g, h \in G$ and all $t \in \mathbb{T}$. In that case the group G is commutative (see [24, p. 284] and [10, § 2]) and therefore amenable [17]. Then the group G acts on the contour $\mathbb{T}$ *topologically freely* [1], that is, for each finite set $F \subset G$ and each open arc $\gamma \subset \mathbb{T}$ there exists a point $t \in \gamma$ such that the points $g(t)$ for $g \in F$ are pairwise distinct.

The aim of this paper is to construct a Fredholm symbol calculus (in other words, a faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$ in a Hilbert space) and to establish a Fredholm criterion for the operators in the nonlocal C^*-subalgebra

$$\mathfrak{B} := \mathrm{alg}(\mathfrak{A}, U_G) = \mathrm{alg}(PQC(\mathbb{T}), S_\mathbb{T}, U_G) \tag{1.2}$$

of $\mathcal{B}(L^2(\mathbb{T}))$ generated by all singular integral operators $A \in \mathfrak{A}$ and by all unitary weighted shift operators in the group $U_G := \{U_g : g \in G\}$, where

$$(U_g\varphi)(t) := |g'(t)|^{1/2}\varphi(g(t)) \quad \text{for} \quad t \in \mathbb{T}. \tag{1.3}$$

To study the C^*-algebra $\mathfrak{B}$, we apply the local-trajectory method combined with using suitable spectral measures (see [18], [19] and [5]). Other approaches for studying nonlocal operator algebras see in [1]–[3].

The C^*-algebra $\mathfrak{B}$ in the case of piecewise slowly oscillating coefficients, a subset of the $PQC(\mathbb{T})$ coefficients, was studied in [5] (see also [6]–[9] for other classes of groups G with more complicate actions). The C^*-algebra $\mathfrak{B}$ with $PQC(\mathbb{T})$ coefficients was studied in [11] in the case of groups G acting freely on $\mathbb{T}$, making use of the local-trajectory method. In the present paper we assume that the shifts $g \in G$ have the same finite set Λ of fixed points, which essentially complicates the study of the C^*-algebra (1.2) because the action of the group G on the maximal ideal space of a central subalgebra $\mathcal{Z}^\pi$ of the quotient C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$ of $\mathfrak{A}$ given by (1.1) stops to be topologically free. To overcome arising difficulties, we need to apply the technique of spectral measures combined with the local-trajectory method and to use deep properties of quasicontinuous functions investigated in [25] and [26].

The paper is organized as follows. In Section 2, following [25]–[26], we consider the C^*-algebra $PQC(\mathbb{T})$ of piecewise quasicontinuous functions, collect their properties and describe the fibers and the Gelfand topology on the maximal ideal space $M(PQC(\mathbb{T}))$ of $PQC(\mathbb{T})$. Section 3 contains main results of the paper: the Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ and a Fredholm criterion for the operators $B \in \mathfrak{B}$. In Section 4 we recall the local-trajectory method and its generalization on the basis of spectral measures. In Section 5, following [26] and [14], we describe the Fredholm symbol calculus for the C^*-algebra (1.1) and study the central C^*-algebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ generated by the cosets $(aI)^\pi$ with $a \in QC(\mathbb{T})$ and by the cosets $H^\pi_{P,t}$ related to singular integral operators with fixed singularities at points $t \in \mathbb{T}$. We describe here the maximal ideal space $M(\mathcal{Z}^\pi)$ of $\mathcal{Z}^\pi$ and its Gelfand transform, and calculate symbols of important operators $U_g V_t \in \mathfrak{A}$ for $t \in \mathbb{T}$, where the operators U_g are related to shifts $g \in G$ with a fixed point at t and the operator V_t has a fixed singularity at t.

In Section 6 we construct a spectral decomposition of the quotient C^*-algebra $\mathfrak{B}^\pi$ into the orthogonal sum of three operator C^*-subalgebras $\mathfrak{B}_{arc}$, $\mathfrak{B}^\circ_\Lambda$ and $\mathfrak{B}^\diamond_\Lambda$ invariant with respect to the action of the group G on $M(\mathcal{Z}^\pi)$, and present an abstract Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of invertibility of their images B_{arc}, B°_Λ and $B^\diamond_\Lambda$ in the C^*-algebras $\mathfrak{B}_{arc}$, $\mathfrak{B}^\circ_\Lambda$ and $\mathfrak{B}^\diamond_\Lambda$, respectively. In Section 7 we study the invertibility in the C^*-algebra $\mathfrak{A}_{arc} \subset \mathfrak{B}_{arc}$. Applying the local-trajectory method and the results of Section 7, in Section 8 we obtain the invertibility criterion for the operators $B_{arc} \in \mathfrak{B}_{arc}$. In Section 9 we elaborate the invertibility criterion for the operators $B^\circ_\Lambda \in \mathfrak{B}^\circ_\Lambda$. Section 10 is devoted to studying the invertibility of functional operators with piecewise quasicontinuous coefficients and shifts in the group G. Finally, in Section 11 we prove that the invertibility of the operators $B_{arc} \in \mathfrak{B}_{arc}$ implies the invertibility of the operators $B^\diamond_\Lambda \in \mathfrak{B}^\diamond_\Lambda$. Combining the results of Sections 8–11, we obtain the Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols (see Section 3).

2. The C^*-algebra $PQC(\mathbb{T})$

Let $L^\infty(\mathbb{T})$ be the C^*-algebra of all bounded measurable functions on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Let $C(\mathbb{T})$ and $PC(\mathbb{T})$ denote the C^*-subalgebras of $L^\infty(\mathbb{T})$ consisting, respectively, of all continuous functions on $\mathbb{T}$ and all piecewise continuous functions on $\mathbb{T}$, that is, the functions having finite one-sided limits at each point $t \in \mathbb{T}$.

For a Lebesgue measurable set $E \subset \mathbb{T}$, we will denote by $|E|$ its measure, $|E| := \int_E dm$. For each subarc I of $\mathbb{T}$ and each function $f \in L^1(\mathbb{T})$, the *average* of f over I is given by

$$I(f) := \frac{1}{|I|} \int_I f\, dm.$$

Taking $\delta > 0$ and considering arcs $I \subset \mathbb{T}$, we let

$$M_\delta(f) := \sup_{I \subset \mathbb{T},\, |I| \le \delta} \frac{1}{|I|} \int_I |f(t) - I(f)|\, dm(t).$$

Obviously, $0 \le M_{\delta_1}(f) \le M_{\delta_2}(f)$ if $0 < \delta_1 \le \delta_2 \le 2\pi$. Let

$$M_0(f) := \lim_{\delta \to 0} M_\delta(f).$$

According to [25], a function $f \in L^1(\mathbb{T})$ is said to have *vanishing mean oscillation* on $\mathbb{T}$ if $M_0(f) = 0$. The set of functions of vanishing mean oscillation on $\mathbb{T}$ is denoted by $VMO(\mathbb{T})$.

Let $H^\infty(\mathbb{D})$ denote the Hardy space of bounded analytic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let H^∞ denote the set of all functions in $L^\infty(\mathbb{T})$ that are non-tangential limits on $\mathbb{T}$ of functions in $H^\infty(\mathbb{D})$. Equivalently, H^∞ consists of the functions $f \in L^\infty(\mathbb{T})$ for which the Fourier coefficients f_n defined for $n \in \mathbb{Z}$ by

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

equal zero for all $n < 0$.

The C^*-subalgebra of $L^\infty(\mathbb{T})$ defined by

$$QC(\mathbb{T}) := (C(\mathbb{T}) + H^\infty) \cap (C(\mathbb{T}) + \overline{H^\infty})$$

is referred to as the C^*-algebra of quasicontinuous functions. By [25],

$$QC(\mathbb{T}) = VMO(\mathbb{T}) \cap L^\infty(\mathbb{T}).$$

Let $PQC(\mathbb{T}) := \mathrm{alg}(QC(\mathbb{T}), PC(\mathbb{T}))$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the C^*-algebras $QC(\mathbb{T})$ and $PC(\mathbb{T})$. The functions in $PQC(\mathbb{T})$ are referred to as the piecewise quasicontinuous functions.

It is known that the maximal ideal space of $C(\mathbb{T})$ and $PC(\mathbb{T})$ can be identified, respectively, with $\mathbb{T}$ and $\mathbb{T} \times \{0, 1\}$,

$$M(C(\mathbb{T})) = \mathbb{T}, \quad M(PC(\mathbb{T})) = \mathbb{T} \times \{0, 1\},$$

where the points $t \in \mathbb{T}$ are identified with the evaluation functionals δ_t given for $f \in C(\mathbb{T})$ by $\delta_t(f) = f(t)$, and the pairs $(t, 0)$ and $(t, 1)$ are the multiplicative linear functionals defined for $a \in PC(\mathbb{T})$ by $(t, 0)a = a(t-0)$ and $(t, 1)a = a(t+0)$, where $a(t-0)$ and $a(t+0)$ are the left and right one-sided limits of a at the point $t \in \mathbb{T}$. The base of open sets on $\mathbb{T} \times \{0, 1\}$ consists of all sets of the form

$$(t, \tau) \times \{0, 1\}, \quad ((t, \tau] \times \{0\}) \cup ((t, \tau) \times \{1\}), \quad ((t, \tau) \times \{0\}) \cup ([t, \tau) \times \{1\}),$$

where $t, \tau \in \mathbb{T}$. Since $C(\mathbb{T}) \subset QC(\mathbb{T}) \subset PQC(\mathbb{T})$, it follows from [26] that

$$M(QC(\mathbb{T})) = \bigcup_{t\in\mathbb{T}} M_t(QC(\mathbb{T})), \quad M(PQC(\mathbb{T})) = \bigcup_{\xi\in M(QC(\mathbb{T}))} M_\xi(PQC(\mathbb{T})), \tag{2.1}$$

where the corresponding fibers are given for $t \in \mathbb{T}$ and $\xi \in M(QC(\mathbb{T}))$ by

$$M_t(QC(\mathbb{T})) = \{\xi \in M(QC(\mathbb{T})) : \xi|_{C(\mathbb{T})} = t\},$$
$$M_\xi(PQC(\mathbb{T})) = \{y \in M(PQC(\mathbb{T})) : y|_{QC(\mathbb{T})} = \xi\}.$$

For a function $f \in L^1(\mathbb{T})$ and points $t = e^{i\theta} \in \mathbb{T}$ and $\lambda \in (1, \infty)$, put

$$\delta_{(\lambda,t)}(f) = \frac{\lambda}{2\pi} \int_{\theta-\frac{\pi}{\lambda}}^{\theta+\frac{\pi}{\lambda}} f(e^{ix})dx.$$

Observe that, for each $(\lambda, t) \in (1, \infty) \times \mathbb{T}$, the map

$$\delta_{(\lambda,t)} : QC(\mathbb{T}) \to \mathbb{C}, \quad a \mapsto \delta_{(\lambda,t)}(a),$$

defines a linear functional in $QC(\mathbb{T})^*$ which allows one to identify $(\lambda, t) \in (1, \infty) \times \mathbb{T}$ with a subset of $QC(\mathbb{T})^*$. Let $M_t^0(QC(\mathbb{T}))$ denote the set of functionals in $M_t(QC(\mathbb{T}))$ that lie in the weak-star closure of $(1, \infty) \times \{t\}$,

$$M_t^0(QC(\mathbb{T})) := M_t(QC(\mathbb{T})) \cap \operatorname{clos}_{QC(\mathbb{T})^*} ((1, \infty) \times \{t\}).$$

For each $t \in \mathbb{T}$, we also consider the sets

$$M_t^+(QC(\mathbb{T})) := \{\xi \in M_t(QC(\mathbb{T})) : \xi(f) = 0 \text{ whenever } f \in QC \text{ and } \limsup_{z\to t^+} |f(z)| = 0\},$$
$$M_t^-(QC(\mathbb{T})) := \{\xi \in M_t(QC(\mathbb{T})) : \xi(f) = 0 \text{ whenever } f \in QC \text{ and } \limsup_{z\to t^-} |f(z)| = 0\}.$$

According to [26, Lemma 8], we have the following.

Lemma 2.1. *For each $t \in \mathbb{T}$,*

$$M_t^+(QC(\mathbb{T})) \cap M_t^-(QC(\mathbb{T})) = M_t^0(QC(\mathbb{T})),$$
$$M_t^+(QC(\mathbb{T})) \cup M_t^-(QC(\mathbb{T})) = M_t(QC(\mathbb{T})).$$

By Lemma 2.1, for each $t \in \mathbb{T}$, we can split the fiber $M_t(QC(\mathbb{T}))$ into the three disjoint sets: $M_t^0(QC(\mathbb{T}))$ and

$$\begin{aligned} \widetilde{M}_t^+(QC(\mathbb{T})) &:= M_t^+(QC(\mathbb{T})) \setminus M_t^0(QC(\mathbb{T})),\\ \widetilde{M}_t^-(QC(\mathbb{T})) &:= M_t^-(QC(\mathbb{T})) \setminus M_t^0(QC(\mathbb{T})). \end{aligned} \tag{2.2}$$

Hence, letting

$$\begin{aligned} M^\pm(QC(\mathbb{T})) := \bigcup_{t\in\mathbb{T}} M_t^\pm(QC(\mathbb{T})), \quad M^0(QC(\mathbb{T})) := \bigcup_{t\in\mathbb{T}} M_t^0(QC(\mathbb{T})),\\ \widetilde{M}^\pm(QC(\mathbb{T})) := \bigcup_{t\in\mathbb{T}} \widetilde{M}_t^\pm(QC(\mathbb{T})), \end{aligned} \tag{2.3}$$

we obtain from (2.1) the following partition of $M(QC(\mathbb{T}))$:

$$M(QC(\mathbb{T})) = \widetilde{M}^-(QC(\mathbb{T})) \cup M^0(QC(\mathbb{T})) \cup \widetilde{M}^+(QC(\mathbb{T})). \tag{2.4}$$

There is a natural one-to-one mapping

$$w : M(PQC(\mathbb{T})) \to M(QC(\mathbb{T})) \times \{0,1\}$$

into a subset of $M(QC(\mathbb{T})) \times \{0,1\}$, which is given as follows: for every $y \in M(PQC(\mathbb{T}))$, defining $\xi = y|_{QC(\mathbb{T})}$, $t = y|_{C(\mathbb{T})}$ and $v = y|_{PC(\mathbb{T})}$, we conclude that $w(y) = (\xi, 0)$ if $v = (t, 0)$ and $w(y) = (\xi, 1)$ if $v = (t, 1)$.

We get the following characterization of fibers of $M(PQC(\mathbb{T}))$ from [26, Lemmas 8, 13].

Lemma 2.2. *Let $t \in \mathbb{T}$ and $\xi \in M_t(QC(\mathbb{T}))$. Then*

- (i) $M_\xi(PQC(\mathbb{T})) = \{(\xi,1)\}$ *whenever* $\xi \in \widetilde{M}_t^+(QC(\mathbb{T}))$;
- (ii) $M_\xi(PQC(\mathbb{T})) = \{(\xi,0)\}$ *whenever* $\xi \in \widetilde{M}_t^-(QC(\mathbb{T}))$;
- (iii) $M_\xi(PQC(\mathbb{T})) = \{(\xi,0),(\xi,1)\}$ *whenever* $\xi \in M_t^0(QC(\mathbb{T}))$. *In this case, if* $\{\lambda_n\} \subset (1,\infty)$ *is such that* $(\lambda_n, t) \to \xi$ *in the weak-star topology on* $QC(\mathbb{T})^*$, *then*

$$(\xi,1)f = \lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_\theta^{\theta+\frac{\pi}{\lambda_n}} f(e^{ix})dx, \quad (\xi,0)f = \lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_{\theta-\frac{\pi}{\lambda_n}}^{\theta} f(e^{ix})dx$$

for every $f \in PQC(\mathbb{T})$, *where* $t = e^{i\theta}$.

The Gelfand topology on $M(PQC(\mathbb{T}))$ can be described as follows. If $\xi \in M_t(QC(\mathbb{T}))$ with $t \in \mathbb{T}$, then a base of neighborhoods for $(\xi,\mu) \in M(PQC(\mathbb{T}))$ consists of all open sets of the form

$$W_{(\xi,\mu)} = \begin{cases} \big[(U_{\xi,t} \times \{0\}) \cup (U_{\xi,t}^- \times \{0,1\})\big] \cap M(PQC(\mathbb{T})) & \text{if } \ \mu = 0, \\ \big[(U_{\xi,t} \times \{1\}) \cup (U_{\xi,t}^+ \times \{0,1\})\big] \cap M(PQC(\mathbb{T})) & \text{if } \ \mu = 1, \end{cases} \tag{2.5}$$

where $U_{\xi,t} = U_\xi \cap M_t(QC(\mathbb{T}))$, U_ξ is an open neighborhood of $\xi \in M(QC(\mathbb{T}))$, and $U_{\xi,t}^-$, $U_{\xi,t}^+$ consists of all $\zeta \in U_\xi$ such that $\tau = \zeta|_{C(\mathbb{T})}$ belong, respectively, to the open arcs $(te^{-i\varepsilon}, t)$ and $(t, te^{i\varepsilon})$ of $\mathbb{T}$ for some $\varepsilon \in (0, 2\pi)$.

In order to study the C^*-algebra $\mathfrak{B}$ defined in (1.2), we need to know how the operators of multiplication aI ($a \in PQC(\mathbb{T})$), the Cauchy singular integral operator $S_\mathbb{T}$ and the unitary shift operators U_g ($g \in G$) interact modulo compact operators in $\mathcal{B}(L^2(\mathbb{T}))$.

It follows from the Hartman theorem (see, e.g., [13, Theorem 2.18]) that

$$aS_\mathbb{T} \simeq S_\mathbb{T} aI \quad \text{for all} \ \ a \in QC(\mathbb{T}). \tag{2.6}$$

As all shifts $g \in G$ are orientation-preserving diffeomorphisms, we get

$$U_g S_\mathbb{T} \simeq S_\mathbb{T} U_g \quad \text{for all} \ \ g \in G,$$

which allows us to conclude that

$$U_g S_\mathbb{T} U_g^* \simeq S_\mathbb{T} \quad \text{for all} \ \ g \in G, \tag{2.7}$$

(see, e.g., [22, Theorem 4.1]). An easy computation shows that

$$U_g a U_g^* = (a \circ g)I \quad \text{for all} \quad a \in L^\infty(\mathbb{T}). \tag{2.8}$$

Obviously, for every $a \in L^\infty(\mathbb{T})$ the function $a \circ g$ is also in $L^\infty(\mathbb{T})$. It was observed in [14] that $a \circ g \in QC(\mathbb{T})$ whenever $a \in QC(\mathbb{T})$. Consequently, we have the following fact.

Lemma 2.3. *If g is a diffeomorphism of $\mathbb{T}$ onto itself, $a \in QC(\mathbb{T})$ and $b \in PQC(\mathbb{T})$, then $a \circ g \in QC(\mathbb{T})$ and $b \circ g \in PQC(\mathbb{T})$.*

As a consequence of (2.7)–(2.8) and Lemma 2.3, for every shift $g \in G$ the mapping

$$\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1} \tag{2.9}$$

defines a *-automorphism of the C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$, where $\mathcal{K} = \mathcal{K}(L^2(\mathbb{T}))$ is the ideal of all compact operators on $L^2(\mathbb{T})$.

3. Main results: Fredholmness in the C^*-algebra $\mathfrak{B}$

Let G be a group of orientation-preserving diffeomorphisms of $\mathbb{T}$ onto itself such that all $g \in G \setminus \{e\}$ have the same finite set Λ of fixed points on $\mathbb{T}$. Consider the C^*-algebra

$$\mathfrak{B} = \operatorname{alg}(PQC(\mathbb{T}), S_\mathbb{T}, U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$$

generated by all multiplication operators by the piecewise quasicontinuous functions on $\mathbb{T}$, by the Cauchy singular integral operator $S_\mathbb{T}$ and by all shift operators U_g $(g \in G)$. Let

$$\mathfrak{N}_{arc} := \bigcup_{\tau \in \mathcal{O}_{arc}} M_\tau(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \quad \mathfrak{N}_\Lambda := \bigcup_{\tau \in \Lambda} M_\tau^0(QC(\mathbb{T})) \times \mathbb{R}, \tag{3.1}$$

where $\mathcal{O}_{arc}$ is a subset of $\mathbb{T}\setminus\Lambda$ that contains exactly one point in each G-orbit $G(\tau) = \{g(\tau) : g \in G\}$ defined for $\tau \in \mathbb{T} \setminus \Lambda$ by the group of shifts G. Thus, $\mathbb{T} \setminus \Lambda = \bigcup_{\tau \in \mathcal{O}_{arc}} G(\tau)$. Along with $\mathfrak{N}_{arc}$, we define the sets

$$\mathfrak{N}_{arc}^\pm := \bigcup_{\tau \in \mathcal{O}_{arc}} \widetilde{M}_\tau^\pm(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \quad \mathfrak{N}_{arc}^0 := \bigcup_{\tau \in \mathcal{O}_{arc}} M_\tau^0(QC(\mathbb{T})) \times \overline{\mathbb{R}},$$

where the sets $\widetilde{M}_\tau^\pm(QC(\mathbb{T}))$, $M_t^0(QC(\mathbb{T}))$ are given by (2.3) and Lemma 2.1.

For each $(\xi, x) \in \mathfrak{N}_{arc}$, we introduce the representation

$$\Phi_{\xi,x} : \mathfrak{B} \to \mathcal{B}(l^2(G, \mathbb{C}^2)), \quad B \mapsto \Phi_{\xi,x}(B), \tag{3.2}$$

given on the generators of the C^*-algebra $\mathfrak{B}$ for all $g \in G$ by

$$\begin{aligned} [\Phi_{\xi,x}(aI)f](g) &= \begin{cases} \operatorname{diag}\{(a\circ g)(\xi,0),(a\circ g)(\xi,0)\}f(g) & \text{if } (\xi,x)\in\mathfrak{N}_{arc}^-,\\ \operatorname{diag}\{(a\circ g)(\xi,1),(a\circ g)(\xi,0)\}f(g) & \text{if } (\xi,x)\in\mathfrak{N}_{arc}^0,\\ \operatorname{diag}\{(a\circ g)(\xi,1),(a\circ g)(\xi,1)\}f(g) & \text{if } (\xi,x)\in\mathfrak{N}_{arc}^+,\end{cases}\\ [\Phi_{\xi,x}(S_{\mathbb{T}})f](g) &= \begin{bmatrix} \tanh(\pi x) & 1/\cosh(\pi x)\\ 1/\cosh(\pi x) & -\tanh(\pi x)\end{bmatrix} f(g),\\ [\Phi_{\xi,x}(U_h)f](g) &= f(gh), \end{aligned} \tag{3.3}$$

where $a \in PQC(\mathbb{T})$, $a(\xi,\mu)$ is the value of the Gelfand transform of a at the point $(\xi,\mu) \in M(PQC(\mathbb{T}))$, $h \in G$, and $f \in l^2(G,\mathbb{C}^2)$.

For each $(\xi,x) \in \mathfrak{N}_\Lambda$, we introduce the representation

$$\Phi_{\xi,x} : \mathfrak{B} \to \mathcal{B}(\mathbb{C}^2), \quad B \mapsto \Phi_{\xi,x}(B), \tag{3.4}$$

given on the generators of $\mathfrak{B}$ by

$$\begin{aligned} [\Phi_{\xi,x}(aI)]f &= \operatorname{diag}\big\{a(\xi,1),a(\xi,0)\big\}f,\\ [\Phi_{\xi,x}(S_{\mathbb{T}})]f &= \begin{bmatrix} \tanh(\pi x) & 1/\cosh(\pi x)\\ 1/\cosh(\pi x) & -\tanh(\pi x)\end{bmatrix} f,\\ [\Phi_{\xi,x}(U_h)]f &= \operatorname{diag}\big\{e^{ix\ln h'(\xi)},e^{ix\ln h'(\xi)}\big\}f, \end{aligned} \tag{3.5}$$

where $a \in PQC(\mathbb{T})$, $a(\xi,\mu)$ is the value of the Gelfand transform of a at the point $(\xi,\mu) \in M(PQC(\mathbb{T}))$, $h \in G$, and $f \in \mathbb{C}^2$.

We establish below the following Fredholm criterion for the operators B in the C^*-algebra $\mathfrak{B}$.

Theorem 3.1. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if the following two conditions are satisfied:*

(i) *for all $(\xi,x) \in \mathfrak{N}_{arc}$ the operators $\Phi_{\xi,x}(B)$ are invertible on the space $l^2(G,\mathbb{C}^2)$ and*

$$\sup_{(\xi,x)\in\mathfrak{N}_{arc}} \big\|(\Phi_{\xi,x}(B))^{-1}\big\| < \infty;$$

(ii) *for all $(\xi,x) \in \mathfrak{N}_\Lambda$ the operators $\Phi_{\xi,x}(B)$ are invertible on the space $\mathbb{C}^2$ and*

$$\inf_{(\xi,x)\in\mathfrak{N}_\Lambda} \big\|(\Phi_{\xi,x}(B))^{-1}\big\| < \infty$$

or, equivalently,

$$\inf_{(\xi,x)\in\mathfrak{N}_\Lambda} |\det(\Phi_{\xi,x}(B))| > 0,$$

where the operators $\Phi_{\xi,x}(B)$ are identified with their 2×2 matrices.

The Fredholm criterion presented in Theorem 3.1 does not depend on the concrete choice of $\mathfrak{N}_{arc}$ because, for every operator $B \in \mathfrak{B}$, its representations associated with different points of the same orbit $G(\tau)$ are unitarily equivalent, and therefore it suffices to use representations related to only one point in each orbit. Moreover, one can easily see that assertion (i) of Theorem 3.1 remains true with $\mathfrak{N}_{arc}$ replaced by

$$\mathfrak{N}_{\mathbb{T}\setminus\Lambda} := \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(QC(\mathbb{T})) \times \overline{\mathbb{R}}.$$

Consider the Hilbert space

$$\mathcal{H} := \Big(\bigoplus_{(\xi,x)\in\mathfrak{N}_{arc}} l^2(G,\mathbb{C}^2)\Big) \oplus \Big(\bigoplus_{(\xi,x)\in\mathfrak{N}_\Lambda} \mathbb{C}^2 \Big) \tag{3.6}$$

and the operator function $\Phi(B) : (\xi,x) \mapsto \Phi_{\xi,x}(B)$ defined on $\mathfrak{N}_{arc} \cup \mathfrak{N}_\Lambda$ by (3.2)–(3.5) and equipped with the norm

$$\|\Phi(B)\|_{\mathcal{B}(\mathcal{H})} = \max\Big\{ \sup_{(\xi,x)\in\mathfrak{N}_{arc}} \big\|\Phi_{\xi,x}(B)\big\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))}, \sup_{(\xi,x)\in\mathfrak{N}_\Lambda} \big\|\Phi_{\xi,x}(B)\big\|_{\mathcal{B}(\mathbb{C}^2)}\Big\}.$$

The operator function $\Phi(B)$ is referred to as the *Fredholm symbol* of an operator $B \in \mathfrak{B}$. Clearly, the set $\Phi(\mathfrak{B}) := \{\Phi(B) : B \in \mathfrak{B}\}$ is a C^*-algebra, and the mapping

$$\Phi : \ \mathfrak{B} \to \mathcal{B}(\mathcal{H}), \quad B \mapsto \Phi(B) \tag{3.7}$$

is a C^*-algebra homomorphism of the C^*-algebra $\mathfrak{B}$ onto the C^*-algebra $\Phi(\mathfrak{B})$ with kernel $\operatorname{Ker}\Phi = \mathcal{K}$. Hence, for the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$, it follows that $\mathfrak{B}^\pi \cong \Phi(\mathfrak{B})$, that is, the map $\mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H})$ is a faithful representation in the Hilbert space (3.6). Making use of this symbol calculus, Theorem 3.1 can be rewritten in the following form.

Theorem 3.2. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if its symbol $\Phi(B)$ is invertible.*

4. The local-trajectory method and spectral measures

In order to proceed to the study of the C^*-algebra $\mathfrak{B}$ of singular integral operators with $PQC(\mathbb{T})$ coefficients and shifts $g \in G$, we recall here the local-trajectory method and its generalization on the basis of spectral measures (cf. [18], [19], [5]).

Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$ with the same unit I, G a discrete group with unit e, $U : g \mapsto U_g$ a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements such that $U_{g_1 g_2} = U_{g_1} U_{g_2}$ and $U_e = I$. Suppose that $\mathcal{A}$ and U_G are contained in a C^*-algebra $\mathcal{D}$. Let

$$\mathcal{B} := \operatorname{alg}(\mathcal{A}, U_G) \tag{4.1}$$

be the minimal C^*-subalgebra of $\mathcal{D}$ containing the C^*-algebra $\mathcal{A}$ and the group U_G. Assume that

(A1) *for every $g \in G$, the mappings $\alpha_g : a \mapsto U_g\, a\, U_g^*$ are *-automorphisms of the C^*-algebras $\mathcal{A}$ and $\mathcal{Z}$;*

(A2) *G is an amenable discrete group.*

According to [17, § 1.2], a discrete group G is called *amenable* if the C^*-algebra $l^\infty(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 such that

$$\rho(f) = \rho({}_s f) = \rho(f_s) \quad \text{for all } \ s \in G \ \text{ and all } f \in l^\infty(G),$$

where $({}_s f)(g) = f(s^{-1}g),\ (f_s)(g) = f(gs),\ g \in G$.

By virtue of (A1), the C^*-algebra $\mathcal{B} = \operatorname{alg}(\mathcal{A}, U_G)$ is the closure of the set $\mathcal{B}^0$ of elements $b = \sum a_g U_g$, where $a_g \in \mathcal{A}$ and g runs through finite subsets of G.

Let $M(\mathcal{Z})$ be the maximal ideal space of the commutative C^*-algebra $\mathcal{Z}$. By the Gelfand-Naimark theorem [23, § 16], $\mathcal{Z} \cong C(M(\mathcal{Z}))$, where $C(M(\mathcal{Z}))$ is the C^*-algebra of all continuous complex-valued functions on $M(\mathcal{Z})$. Under assumption (A1), identifying the characters φ_m of the commutative C^*-algebra $\mathcal{Z}$ and the maximal ideals $m = \operatorname{Ker} \varphi_m \in M(\mathcal{Z})$, we see that each *-automorphism $\alpha_g : \mathcal{Z} \to \mathcal{Z}$ induces a homeomorphism $\beta_g : M(\mathcal{Z}) \to M(\mathcal{Z})$ given by the rule

$$z(\beta_g(m)) = (\alpha_g(z))(m), \quad z \in \mathcal{Z},\ m \in M(\mathcal{Z}),\ g \in G, \tag{4.2}$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of the element $z \in \mathcal{Z}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the G-orbit of a point $m \in M(\mathcal{Z})$.

Let $\mathcal{P}_\mathcal{A}$ denote the set of all pure states of the C^*-algebra $\mathcal{A}$ with the induced weak* topology, and let J_m be the closed two-sided ideal of $\mathcal{A}$ generated by the maximal ideal $m \in M(\mathcal{Z})$ of the central C^*-algebra $\mathcal{Z} \subset \mathcal{A}$. If $\mu \in \mathcal{P}_\mathcal{A}$, then $\operatorname{Ker} \mu \supset J_m$ where $m := \mathcal{Z} \cap \operatorname{Ker} \mu \in M(\mathcal{Z})$ (see, e.g., [12, Lemma 4.1]), and hence $\mathcal{P}_\mathcal{A} = \bigcup_{m \in M} \{\nu \in \mathcal{P}_\mathcal{A} : \operatorname{Ker} \nu \supset J_m\}$.

Let the following version of *topologically free* action of the group G hold:

(A3) *there is a set $M_0 \subset M(\mathcal{Z})$ such that for every finite set $G_0 \subset G \setminus \{e\}$ and every nonempty open set $W \subset \mathcal{P}_\mathcal{A}$ there exists a state $\nu \in W$ such that $\beta_g(m_\nu) \neq m_\nu$ for all $g \in G_0$, where $m_\nu = \mathcal{Z} \cap \operatorname{Ker} \nu$ belongs to the G-orbit $G(M_0) := \{\beta_g(m) :\ g \in G,\ m \in M_0\}$ of the set M_0.*

For every $m \in M(\mathcal{Z})$, let $\widetilde{\pi}_m$ be an isometric representation

$$\widetilde{\pi}_m :\ \mathcal{A}/J_m \to \mathcal{B}(\mathcal{H}_m) \tag{4.3}$$

of the quotient C^*-algebra $\mathcal{A}/J_m$ in a Hilbert space $\mathcal{H}_m$. Consider the canonical *-homomorphism $\varrho_m : \mathcal{A} \to \mathcal{A}/J_m$ and the representation

$$\pi'_m :\ \mathcal{A} \to \mathcal{B}(\mathcal{H}_m), \quad A \mapsto (\widetilde{\pi}_m \circ \varrho_m)(A). \tag{4.4}$$

Since $\alpha_g(J_{\beta_g(m)}) = J_m$ for all $g \in G$ and all $m \in M(\mathcal{Z})$ in view of (A1), it follows that the quotient C^*-algebras $\mathcal{A}/J_{\beta_g(m)}$ and $\mathcal{A}/J_m$ are *-isomorphic, and therefore the spaces $\mathcal{H}_{\beta_g(m)}$ can be chosen equal for all $g \in G$.

Let $\Omega(M_0)$ be the set of G-orbits of all points $m \in M_0$. For each G-orbit $\omega \in \Omega(M_0)$, fix a point $m = m_\omega \in \omega$, put $\mathcal{H}_\omega = \mathcal{H}_m$, and let $l^2(G, \mathcal{H}_\omega)$ be the Hilbert space of all functions $f : G \mapsto \mathcal{H}_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\|f\| := (\sum_g \|f(g)\|^2)^{1/2} < \infty$. For every $\omega \in \Omega(M_0)$ we consider the representation $\pi_\omega : \mathcal{B} \to \mathcal{B}(l^2(G, \mathcal{H}_\omega))$ defined by

$$[\pi_\omega(a)f](g) = \pi'_{m_\omega}(\alpha_g(a))f(g), \quad [\pi_\omega(U_h)f](g) = f(gh) \tag{4.5}$$

for all $a \in \mathfrak{A}$, all $g, h \in G$ and all $f \in l^2(G, \mathcal{H}_\omega)$.

Under the above three conditions we have the following result ([19, Theorems 4.1, 4.12], [5, Theorem 3.1]).

Theorem 4.1. *If assumptions* (A1)–(A3) *are satisfied, then an element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega(M_0)$ the operator $\pi_\omega(b)$ is invertible on the space $l^2(G, \mathcal{H}_\omega)$ and, in the case of infinite set $\Omega(M_0)$,*

$$\sup\left\{\|(\pi_\omega(b))^{-1}\|_{\mathcal{B}(l^2(G,\mathcal{H}_\omega))} : \ \omega \in \Omega(M_0)\right\} < \infty.$$

Note that Theorem 4.1 remains true with $M_0 = M(\mathcal{Z})$, and then $\Omega(M_0) = \Omega(M(\mathcal{Z}))$ is the set of all G-orbits on $M(\mathcal{Z})$.

We will present now a generalization of the local-trajectory method for the case when condition (A3) is not fulfilled. Such generalization based on the notion of spectral measures was developed in [19] and [5].

Let M be a compact Hausdorff space and H a Hilbert space. By [23, p. 249], a *spectral measure* $P(\cdot)$ is a map from the σ-algebra of all Borel sets of M into the set of orthogonal projections in $\mathcal{B}(H)$ such that for every $\xi \in H$ the function $\Delta \to (P(\Delta)\xi, \xi)$ is the restriction to Borel sets of a measure on M defined by some integral on $C(M)$. Hence, $P(\emptyset) = 0$, $P(M) = I$, $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$ for all Borel sets $\Delta_1, \Delta_2 \subset M$, and $P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2)$ if $\Delta_1 \cap \Delta_2 = \emptyset$.

Suppose the C^*-algebra $\mathcal{B} = \operatorname{alg}(\mathcal{A}, U_G)$ satisfies only the conditions (A1)–(A2) of the local-trajectory method. Let $\mathfrak{R}(M(\mathcal{Z}))$ denote the σ-algebra of all Borel subsets of $M(\mathcal{Z})$, and let

$$\mathfrak{R}_G(M(\mathcal{Z})) := \left\{\Delta \in \mathfrak{R}(M(\mathcal{Z})) : \ \beta_g(\Delta) = \Delta \ \text{ for all } \ g \in G\right\}, \tag{4.6}$$

where the homeomorphisms β_g are given by (4.2). As is well known (see, e.g., [15, Theorem 2.6.1]), there exists an isometric representation $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ of the C^*-algebra $\mathcal{B}$ in a Hilbert space $\mathcal{H}$. According to [23, § 17], for the representation $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{B}(\mathcal{H})$ of the unital commutative C^*-algebra $\mathcal{Z}$, there is a unique spectral measure $P_\pi : \mathfrak{R}(M(\mathcal{Z})) \to \mathcal{B}(\mathcal{H})$ that commutes with all operators in the C^*-algebra $\pi(\mathcal{Z})$ and in its commutant $\pi(\mathcal{Z})'$, and such that

$$\pi(z) = \int_{M(\mathcal{Z})} z(m)\, dP_\pi(m) \quad \text{for all} \ \ z \in \mathcal{Z}, \tag{4.7}$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of the element $z \in \mathcal{Z}$.

The integral in (4.7) is defined as follows. Since

$$\int_{M(\mathcal{Z})} z(m)\, dP_\pi(m) = \int_{M(\mathcal{Z})} \operatorname{Re} z(m)\, dP_\pi(m) + i \int_{M(\mathcal{Z})} \operatorname{Im} z(m)\, dP_\pi(m),$$

it is sufficient to define such integral for real-valued functions $f \in C(M(\mathcal{Z}))$. Any real-valued function $f \in C(M(\mathcal{Z}))$ is the uniform limit in $L^\infty(M(\mathcal{Z}))$ of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple Borel real-valued functions $f_n = \sum_{k=1}^n c_{k,n}\chi_{\Delta_{k,n}}$, where $\Delta_{1,n},\dots,\Delta_{n,n}$ are pairwise disjoint Borel subsets of $M(\mathcal{Z})$ for every $n\in\mathbb{N}$, $c_{k,n}$ are real constants that can be chosen equal to values of f at some points $m_{k,n}\in\Delta_{k,n}$, and $\chi_{\Delta_{k,n}}$ are the characteristic functions of $\Delta_{k,n}$. Then

$$\int_{M(\mathcal{Z})} f(m)\,dP_\pi(m)=\lim_{n\to\infty}\sum_{k=1}^{n} f(m_{k,n})P_\pi(\Delta_{k,n})\in\mathcal{B}(\mathcal{H}),$$

where the convergence is uniform in $\mathcal{B}(\mathcal{H})$ because $\|f-f_n\|_{L^\infty(M(\mathcal{Z}))}\to 0$ as $n\to\infty$. Hence, for every $\xi\in\mathcal{H}$,

$$\Big(\Big(\int_{M(\mathcal{Z})} f(m)\,dP_\pi(m)\Big)\xi,\xi\Big)=\int_{M(\mathcal{Z})} f(m)\,d\mu_\xi(m),$$

where $\mu_\xi(\cdot):=(P_\pi(\cdot)\xi,\xi)$ is a Borel measure, and

$$\mu_\xi(\Delta)=(P_\pi(\Delta)\xi,\xi)=\int_{M(\mathcal{Z})}\chi_\Delta(m)d\mu_\xi(m)\quad\text{for all}\quad\Delta\in\mathfrak{R}(M(\mathcal{Z})).$$

Since (A1) holds and since $az=za$ for all $a\in\mathcal{A}$ and all $z\in\mathcal{Z}$, we deduce from [19, Lemma 4.6] and (4.6)–(4.7) that

$$\pi(b)P_\pi(\Delta)=P_\pi(\Delta)\pi(b)\quad\text{for all}\ \ b\in\mathcal{B}\ \ \text{and all}\ \ \Delta\in\mathfrak{R}_G(M(\mathcal{Z})).\tag{4.8}$$

Given $\Delta\in\mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta)\neq 0$, let us define the Hilbert space $\mathcal{H}_\Delta:=P_\pi(\Delta)\mathcal{H}=\{P_\pi(\Delta)\xi:\xi\in\mathcal{H}\}$ and introduce the following three C^*-subalgebras of $\mathcal{B}(\mathcal{H}_\Delta)$:

$$\begin{aligned}\mathcal{B}_\Delta:=&\{P_\pi(\Delta)\pi(b):b\in\mathcal{B}\},\\ \mathcal{A}_\Delta:=&\{P_\pi(\Delta)\pi(a):a\in\mathcal{A}\},\\ \mathcal{Z}_\Delta:=&\{P_\pi(\Delta)\pi(z):z\in\mathcal{Z}\}.\end{aligned}\tag{4.9}$$

Since $\mathcal{Z}$ is a central C^*-subalgebra of $\mathcal{A}$, it follows from (4.8) that $\mathcal{Z}_\Delta$ is a central C^*-subalgebra of $\mathcal{A}_\Delta$, where $\mathcal{A}_\Delta\subset\mathcal{B}_\Delta$.

For each Borel set $\Delta\in\mathfrak{R}(M(\mathcal{Z}))$, let $\operatorname{Int}\Delta$ and $\overline{\Delta}$ denote the interior and the closure of Δ, respectively, and let $\widetilde{\Delta}$ be the closed subset of $\overline{\Delta}$ given by

$$\widetilde{\Delta}:=\{m\in M(\mathcal{Z})\colon P_\pi(W_m\cap\Delta)\neq 0 \text{ for each open neighborhood } W_m \text{ of } m\}.$$

The next lemma summarizes some important properties of sets $\widetilde{\Delta}$.

Lemma 4.2. [19, Lemmas 5.1–5.2] *If* $\Delta\in\mathfrak{R}(M(\mathcal{Z}))$ *and* $\operatorname{Int}\Delta\neq 0$, *then*

(i) $P_\pi(\Delta)\neq 0$,
(ii) $\mathcal{Z}_\Delta\cong C(\widetilde{\Delta})$,
(iii) $\widetilde{\Delta}=\overline{\operatorname{Int}\Delta}$ *in case* $P_\pi(\Delta\setminus\overline{\operatorname{Int}\Delta})=0$.

Fix $\Delta\in\mathfrak{R}_G(M(\mathcal{Z}))$. For each $g\in G$, consider the unitary operator $U_{g,\Delta}:=P_\pi(\Delta)\pi(U_g)$ on $\mathcal{H}_\Delta$. As condition (A1) holds, the mappings

$$\alpha_{g,\Delta}:P_\pi(\Delta)\pi(a)\mapsto U_{g,\Delta}P_\pi(\Delta)\pi(a)U_{g,\Delta}^*=P_\pi(\Delta)\pi(U_gaU_g^*)\ \ (g\in G),$$

are *-automorphisms of the C^*-algebras $\mathcal{Z}_\Delta$ and $\mathcal{A}_\Delta$ defined by (4.9). Since $\mathcal{Z}_\Delta\cong C(\widetilde{\Delta})$, where $\widetilde{\Delta}\in\mathfrak{R}_G(M(\mathcal{Z}))$ and the isomorphism $\mathcal{Z}_\Delta\to C(\widetilde{\Delta})$ is given by $P_\pi(\Delta)\pi(z)\mapsto z(\cdot)|_{\widetilde{\Delta}}$, with restriction $z(\cdot)|_{\widetilde{\Delta}}$ of the Gelfand transform of $z \in \mathcal{Z}$, we conclude that each *-automorphism $\alpha_{g,\Delta}$ induces on $\widetilde{\Delta}$ the homeomorphism $\beta_{g,\Delta} := \beta_g|_{\widetilde{\Delta}}$, where β_g is defined by (4.2).

We will need below the following decomposition result.

Theorem 4.3. [6, Proposition 3.3] *Let $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be an isometric representation of the C^*-algebra $\mathcal{B} = \mathrm{alg}(\mathcal{A}, U_G)$ in a Hilbert space $\mathcal{H}$ and let $\{\Delta_i\}$ be an at most countable family of disjoint Borel sets in $\mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta_i) \neq 0$ for all i and $P_\pi(M(\mathcal{Z}) \setminus \bigcup_i \Delta_i) = 0$. If condition* (A1) *holds, then the mapping*

$$\Theta : \ \mathcal{B} \to \bigoplus_i \mathcal{B}_{\Delta_i}, \quad b \mapsto \bigoplus_i P_\pi(\Delta_i)\pi(b),$$

is an isometric C^*-algebra homomorphism from the C^*-algebra $\mathcal{B}$ into the C^*-algebra $\widetilde{\mathcal{B}} := \bigoplus_i \mathcal{B}_{\Delta_i}$. Then an element $b \in \mathcal{B}$ is invertible if and only if for each i the operator $P_\pi(\Delta_i)\pi(b)$ is invertible on the Hilbert space $\mathcal{H}_{\Delta_i}$ and

$$\sup_i \left\| (P_\pi(\Delta_i)\pi(b))^{-1} \right\| < \infty \quad \text{in case } \ \{\Delta_i\} \ \text{ is countable.}$$

It should be noted that, for the C^*-algebra $\mathcal{B} = \mathrm{alg}(\mathcal{A}, U_G)$, a G-invariant decomposition considered in Theorem 4.3 does not always exist. On the other hand, such decompositions always exist for the C^*-algebras $\mathcal{B} = \mathrm{alg}(\mathcal{A}, U_G)$ with groups G containing nonempty sets of common fixed points for all $g \in G$ (see Theorem 6.3 below).

5. The C^*-algebra $\mathfrak{A}$

Consider the C^*-algebra

$$\mathfrak{A} = \mathrm{alg}(PQC(\mathbb{T}), S_\mathbb{T}) \subset \mathcal{B}(L^2(\mathbb{T}))$$

of singular integral operators on $L^2(\mathbb{T})$ with $PQC(\mathbb{T})$ coefficients. With the C^*-algebra $\mathfrak{A}$ we associate the set

$$\mathfrak{M} := M(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \tag{5.1}$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$ is the two-point compactification of the real line $\mathbb{R} = (-\infty, +\infty)$. Let $B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ stand for the C^*-algebra of all bounded matrix functions $f : \mathfrak{M} \to \mathbb{C}^{2\times 2}$. Put

$$\mathfrak{M}^\pm := \widetilde{M}^\pm(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \quad \mathfrak{M}^0 := M^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \tag{5.2}$$

where the sets $\widetilde{M}^\pm(QC(\mathbb{T}))$ and $M^0(QC(\mathbb{T}))$ are defined by (2.3).

According to [14, Section 7.4] and [5, Theorem 5.1], we obtain the following symbol calculus for the C^*-algebra $\mathfrak{A}$.

Theorem 5.1. *The map* Sym : $\{aI : a \in PQC(\mathbb{T})\} \cup \{S_{\mathbb{T}}\} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ *given by the matrix functions*

$$(\mathrm{Sym}\, aI)(\xi, x) := \begin{cases} \mathrm{diag}\{a(\xi,0), a(\xi,0)\} & \textit{for all } (\xi,x) \in \mathfrak{M}^-, \\ \mathrm{diag}\{a(\xi,1), a(\xi,0)\} & \textit{for all } (\xi,x) \in \mathfrak{M}^0, \\ \mathrm{diag}\{a(\xi,1), a(\xi,1)\} & \textit{for all } (\xi,x) \in \mathfrak{M}^+, \end{cases} \tag{5.3}$$

$$(\mathrm{Sym}\, S_{\mathbb{T}})(\xi, x) := \begin{bmatrix} u(x) & w(x) \\ w(x) & -u(x) \end{bmatrix} \quad \textit{for all } (\xi, x) \in \mathfrak{M},$$

where $a(\xi, \mu)$ *is the Gelfand transform of a function* $a \in PQC(\mathbb{T})$ *at the point* $(\xi, \mu) \in M(PQC(\mathbb{T}))$*, the sets* $\mathfrak{M}^{\pm}$ *and* $\mathfrak{M}^0$ *are given by* (5.2)*, and*

$$u(x) := \tanh(\pi x), \quad w(x) := 1/\cosh(\pi x) \quad \textit{for all } x \in \overline{\mathbb{R}}, \tag{5.4}$$

extends to a C^**-algebra homomorphism*

$$\mathrm{Sym} : \mathfrak{A} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2}) \tag{5.5}$$

whose kernel consists of all compact operators on $L^2(\mathbb{T})$*. An operator* $A \in \mathfrak{A}$ *is Fredholm on the space* $L^2(\mathbb{T})$ *if and only if*

$$\det((\mathrm{Sym}\, A)(\xi, x)) \neq 0 \quad \textit{for all } (\xi, x) \in \mathfrak{M}.$$

Proof. Let $M := \bigcup_{t\in\mathbb{T}} M_t$, where, with $\widetilde{M}_t^{\pm}(QC(\mathbb{T}))$ given by (2.2),

$$M_t := \widetilde{M}_t^-(QC(\mathbb{T})) \cup \big(M_t^0(QC(\mathbb{T})) \times [0,1]\big) \cup \widetilde{M}_t^+(QC(\mathbb{T})), \quad t \in \mathbb{T}.$$

For the C^*-algebra $\mathfrak{A}$, we know from [14, Section 7] that the mapping

$$\widetilde{\mathrm{Sym}} : \{aI : a \in PQC(\mathbb{T})\} \cup \{S_{\mathbb{T}}\} \to B(M, \mathbb{C}^{2\times 2}),$$

defined by the matrix functions

$$(\widetilde{\mathrm{Sym}}\, S_{\mathbb{T}})(\zeta) = \mathrm{diag}\{1, -1\} \quad \text{for all } \zeta \in M,$$

$$(\widetilde{\mathrm{Sym}}\, aI)(\zeta) = \begin{cases} \mathrm{diag}\{a(\xi,0), a(\xi,0)\} \quad \text{for all } \zeta = \xi \in \widetilde{M}^-(QC(\mathbb{T})), \\ \begin{bmatrix} a(\xi,1)\mu + a(\xi,0)(1-\mu) & [a(\xi,1) - a(\xi,0)]\varrho(\mu) \\ [a(\xi,1) - a(\xi,0)]\varrho(\mu) & a(\xi,1)(1-\mu) + a(\xi,0)\mu \end{bmatrix} \\ \qquad \text{for all } \zeta = (\xi,\mu) \in M^0(QC(\mathbb{T})) \times [0,1], \\ \mathrm{diag}\{a(\xi,1), a(\xi,1)\} \quad \text{for all } \zeta = \xi \in \widetilde{M}^+(QC(\mathbb{T})), \end{cases}$$

where $\varrho(\mu) = \sqrt{\mu(1-\mu)}$ for all $\mu \in [0,1]$ and the sets $\widetilde{M}^{\pm}(QC(\mathbb{T}))$ and $M^0(QC(\mathbb{T}))$ in the partition (2.4) of $M(QC(\mathbb{T}))$ are defined by (2.3), extends to a C^*-algebra homomorphism $\widetilde{\mathrm{Sym}} : \mathfrak{A} \to B(M, \mathbb{C}^{2\times 2})$ whose kernel consists of all compact operators in the C^*-algebra $\mathcal{B}(L^2(\mathbb{T}))$, and an operator $A \in \mathfrak{A}$ is Fredholm on space $L^2(\mathbb{T})$ if and only if

$$\det((\widetilde{\mathrm{Sym}}\, A)(\zeta)) \neq 0 \quad \text{for all } \zeta \in M.$$

Let $\mathcal{M} := M(QC(\mathbb{T})) \times [0,1]$ and consider the mapping

$$\mathrm{Sym}_{[0,1]} : \mathfrak{A} \to B(\mathcal{M}, \mathbb{C}^{2\times 2})$$

defined for all operator $A \in \mathfrak{A}$ by

$$(\mathrm{Sym}_{[0,1]}A)(\xi,\mu) := \begin{cases} (\widetilde{\mathrm{Sym}}\,A)(\xi,\mu) & \text{if } (\xi,\mu) \in M^0(QC(\mathbb{T})) \times [0,1], \\ (\widetilde{\mathrm{Sym}}\,A)(\xi) & \text{if } (\xi,\mu) \in \mathcal{M} \setminus (M^0(QC(\mathbb{T})) \times [0,1]). \end{cases}$$

Clearly, $\mathrm{Sym}_{[0,1]}$ is a C^*-algebra homomorphism with the same kernel as $\widetilde{\mathrm{Sym}}$, and such that an operator $A \in \mathfrak{A}$ is Fredholm on space $L^2(\mathbb{T})$ if and only if

$$\det((\mathrm{Sym}_{[0,1]}A)(\xi,\mu)) \neq 0 \quad \text{for all} \quad (\xi,\mu) \in \mathcal{M}.$$

Since $\mu(x) = (1+u(x))/2 \in [0,1]$ for all $x \in \overline{\mathbb{R}}$, it remains to see that

$$(\mathrm{Sym}\,A)(\xi,x) =$$
$$\begin{bmatrix} \sqrt{\mu(x)} & \sqrt{1-\mu(x)} \\ \sqrt{1-\mu(x)} & -\sqrt{\mu(x)} \end{bmatrix} (\mathrm{Sym}_{[0,1]}A)(\xi,\mu(x)) \begin{bmatrix} \sqrt{\mu(x)} & \sqrt{1-\mu(x)} \\ \sqrt{1-\mu(x)} & -\sqrt{\mu(x)} \end{bmatrix}^{-1}$$

for all $(\xi,x) \in \mathfrak{M}$. □

By Theorem 5.1,

$$|A| = \|(\mathrm{Sym}\,A)I\|_{\mathcal{B}(l^2(\mathfrak{M},\mathbb{C}^2))} \quad \text{for all} \quad A \in \mathfrak{A}.$$

For each point $t \in \mathbb{T}$, we consider the operator $V_t \in \mathcal{B}(L^2(\mathbb{T}))$, with fixed singularity at t, given by

$$(V_t\varphi)(z) := \frac{\chi_t^+(z)}{\pi i}\int_{\mathbb{T}} \frac{\varphi(y)\chi_t^+(y)}{y+z-2t}\,dy - \frac{\chi_t^-(z)}{\pi i}\int_{\mathbb{T}} \frac{\varphi(y)\chi_t^-(y)}{y+z-2t}\,dy \quad \text{for} \quad z \in \mathbb{T}, \tag{5.6}$$

where $\chi_t^\pm$ are the characteristic functions of arcs $\gamma_t^\pm$ such that $\gamma_t := \gamma_t^+ \cup \gamma_t^-$ is a neighborhood of t, $\gamma_t^+ \cap \gamma_t^- = \{t\}$, γ_t is separated from $-t$, and $\gamma_t^+ \cap (-t,t) = \varnothing$, $\gamma_t^- \cap (t,-t) = \varnothing$. Let

$$v(x) := -iw(x) = -i/\cosh(\pi x) \quad \text{for all} \quad x \in \overline{\mathbb{R}}. \tag{5.7}$$

Lemma 5.2. *For each $t \in \mathbb{T}$, the operator V_t given by* (5.6) *belongs to the C^*-algebra $\mathfrak{A}$ and its symbol is given by*

$$(\mathrm{Sym}V_t)(\xi,x) := \begin{cases} \mathrm{diag}\{v(x),v(x)\} & \textit{if} \ (\xi,x) \in M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2\times 2} & \textit{if} \ (\xi,x) \in \mathfrak{M} \setminus (M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}), \end{cases} \tag{5.8}$$

where the function v and the set $\mathfrak{M}$ are defined by (5.7) *and* (5.1), *respectively.*

Proof. For every $t \in \mathbb{T}$, the operator V_t belongs to the C^*-algebra $\mathfrak{S} := \mathrm{alg}(PC(\mathbb{T}), S_{\mathbb{T}}) \subset \mathfrak{A}$ (cf. [5, Lemma 5.3]). Fix a sequence of polynomials $P_n(u) = \sum_{k=0}^n a_{k,n}u^k$ with coefficients $a_{k,n} \in \mathbb{C}$ that uniformly converges on $[-1,1]$ to the function $\sqrt{u^2-1} \in C([-1,1])$. Hence, because $S_{\mathbb{T}}^2 = I$ and therefore the spectrum of the operator $S_{\mathbb{T}}$ consists of the points ± 1, we conclude that

$$\lim_{n\to\infty} P_n(S_{\mathbb{T}}) = \sqrt{S_{\mathbb{T}}^2 - I} = 0. \tag{5.9}$$

By the proof of [5, Lemma 5.3],

$$(V_t)^\pi = \lim_{n\to\infty} \left(\chi_{t,n} P_n(\chi_t^+ S_\mathbb{T} \chi_t^+ I - \chi_t^- S_\mathbb{T} \chi_t^- I)\right)^\pi, \tag{5.10}$$

where $\chi_{t,n} \in PC(\mathbb{T})$ are characteristic functions of open neighborhoods of the point t, $\operatorname{supp}\chi_{t,n+1} \subset \operatorname{supp}\chi_{t,n}$ and $\bigcap_{n\in\mathbb{N}} \operatorname{supp}\chi_{t,n} = \{t\}$. Since

$$\left(\mathrm{Sym}(\chi_t^+ S_\mathbb{T} \chi_t^+ I - \chi_t^- S_\mathbb{T} \chi_t^- I)\right)(\xi, x) = \operatorname{diag}\{u(x), u(x)\} \tag{5.11}$$

for all $(\xi, x) \in M_t^0(SO(\mathbb{T})) \times \overline{\mathbb{R}}$, and since the sequence $\{P_n(u(x))\}$ converges uniformly on $\overline{\mathbb{R}}$ to the function $v(x) = \sqrt{u^2(x) - 1}$, where the functions u and v are given by (5.4) and (5.7), respectively, we conclude from (5.10) and (5.11) similarly to [5, Lemma 5.3] that

$$(\mathrm{Sym} V_t)(\xi, x) = \operatorname{diag}\{v(x), v(x)\} \quad \text{for all} \quad (\xi, x) \in M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}. \tag{5.12}$$

On the other hand, if $(\xi, x) \in \widetilde{M}_t^\pm(QC(\mathbb{T})) \times \overline{\mathbb{R}}$, then, respectively,

$$\left(\mathrm{Sym}(\chi_t^+ S_\mathbb{T} \chi_t^+ I - \chi_t^- S_\mathbb{T} \chi_t^- I)\right)(\xi, x) = \left(\mathrm{Sym}(\pm S_\mathbb{T})\right)(\xi, x).$$

Hence, taking into account (5.10), (5.3) and (5.9), we infer that

$$(\mathrm{Sym} V_t)(\xi, x) = \lim_{n\to\infty} \left(\mathrm{Sym}(\chi_{t,n} P_n(\pm S_\mathbb{T}))\right)(\xi, x) = 0 \tag{5.13}$$

for all $(\xi, x) \in \widetilde{M}_t^\pm(QC(\mathbb{T})) \times \overline{\mathbb{R}}$. Finally, since $\bigcap_{n\in\mathbb{N}} \operatorname{supp}\chi_{t,n} = \{t\}$, we obtain (5.8) from (5.12), (5.13) and (5.3). ☐

We now study the product $U_g V_t$ of the shift operator U_g given by (1.3) and the operator V_t with a fixed singularity at a fixed point $t \in \mathbb{T}$ of g.

Lemma 5.3. *Let g be an orientation-preserving diffeomorphism of $\mathbb{T}$ onto itself and let $t \in \mathbb{T}$ be a fixed point of g. Then $U_g V_t \in \mathfrak{A}$, $U_g V_t \simeq V_t U_g$ and*

$$(\mathrm{Sym}(U_g V_t))(\xi, x) := \begin{cases} \operatorname{diag}\{e^{ix\ln g'(t)} v(x), e^{ix\ln g'(t)} v(x)\} & \text{if } (\xi, x) \in M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2\times 2} & \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}). \end{cases} \tag{5.14}$$

Proof. Let $t \in \mathbb{T}$ be a fixed point of a $g \in G$. By the proof of [5, Lemma 5.4], $U_g V_t \in \mathfrak{A}$, $U_g V_t \simeq V_t U_g$ and

$$(U_g V_t)^\pi = \lim_{n\to\infty} \left(\chi_{t,n} P_n(\chi_t^+ S_\mathbb{T} \chi_t^+ I - \chi_t^- S_\mathbb{T} \chi_t^- I)\right)^\pi, \tag{5.15}$$

where the sequence $\{P_n(u(x))\}$ of functions $P_n(u(x)) = \sum_{k=0}^n a_{k,n} u^k(x)$ with coefficients $a_{k,n} \in \mathbb{C}$ converges uniformly on $\overline{\mathbb{R}}$ to the continuous function $x \mapsto e^{ix\ln g'(t)} v(x)$, $u(x) = \tanh(\pi x)$ and $v(x) = -i/\cosh(\pi x)$. Repeating now the arguments used in the proof of Lemma 5.2, we establish (5.14) on the basis of (5.15) and Theorem 5.1. ☐

We know from [5, (4.10)] that

$$aV_t \simeq V_t aI, \quad S_\mathbb{T} V_t \simeq V_t S_\mathbb{T} \quad \text{for all} \quad a \in PC(\mathbb{T}) \text{ and all } t \in \mathbb{T}. \tag{5.16}$$

Since $V_t \in \mathfrak{S} = \mathrm{alg}(PC(\mathbb{T}), S_{\mathbb{T}})$, we infer from (2.6) that

$$aV_t \simeq V_t aI \quad \text{for all} \quad a \in QC(\mathbb{T}) \quad \text{and all} \quad t \in \mathbb{T}. \tag{5.17}$$

Hence, relations (5.16)–(5.17) imply

$$aV_t \simeq V_t aI \quad \text{for all} \quad a \in PQC(\mathbb{T}) \quad \text{and all} \quad t \in \mathbb{T}. \tag{5.18}$$

We now introduce the C^*-algebra

$$\mathcal{Z} := \mathrm{alg}\big\{aI,\ H_{P,t} :\ a \in QC(\mathbb{T}),\ P \in \mathcal{P},\ t \in \mathbb{T}\big\} \subset \mathfrak{A} \subset \mathcal{B}(L^2(\mathbb{T})) \tag{5.19}$$

generated by all multiplication operators aI ($a \in QC(\mathbb{T})$) and by all operators

$$H_{P,t} := P(\chi_t^+ S_{\mathbb{T}} \chi_t^+ I - \chi_t^- S_{\mathbb{T}} \chi_t^- I)V_t \in \mathfrak{S} \subset \mathfrak{A} \quad (P \in \mathcal{P},\ t \in \mathbb{T}), \tag{5.20}$$

where

$$\mathcal{P} := \Big\{P_n(u) = \sum_{k=0}^{n} a_k u^k :\ a_k \in \mathbb{C},\ n \in \mathbb{N}\Big\}. \tag{5.21}$$

By (5.16) and (5.18), we also obtain the relations

$$aH_{P,t} \simeq H_{P,t}aI, \quad S_{\mathbb{T}}H_{P,t} \simeq H_{P,t}S_{\mathbb{T}} \ \text{ for all } \ a \in PQC(\mathbb{T}),\ P \in \mathcal{P},\ t \in \mathbb{T}. \tag{5.22}$$

Moreover, due to (2.6) and (5.22), $\mathcal{Z}^\pi := (\mathcal{Z}+\mathcal{K})/\mathcal{K}$ is a commutative central C^*-subalgebra of the C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$.

Applying Lemma 5.2, we infer the following result by analogy with [5, Lemma 6.1].

Lemma 5.4. *For each $t \in \mathbb{T}$ and each polynomial $P \in \mathcal{P}$, the symbol of the operator $H_{P,t} \in \mathfrak{A} \subset \mathcal{B}(L^2(\mathbb{T}))$ is given by*

$$(\mathrm{Sym}\, H_{P,t})(\xi, x) = \begin{cases} \mathrm{diag}\,\{P(u(x))v(x), P(u(x))v(x)\} & \text{if} \quad (\xi,x) \in M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2\times 2} & \text{if} \quad (\xi,x) \in \mathfrak{M} \setminus (M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}), \end{cases}$$

where the functions u, v and the set $\mathfrak{M}$ are defined by (5.4), (5.7) *and* (5.1), *respectively.*

Let $\mathfrak{A}^0$ be the non-closed subalgebra of $\mathfrak{A}$ generated by the operators aI ($a \in PQC^0(\mathbb{T})$) and $S_{\mathbb{T}}$, where $PQC^0(\mathbb{T})$ is the dense subalgebra of $PQC(\mathbb{T})$ consisting of the functions $\sum_{k=1}^n a_k p_k$, where $a_k \in QC(\mathbb{T})$ and $p_k \in PC^0(\mathbb{T})$ for $k = 1, 2, \dots, n$, and $PC^0(\mathbb{T})$ consists of all functions in $PC(\mathbb{T})$ with finite sets of discontinuities.

Theorem 5.5. *Every operator $A \in \mathfrak{A}$ is uniquely represented in the form*

$$A = a_+ P_{\mathbb{T}}^+ + a_- P_{\mathbb{T}}^- + H_A, \tag{5.23}$$

where $a_\pm \in PQC(\mathbb{T})$, $P_{\mathbb{T}}^\pm = (I \pm S_{\mathbb{T}})/2$ and $(\mathrm{Sym}\, H_A)(\xi,x) = 0_{2\times 2}$ for all $(\xi,x) \in \mathfrak{M} \setminus (M^0(QC(\mathbb{T})) \times \mathbb{R})$. Moreover, the mappings $A \mapsto a_\pm$ are C^*-algebra homomorphisms of the C^*-algebra $\mathfrak{A}$ onto the C^*-algebra $PQC(\mathbb{T})$, and

$$\|a_\pm\|_{L^\infty(\mathbb{T})} \le |A| := \inf_{K \in \mathcal{K}} \|A + K\|. \tag{5.24}$$

Proof. Obviously, every operator $A \in \mathfrak{A}^0$ is of the form (5.23), where $a_\pm \in PQC^0(\mathbb{T})$ and $(\mathrm{Sym}\, H_A)(\xi, \pm\infty) = 0_{2\times 2}$ for all $\xi \in M(QC(\mathbb{T}))$. Moreover, $(\mathrm{Sym}\, H_A)(\xi, x) = 0_{2\times 2}$ for all $(\xi, x) \in \widetilde{M}^\pm(QC(\mathbb{T})) \times \overline{\mathbb{R}}$. Therefore, by (5.3) and (5.4), $(\mathrm{Sym}\, A)(\xi, \pm\infty)$ is a diagonal matrix for every $\xi \in M(QC(\mathbb{T}))$ and its diagonal entries $(\mathrm{Sym}\, A)_{11}(\xi, \pm\infty)$ and $(\mathrm{Sym}\, A)_{22}(\xi, \pm\infty)$ satisfy the relations

$$\begin{aligned} (\mathrm{Sym}\, A)_{11}(\xi, +\infty) &= \begin{cases} a_+(\xi, 1) & \text{if } \ \xi \in M^+(QC(\mathbb{T})), \\ a_+(\xi, 0) & \text{if } \ \xi \in \widetilde{M}^-(QC(\mathbb{T})), \end{cases} \\ (\mathrm{Sym}\, A)_{22}(\xi, -\infty) &= \begin{cases} a_+(\xi, 1) & \text{if } \ \xi \in \widetilde{M}^+(QC(\mathbb{T})), \\ a_+(\xi, 0) & \text{if } \ \xi \in M^-(QC(\mathbb{T})), \end{cases} \\ (\mathrm{Sym}\, A)_{11}(\xi, -\infty) &= \begin{cases} a_-(\xi, 1) & \text{if } \ \xi \in M^+(QC(\mathbb{T})), \\ a_-(\xi, 0) & \text{if } \ \xi \in \widetilde{M}^-(QC(\mathbb{T})), \end{cases} \\ (\mathrm{Sym}\, A)_{22}(\xi, +\infty) &= \begin{cases} a_-(\xi, 1) & \text{if } \ \xi \in \widetilde{M}^+(QC(\mathbb{T})), \\ a_-(\xi, 0) & \text{if } \ \xi \in M^-(QC(\mathbb{T})). \end{cases} \end{aligned} \tag{5.25}$$

Since $\mathfrak{A}^\pi \cong \mathrm{Sym}\, \mathfrak{A}$ according to Theorem 5.1, we deduce from (5.25) in view of topology (2.5) that

$$\begin{aligned} \|a_\pm\|_{L^\infty(\mathbb{T})} &= \max\Big\{ \max_{\xi \in M^+(QC(\mathbb{T}))} |a_\pm(\xi, 1)|, \max_{\xi \in M^-(QC(\mathbb{T}))} |a_\pm(\xi, 0)| \Big\} \\ &\le \max_{\xi \in M(QC(\mathbb{T}))} \max\big\{ |(\mathrm{Sym}\, A)_{11}(\xi, \pm\infty)|, |(\mathrm{Sym}\, A)_{22}(\xi, \mp\infty)| \big\} \le |A|, \end{aligned}$$

respectively. Consequently, the mappings $A \mapsto a_\pm$ extends by continuity to C^*-algebra homomorphisms of the C^*-algebra $\mathfrak{A}$ onto $PQC(\mathbb{T})$ that preserve (5.24). This implies the unique representation (5.23) for every $A \in \mathfrak{A}$. ☐

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Using reasonings similar to those in the proof of [5, Theorem 6.3], we easily infer from (5.19) that

$$\mathcal{Z}^\pi \cong C(\dot{\mathfrak{M}}), \quad \dot{\mathfrak{M}} := M(QC(\mathbb{T})) \times \dot{\mathbb{R}}, \tag{5.26}$$

where $\dot{\mathfrak{M}}$ is the compact Hausdorff space equipped with the Gelfand topology whose neighborhood base of a point $(\xi, x) \in \dot{\mathfrak{M}}$ consists of all open sets of the form

$$W_{\xi,x} = \begin{cases} U^0_{\xi,t} \times (x - \varepsilon, x + \varepsilon) & \text{if } \ (\xi, x) \in M^0(QC(\mathbb{T})) \times \mathbb{R}, \\ (U_\xi \times \dot{\mathbb{R}}) \setminus (U^0_{\xi,t} \times [-\varepsilon, \varepsilon]) & \text{if } \ (\xi, x) \in M^0(QC(\mathbb{T})) \times \{\infty\}, \\ \big(\widetilde{U}^\pm_{\xi,t} \cup U^\pm_{\xi,t}\big) \times \dot{\mathbb{R}} & \text{if } \ (\xi, x) \in \widetilde{M}^\pm(QC(\mathbb{T})) \times \dot{\mathbb{R}}, \end{cases} \tag{5.27}$$

where $\varepsilon > 0$, U_ξ is an open neighborhood of a point $\xi \in M(QC(\mathbb{T}))$ and, for $t = \xi|_{C(\mathbb{T})} \in \mathbb{T}$, $U^0_{\xi,t} := U_\xi \cap M^0_t(QC(\mathbb{T}))$, $\widetilde{U}^\pm_{\xi,t} := U_\xi \cap \widetilde{M}^\pm_t(QC(\mathbb{T}))$ and $U^-_{\xi,t}$, $U^+_{\xi,t}$ consists of all $\zeta \in U_\xi$ such that $\tau = \zeta|_{C(\mathbb{T})}$ belong, respectively, to the open arcs $(te^{-i\varepsilon}, t)$ and $(t, te^{i\varepsilon})$ of $\mathbb{T}$ for some $\varepsilon \in (0, 2\pi)$.

Theorem 5.1 and Lemma 5.4 imply the following.

Theorem 5.6. *The maximal ideal space $M(\mathcal{Z}^\pi)$ of the C^*-algebra $\mathcal{Z}^\pi$ is homeomorphic to the compact Hausdorff set $\dot{\mathfrak{M}}$ given by* (5.26), *and the Gelfand transform of $\mathcal{Z}^\pi$ is defined by*

$$\Gamma: \ \mathcal{Z}^\pi \to C(\dot{\mathfrak{M}}), \quad Z^\pi \mapsto z(\xi, x),$$

where

$$z(\xi,x) = \begin{cases} (\mathrm{Sym}\, Z)_{11}(\xi,x) & \text{if} \quad (\xi,x) \in M(QC(\mathbb{T})) \times \mathbb{R}, \\ (\mathrm{Sym}\, Z)_{11}(\xi,\pm x) & \text{if} \quad (\xi,x) \in M(QC(\mathbb{T})) \times \{\infty\}, \end{cases}$$

and $(\mathrm{Sym}\, Z)(\xi,x)$ *given by* (5.3)–(5.5) *for each operator $Z \in \mathcal{Z}$ has the form*

$$(\mathrm{Sym}\, Z)(\xi,x) = \mathrm{diag}\{(\mathrm{Sym}\, Z)_{11}(\xi,x), (\mathrm{Sym}\, Z)_{22}(\xi,x)\} \quad \text{for all } (\xi,x) \in \mathfrak{M}$$

and possesses the properties

$$(\mathrm{Sym}\, Z)_{11}(\xi,x) = (\mathrm{Sym}\, Z)_{22}(\xi,x), \quad (\mathrm{Sym}\, Z)(\xi,+\infty) = (\mathrm{Sym}\, Z)(\xi,-\infty).$$

It follows from [21, (5.24)] that

$$U_g V_t \simeq V_{g^{-1}(t)} U_g \quad \text{for} \ \ t \in \mathbb{T}, \ g \in G. \tag{5.28}$$

Taking into account (5.20), (2.7)–(2.8) and Lemma 2.3, we infer that

$$U_g H_{P,t} \simeq H_{P,g^{-1}(t)} U_g \quad \text{for all} \ \ t \in \mathbb{T}, \ g \in G, \ P \in \mathcal{P}.$$

Hence, for every $g \in G$, the mapping $\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1}$ introduced in (2.9) is also a *-automorphism of the central C^*-subalgebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ that induces on the set $M(\mathcal{Z}^\pi) = \dot{\mathfrak{M}}$ given by (5.26) the homeomorphism

$$\beta_g : \ \dot{\mathfrak{M}} \to \dot{\mathfrak{M}}, \quad (\xi,x) \mapsto (g(\xi),x), \tag{5.29}$$

where $\xi \mapsto g(\xi)$ is the homeomorphism on $M(QC(\mathbb{T}))$ given by

$$a(g(\xi)) = (a \circ g)(\xi) \quad \text{for all} \ \ a \in QC(\mathbb{T}) \ \ \text{and all} \ \ \xi \in M(QC(\mathbb{T})) \tag{5.30}$$

(as usual $a(\xi) := \xi(a)$).

6. Representations of the C^*-algebra $\mathfrak{B}^\pi$

Consider the C^*-algebra $\mathfrak{B}$ given by (1.2) and fix an isometric representation

$$\varphi : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\varphi) \tag{6.1}$$

of the C^*-algebra $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$ in an abstract Hilbert space $\mathcal{H}_\varphi$.

For each point $t \in \mathbb{T}$, we consider the following subsets of $\dot{\mathfrak{M}}$ and $\mathfrak{M}$ (see (5.26) and (5.1)):

$$\begin{gathered} \mathfrak{M}_t^\circ := M_t^0(QC(\mathbb{T})) \times \mathbb{R}, \quad \overline{\mathfrak{M}_t^\circ} := M_t^0(QC(\mathbb{T})) \times \overline{\mathbb{R}}, \\ \dot{\mathfrak{M}}_t^\infty := M_t(QC(\mathbb{T})) \times \{\infty\}, \quad \mathfrak{M}_t^\infty := M_t(QC(\mathbb{T})) \times \{\pm\infty\}, \\ \dot{\mathfrak{M}}_t := M_t(QC(\mathbb{T})) \times \dot{\mathbb{R}}, \quad \mathfrak{M}_t := M_t(QC(\mathbb{T})) \times \overline{\mathbb{R}}. \end{gathered} \tag{6.2}$$

Let $\Lambda \subset \mathbb{T}$ be the finite set of all fixed points for all shifts $g \in G \setminus \{e\}$. We also introduce the sets

$$\begin{gathered}\dot{\mathfrak{M}}_\Lambda^\infty := \bigcup_{t\in\Lambda} \dot{\mathfrak{M}}_t^\infty, \quad \mathfrak{M}_\Lambda^\infty := \bigcup_{t\in\Lambda} \mathfrak{M}_t^\infty, \quad \mathfrak{M}_\Lambda := \bigcup_{t\in\Lambda} \mathfrak{M}_t, \quad \mathfrak{M}_\Lambda^\circ := \bigcup_{t\in\Lambda} \mathfrak{M}_t^\circ, \\ \dot{\mathfrak{M}}_{arc} := \bigcup_{t\in\mathbb{T}\setminus\Lambda} \dot{\mathfrak{M}}_t, \quad \mathfrak{M}_{arc} := \bigcup_{t\in\mathbb{T}\setminus\Lambda} \mathfrak{M}_t, \quad \mathfrak{M}_\Lambda^\diamond := \bigcup_{t\in\Lambda} (\dot{\mathfrak{M}}_t \setminus \mathfrak{M}_t^\circ),\end{gathered} \tag{6.3}$$

where the sets $\dot{\mathfrak{M}}_{arc}$ and $\mathfrak{M}_\Lambda^\circ$ are open in $\dot{\mathfrak{M}}$, while the sets $\dot{\mathfrak{M}}_\Lambda^\infty$ and $\mathfrak{M}_\Lambda^\diamond$ are closed in $\dot{\mathfrak{M}}$. As a result, we get the following partition of $\dot{\mathfrak{M}}$:

$$\dot{\mathfrak{M}} = \dot{\mathfrak{M}}_{arc} \cup \mathfrak{M}_\Lambda^\circ \cup \mathfrak{M}_\Lambda^\diamond. \tag{6.4}$$

The following simple assertion contains properties of sets (6.2)–(6.3).

Lemma 6.1. *For each $g \in G$ and each $t \in \mathbb{T}$, the homeomorphism $\beta_g : \dot{\mathfrak{M}} \to \dot{\mathfrak{M}}$ defined by* (5.29)–(5.30) *sends the sets $\dot{\mathfrak{M}}_t$ and $\mathfrak{M}_t^\circ$ onto the sets $\dot{\mathfrak{M}}_{g(t)}$ and $\mathfrak{M}_{g(t)}^\circ$, respectively, whence the sets $\dot{\mathfrak{M}}_{arc}$, $\mathfrak{M}_\Lambda^\circ$, $\mathfrak{M}_\Lambda^\diamond$ and $\dot{\mathfrak{M}}_\Lambda^\infty$ belong to the set*

$$\mathfrak{R}_G(\dot{\mathfrak{M}}) := \big\{\Delta \in \mathfrak{R}(\dot{\mathfrak{M}}) : \ \beta_g(\Delta) = \Delta \ \ \textit{for all} \ \ g \in G\big\}, \tag{6.5}$$

where $\mathfrak{R}(\dot{\mathfrak{M}})$ denotes the σ-algebra of all Borel subsets of $\dot{\mathfrak{M}} = M(\mathcal{Z}^\pi)$.

Let $\mathcal{H}_\phi$ denote the Hilbert space

$$\mathcal{H}_\phi := l^2(\mathfrak{M}_{arc}, \mathbb{C}^2) \oplus l^2(M_\Lambda^0(QC(\mathbb{T})), L_2^2(\mathbb{R})), \tag{6.6}$$

where

$$M_\Lambda^0(QC(\mathbb{T})) := \bigcup_{t\in\Lambda} M_t^0(QC(\mathbb{T})), \tag{6.7}$$

Consider the C^*-subalgebra $\phi(\mathfrak{A}^\pi)$ of $\mathcal{B}(\mathcal{H}_\phi)$ consisting of the operators

$$\phi(A^\pi) := \Bigg(\bigoplus_{(\xi,x)\in\mathfrak{M}_{arc}} (\mathrm{Sym}\, A)(\xi,x)I\Bigg) \oplus \Bigg(\bigoplus_{\xi\in M_\Lambda^0(QC(\mathbb{T}))} (\mathrm{Sym}\, A)(\xi,\cdot)I\Bigg), \tag{6.8}$$

where $A \in \mathfrak{A}$ and $(\mathrm{Sym}\, A)(\xi,\cdot)$ for every $\xi \in M_\Lambda^0(QC(\mathbb{T}))$ is the matrix function $x \mapsto (\mathrm{Sym}\, A)(\xi,x)$ defined for all $x \in \overline{\mathbb{R}}$.

Consider the C^*-algebras $(\mathrm{Sym}\,\mathfrak{A})|_{\mathfrak{M}_{arc}}$ and $(\mathrm{Sym}\,\mathfrak{A})|_{\overline{\mathfrak{M}}_{arc}}$ consisting, respectively, of the restrictions $(\mathrm{Sym}\, A)|_{\mathfrak{M}_{arc}}$ and $(\mathrm{Sym}\, A)|_{\overline{\mathfrak{M}}_{arc}}$ of $\mathrm{Sym}\, A$ for all operators $A \in \mathfrak{A}$ to the sets $\mathfrak{M}_{arc}$ and

$$\overline{\mathfrak{M}}_{arc} := \mathfrak{M} \setminus \mathfrak{M}_\Lambda^\circ = \mathfrak{M}_{arc} \cup (\mathfrak{M}_\Lambda \setminus \mathfrak{M}_\Lambda^\circ), \tag{6.9}$$

where the sets $\mathfrak{M}_{arc}$, $\mathfrak{M}_\Lambda$ and $\mathfrak{M}_\Lambda^\circ$ are given by (6.2)–(6.3).

Lemma 6.2. $(\mathrm{Sym}\,\mathfrak{A})|_{\mathfrak{M}_{arc}} \cong (\mathrm{Sym}\,\mathfrak{A})|_{\overline{\mathfrak{M}}_{arc}}$.

Proof. By (5.3) and (5.4), $(\mathrm{Sym}\, A)(\xi,\pm\infty)$ is a diagonal matrix for every $\xi \in M(QC(\mathbb{T}))$ and every $A \in \mathfrak{A}$, and its entries $(\mathrm{Sym}\, A)_{11}(\xi,\pm\infty)$ and $(\mathrm{Sym}\, A)_{22}(\xi,\pm\infty)$ for all $\xi \in M_t(QC(\mathbb{T}))$ with $t \in \Lambda$ are values of two $PQC(\mathbb{T})$ functions due to (5.25). Since the action of the group G on the maximal ideal space $M(QC(\mathbb{T}))$ is topologically free according to [11, Lemma 4.2], the mentioned entries for $\xi \in M_t(QC(\mathbb{T}))$ with $t \in \Lambda$ can be approximated, in view of (5.25) and the Gelfand topology (2.5) on $M(PQC(\mathbb{T}))$, by the corresponding entries of the matrices $(\mathrm{Sym}\, A)(\zeta, \pm\infty)$, where $\zeta \in \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(QC(\mathbb{T}))$ and, for $(\mathrm{Sym}\, A)_{11}(\xi, \pm\infty)$, τ belong to the right semi-neighborhood of t if $\xi \in M^+(QC(\mathbb{T}))$ and τ are in the left semi-neighborhood of t if $\xi \in \widetilde{M}^-(QC(\mathbb{T}))$, while for $(\mathrm{Sym}\, A)_{22}(\xi, \pm\infty)$, τ belong to the right semi-neighborhood of t if $\xi \in \widetilde{M}^+(QC(\mathbb{T}))$ and τ belong to the left semi-neighborhood of t if $\xi \in M^-(QC(\mathbb{T}))$. Hence, for every $A \in \mathfrak{A}$,

$$\sup_{(\xi,x)\in\mathfrak{M}_{arc}} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp} = \sup_{(\xi,x)\in\mathfrak{M}_{arc}\cup\mathfrak{M}_\Lambda^\infty} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp}, \tag{6.10}$$

where $\|\cdot\|_{sp}$ is the spectral norm, and the set $\mathfrak{M}_\Lambda^\infty$ is given by (6.2)–(6.3).

On the other hand, the singular values of the matrices $(\mathrm{Sym}\, A)(\xi, x)$ for all $(\xi, x) \in \mathfrak{M}^- \cup \mathfrak{M}^+$ coincide with the singular values

$$|(\mathrm{Sym}\, A)_{11}(\xi, \pm\infty)| \quad \text{and} \quad |(\mathrm{Sym}\, A)_{22}(\xi, \pm\infty)|$$

of the matrices $(\mathrm{Sym}\, A)(\xi, \pm\infty)$, respectively, where

$$(\mathrm{Sym}\, A)_{11}(\xi, \pm\infty) = (\mathrm{Sym}\, A)_{22}(\xi, \mp\infty)$$

and the sets $\mathfrak{M}^\pm$ are given by (5.2). Hence,

$$\sup_{(\xi,x)\in\mathfrak{M}_\Lambda^\infty} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp} = \sup_{(\xi,x)\in\mathfrak{M}_\Lambda\setminus\mathfrak{M}_\Lambda^\circ} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp}.$$

This together with (6.9) and (6.10) implies the equality

$$\sup_{(\xi,x)\in\mathfrak{M}_{arc}} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp} = \sup_{(\xi,x)\in\overline{\mathfrak{M}}_{arc}} \|(\mathrm{Sym}\, A)(\xi, x)\|_{sp},$$

which completes the proof. □

Since $\mathrm{Sym}\,\mathfrak{A} \cong (\mathrm{Sym}\,\mathfrak{A})|_{\mathfrak{M}_{arc}\cup\mathfrak{M}_\Lambda^\circ}$ according to Lemma 6.2, we infer from (6.6)–(6.8) and Theorem 5.1 that the homomorphism

$$\phi : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_\phi), \quad A^\pi \mapsto \phi(A^\pi), \tag{6.11}$$

is an isometric representation of $\mathfrak{A}^\pi$ in the Hilbert space $\mathcal{H}_\phi$. Let

$$P_\varphi : \mathfrak{R}(\dot{\mathfrak{M}}) \to \mathcal{B}(\mathcal{H}_\varphi), \quad P_\phi : \mathfrak{R}(\dot{\mathfrak{M}}) \to \mathcal{B}(\mathcal{H}_\phi) \tag{6.12}$$

be the unique spectral measures associated to the representations (6.1) and (6.11) in the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\phi$, respectively, which are restricted to the unital commutative C^*-algebra $\mathcal{Z}^\pi$.

For the representation (6.11) given by (6.6)–(6.8) we get the equalities

$$P_\phi(\dot{\mathfrak{M}}_{arc}) = I_{arc} \oplus O_\Lambda, \quad P_\phi(\dot{\mathfrak{M}}_\Lambda) = O_{arc} \oplus I_\Lambda, \tag{6.13}$$

where O_{arc} and I_{arc} are, respectively, the zero and the identity operators on the Hilbert space $l^2(\mathfrak{M}_{arc}, \mathbb{C}^2)$, while O_Λ and I_Λ are the zero and identity operators on Hilbert space $l^2(M_\Lambda^0(QC(\mathbb{T})), L_2^2(\mathbb{R}))$.

With partition (6.4) we associate the following C^*-subalgebras of $\varphi(\mathfrak{B}^\pi)$:

$$\mathfrak{B}_{arc} := \mathrm{alg}\{P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi),\ P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_g^\pi) :\ A \in \mathfrak{A},\ g \in G\}; \tag{6.14}$$

$$\mathfrak{B}_\Lambda^\circ := \mathrm{alg}\{P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(A^\pi),\ P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(U_g^\pi) :\ A \in \mathfrak{A},\ g \in G\}; \tag{6.15}$$

$$\mathfrak{B}_\Lambda^\diamond := \mathrm{alg}\{P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(A^\pi),\ P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(U_g^\pi) :\ A \in \mathfrak{A},\ g \in G\}. \tag{6.16}$$

Since the sets $\dot{\mathfrak{M}}_{arc}$, $\mathfrak{M}_\Lambda^\circ$ and $\mathfrak{M}_\Lambda^\diamond$ belong to $\mathfrak{R}_G(\dot{\mathfrak{M}})$, we infer from Theorem 4.3 and partition (6.4) the following abstract Fredholm criterion.

Theorem 6.3. *An operator $B \in \mathfrak{B}$ is Fredholm on the Lebesgue space $L^2(\mathbb{T})$ if and only if*

(i) *the operator $B_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$;*
(ii) *the operator $B_\Lambda^\circ := P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\mathfrak{M}_\Lambda^\circ)\mathcal{H}_\varphi$;*
(iii) *the operator $B_\Lambda^\diamond := P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$.*

7. An invertibility symbol calculus for the C^*-algebra $\mathfrak{A}_{arc}$

Let us study the invertibility in the C^*-subalgebra

$$\mathfrak{A}_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(\mathfrak{A}^\pi) \subset \mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi) \tag{7.1}$$

consisting of the operators $P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)$ for all $A \in \mathfrak{A}$. The C^*-algebra $\mathcal{Z}_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(\mathcal{Z}^\pi)$, which consists of the operators $P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(Z^\pi)$ with $Z \in \mathcal{Z}$, is a central C^*-subalgebra of $\mathfrak{A}_{arc}$. Since the set $\dot{\mathfrak{M}}_{arc}$ given in (6.3) is open in the Gelfand topology of $\dot{\mathfrak{M}}$ (see (5.27)), we infer from Lemma 4.2 that $\mathcal{Z}_{arc} \cong C(\widetilde{\dot{\mathfrak{M}}}_{arc})$, where $\widetilde{\dot{\mathfrak{M}}}_{arc}$ is the closure of $\dot{\mathfrak{M}}_{arc}$ in $\dot{\mathfrak{M}}$,

$$\widetilde{\dot{\mathfrak{M}}}_{arc} := \dot{\mathfrak{M}} \setminus \mathfrak{M}_\Lambda^\circ = \dot{\mathfrak{M}}_{arc} \cup \mathfrak{M}_\Lambda^\diamond. \tag{7.2}$$

We now consider the Hilbert space $\mathcal{H}_\phi$ given by (6.6) and its subspace $P_\phi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\phi$, which is isometrically isomorphic to the Hilbert space $l^2(\mathfrak{M}_{arc}, \mathbb{C}^2)$ by (6.13). Along with the C^*-algebra $\mathfrak{A}_{arc}$, we consider the C^*-algebra

$$\widetilde{\mathfrak{A}}_{arc} := P_\phi(\dot{\mathfrak{M}}_{arc})\phi(\mathfrak{A}^\pi) \subset \mathcal{B}(l^2(\mathfrak{M}_{arc}, \mathbb{C}^2)),$$

which consists of the operators $P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi)$ with $A \in \mathfrak{A}$. Comparing the images of spectral measures (6.12), we obtain the following.

Theorem 7.1. *The mapping given by*

$$P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) \mapsto P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi) \quad \textit{for all} \ \ A \in \mathfrak{A} \tag{7.3}$$

is a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{A}_{arc}$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_{arc}$.

Proof. According to [5, Lemma 3.5], for the open Borel set $\dot{\mathfrak{M}}_{arc} \subset \dot{\mathfrak{M}}$ and each operator $A \in \mathfrak{A}$, we have the equalities

$$\|P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \sup_{Z \in \mathcal{Z}(\dot{\mathfrak{M}}_{arc})} \|\varphi(Z^\pi A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \tag{7.4}$$

$$\|P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} = \sup_{Z \in \mathcal{Z}(\dot{\mathfrak{M}}_{arc})} \|\phi(Z^\pi A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)}, \tag{7.5}$$

where the set $\mathcal{Z}(\dot{\mathfrak{M}}_{arc})$ consists of the operators $Z \in \mathcal{Z}$ for which the Gelfand transform of the coset Z^π is a function $z(\cdot,\cdot) \in C(\dot{\mathfrak{M}})$ with values in $[0,1]$ and with support contained in $\widetilde{\dot{\mathfrak{M}}_{arc}}$. Since φ and ϕ are isometric representations of the C^*-algebra $\mathfrak{A}^\pi$, we conclude that the right parts of (7.4) and (7.5) are equal, and therefore

$$\|P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \quad \text{for all} \quad A \in \mathfrak{A} \tag{7.6}$$

or, equivalently, for all $A \in \mathfrak{A}$,

$$\|P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)\|_{\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi)} = \|P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi)\|_{\mathcal{B}(P_\phi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\phi)}$$

because $\dot{\mathfrak{M}}_{arc} \in \mathfrak{R}(\dot{\mathfrak{M}})$. This implies that the mapping (7.3) is a well-defined isometric *-isomorphism of the C^*-algebra $\mathfrak{A}_{arc}$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_{arc}$. ☐

Combining Theorem 7.1 and Lemma 6.2, we obtain an invertibility symbol calculus for the abstract C^*-algebra $\mathfrak{A}_{arc}$ given by (7.1).

Theorem 7.2. *The mapping*

$$\mathrm{Sym}_{arc} : \mathfrak{A}_{arc} \to B(\overline{\mathfrak{M}}_{arc}, \mathbb{C}^{2\times 2}), \quad P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) \mapsto (\mathrm{Sym}\, A)|_{\overline{\mathfrak{M}}_{arc}}, \tag{7.7}$$

is an isometric C^*-algebra homomorphism. For every $A \in \mathfrak{A}$ the operator $P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ if and only if

$$\det((\mathrm{Sym}\, A)(\xi, x)) \neq 0 \quad \textit{for all} \quad (\xi, x) \in \overline{\mathfrak{M}}_{arc}.$$

Proof. By (6.8) and (6.13), the C^*-algebra $\widetilde{\mathfrak{A}}_{arc}$ is isometrically *-isomorphic to the C^*-algebra of all matrix functions $(\mathrm{Sym}\,\mathfrak{A})|_{\mathfrak{M}_{arc}}$, which, in its turn, is isometrically *-isomorphic to the C^*-algebra $(\mathrm{Sym}\,\mathfrak{A})|_{\overline{\mathfrak{M}}_{arc}}$ according to Lemma 6.2. Hence, the mapping

$$\widetilde{\mathfrak{A}}_{arc} \to B(\overline{\mathfrak{M}}_{arc}, \mathbb{C}^{2\times 2}), \quad P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi) \mapsto (\mathrm{Sym}\, A)|_{\overline{\mathfrak{M}}_{arc}} \tag{7.8}$$

is an isometric C^*-algebra homomorphism. Since $\mathfrak{A}_{arc} \cong \widetilde{\mathfrak{A}}_{arc}$ in virtue of Theorem 7.1, we conclude from (7.8) that the map (7.7) taken in the form

$$P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) \mapsto P_\phi(\dot{\mathfrak{M}}_{arc})\phi(A^\pi) \mapsto (\mathrm{Sym}\, A)|_{\overline{\mathfrak{M}}_{arc}}$$

is an isometric C^*-algebra homomorphism of $\mathfrak{A}_{arc}$ into $B(\overline{\mathfrak{M}}_{arc}, \mathbb{C}^{2\times 2})$, and its image coincides with $(\mathrm{Sym}\,\mathfrak{A})|_{\overline{\mathfrak{M}}_{arc}}$. Then the invertibility criterion for elements in $\mathfrak{A}_{arc}$ follows immediately. ☐

8. The C^*-algebra $\mathfrak{B}_{arc}$

The local-trajectory method will allow us to establish in this section an invertibility criterion for the C^*-algebra $\mathfrak{B}_{arc}$ given by (6.14). Observe that C^*-algebra $\mathfrak{B}_{arc}$ can be viewed similarly to (4.1) as

$$\mathfrak{B}_{arc} := \operatorname{alg}(\mathfrak{A}_{arc}, (U_G)_{arc}) \subset \mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi),$$

the C^*-algebra generated by all operators $P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) \in \mathfrak{A}_{arc}$ and by all unitary operators $U_{g,\,arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_g^\pi)$, with $g \in G$, in the commutative (and therefore amenable) group $(U_G)_{arc}$.

For each $g \in G$, the mapping

$$\alpha_{g,\,arc} : P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(\mathfrak{A}^\pi) \mapsto U_{g,\,arc}(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(\mathfrak{A}^\pi))U^*_{g,\,arc}$$

is a *-automorphism of the C^*-algebra $\mathfrak{A}_{arc}$ and its central C^*-subalgebra $\mathcal{Z}_{arc}$ because

$$U_{g,\,arc}(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(\mathfrak{A}^\pi))U^*_{g,\,arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_g^\pi\mathfrak{A}^\pi(U_g^\pi)^*)$$

and mapping (2.9) is a *-automorphism of the C^*-algebras $\mathcal{Z}^\pi$ and $\mathfrak{A}^\pi$.

Thus, conditions (A1)–(A2) of the local-trajectory method are satisfied for the C^*-algebra $\mathfrak{B}_{arc}$. For every $g \in G$, the *-automorphism $\alpha_{g,\,arc}$ induces on the maximal ideal space $M(\mathcal{Z}_{arc}) = \widetilde{\dot{\mathfrak{M}}}_{arc}$ of $\mathcal{Z}_{arc}$ the homeomorphism

$$\beta_{g,\,arc} : \widetilde{\dot{\mathfrak{M}}}_{arc} \to \widetilde{\dot{\mathfrak{M}}}_{arc}, \quad (\xi, x) \mapsto \beta_g(\xi, x), \tag{8.1}$$

where β_g and $\widetilde{\dot{\mathfrak{M}}}_{arc}$ are given by (5.29) and (7.2), respectively. It follows from (5.29)–(5.30) that for every $g \in G \setminus \{e\}$ the set of fixed points of $\beta_{g,\,arc}$ is contained in the set $\dot{\mathfrak{M}}_\Lambda := \bigcup_{t\in\Lambda} \dot{\mathfrak{M}}_t$.

Let us verify the fulfillment of condition (A3) of the local-trajectory method for the C^*-algebra $\mathfrak{B}_{arc}$. For each $(\xi, x) \in \widetilde{\dot{\mathfrak{M}}}_{arc}$, let $J_{(\xi,x)}$ be the smallest closed two-sided ideal of $\mathfrak{A}_{arc}$ that contains the set

$$\big\{P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(Z^\pi) : \ Z \in \mathcal{Z}, \ (\operatorname{Sym} Z)(\xi, x) = 0_{2\times 2}\big\}.$$

The set $\mathcal{P}_{arc}$ of all pure states of the C^*-algebra $\mathfrak{A}_{arc}$ can be written as

$$\mathcal{P}_{arc} = \bigcup_{(\xi,x)\in\widetilde{\dot{\mathfrak{M}}}_{arc}} \mathcal{P}_{(\xi,x)}, \quad \mathcal{P}_{(\xi,x)} := \{\rho \in \mathcal{P}_{arc} : \ \operatorname{Ker}\rho \supset J_{(\xi,x)}\}.$$

According to [15, Theorem 2.11.8(i)] (see also [19]), the set $\mathcal{P}_{(\xi,x)}$ can be identified with the set of all pure states of the quotient C^*-algebra $\mathfrak{A}_{arc}/J_{(\xi,x)}$.

By Theorem 7.2, for every $(\xi, x) \in \mathfrak{M}^\circ_{arc} := \bigcup_{t\in\mathbb{T}\setminus\Lambda} \mathfrak{M}^\circ_t$, the quotient C^*-algebra $\mathfrak{A}_{arc}/J_{(\xi,x)}$ is isometrically *-isomorphic to the C^*-subalgebra $(\operatorname{Sym}\mathfrak{A})(\xi, x)$ of $\mathbb{C}^{2\times 2}$, and this isomorphism is given by

$$\widetilde{\pi}_{(\xi,x)} : P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) + J_{(\xi,x)} \mapsto (\operatorname{Sym} A)(\xi, x) \quad \text{for all} \ \ A \in \mathfrak{A}. \tag{8.2}$$

For each $(\xi, x) \in \widetilde{\dot{\mathfrak{M}}}_{arc} \setminus \mathfrak{M}^\circ_{arc}$, we infer from Theorems 5.1, 7.2 and the proof of Lemma 6.2 that the C^*-algebra $\mathfrak{A}_{arc}/J_{(\xi,x)}$ is isometrically *-isomorphic to the C^*-subalgebra $\operatorname{diag}\big\{(\operatorname{Sym}\mathfrak{A})(\xi, +\infty), \ (\operatorname{Sym}\mathfrak{A})(\xi, -\infty)\big\}$ of $\mathbb{C}^{4\times 4}$ if

$(\xi, x) \in \mathfrak{M}^{0,\infty} := M^0(QC(\mathbb{T})) \times \{\infty\}$, and to the C^*-algebra $(\operatorname{Sym}\mathfrak{A})(\xi, +\infty) \subset \mathbb{C}^{2\times 2}$ if $(\xi, x) \in \dot{\mathfrak{M}}^{\pm} := \widetilde{M}^{\pm}(QC(\mathbb{T})) \times \dot{\mathbb{R}}$, where the isomorphism is defined for all $A \in \mathfrak{A}$ by

$$\widetilde{\pi}_{(\xi,x)} : P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) + J_{(\xi,x)} \mapsto \begin{cases} \operatorname{diag}\{(\operatorname{Sym} A)(\xi, +\infty),\ (\operatorname{Sym} A)(\xi, -\infty)\} \\ \qquad\qquad\qquad\qquad \text{if } \ (\xi, x) \in \mathfrak{M}^{0,\infty}, \\ (\operatorname{Sym} A)(\xi, +\infty) \quad \text{if } \ (\xi, x) \in \dot{\mathfrak{M}}^{\pm}. \end{cases} \tag{8.3}$$

Since the matrices $(\operatorname{Sym} A)(\xi, \pm\infty)$ are diagonal for all $A \in \mathfrak{A}$ and all $\xi \in M(QC(\mathbb{T}))$, we conclude from (8.3) that the C^*-algebras $\mathfrak{A}_{arc}/J_{(\xi,x)}$ are commutative, and therefore for every $(\xi, x) \in \widetilde{\dot{\mathfrak{M}}}_{arc} \setminus \mathfrak{M}^\circ_{arc}$ the set $\mathcal{P}_{(\xi,x)} = \mathcal{P}_{\mathfrak{A}_{arc}/J_{(\xi,x)}}$ of pure states of the C^*-algebra $\mathfrak{A}_{arc}/J_{(\xi,x)}$ consists of four multiplicative linear functionals whose values coincide with the diagonal entries of matrices $(\operatorname{Sym} A)(\xi, \pm\infty)$ if $(\xi, x) \in \mathfrak{M}^{0,\infty}$, and consists of two multiplicative linear functionals whose values coincide with the diagonal entries of the matrix $(\operatorname{Sym} A)(\xi, +\infty)$ if $(\xi, x) \in \dot{\mathfrak{M}}^{\pm}$.

Hence, for every $\xi \in M(QC(\mathbb{T}))$ and every $(\xi, x) \in \widetilde{\dot{\mathfrak{M}}}_{arc} \setminus \mathfrak{M}^\circ_{arc}$,

$$\mathcal{P}_{(\xi,x)} = \mathcal{P}_{(\xi,\infty)} := \big\{\rho^{(1)}_{\xi,+\infty}, \rho^{(2)}_{\xi,+\infty}, \rho^{(1)}_{\xi,-\infty}, \rho^{(2)}_{\xi,-\infty}\big\},$$

where the pure states $\rho^{(j)}_{\xi,\pm\infty}$ for $j = 1, 2$ are given by

$$\rho^{(j)}_{\xi,\pm\infty} : \mathfrak{A}_{arc} \to \mathbb{C}, \quad P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi) \mapsto (\operatorname{Sym} A)_{jj}(\xi, \pm\infty), \tag{8.4}$$

$(\operatorname{Sym} A)_{jj}(\xi, \pm\infty)$ is the (j, j)-entry of the matrix $(\operatorname{Sym} A)(\xi, \pm\infty)$, and the set $\mathcal{P}_{(\xi,\infty)}$ consists of two elements for all $(\xi, x) \in \dot{\mathfrak{M}}^{\pm}$ because in that case $\rho^{(1)}_{\xi,+\infty} = \rho^{(2)}_{\xi,-\infty}$ and $\rho^{(2)}_{\xi,+\infty} = \rho^{(1)}_{\xi,-\infty}$.

Since $\beta_g(\xi, x) \neq (\xi, x)$ for all $(\xi, x) \in \dot{\mathfrak{M}}_{arc}$, to verify the fulfillment of condition (A3) of the local-trajectory method for the C^*-algebra $\mathfrak{B}_{arc}$, it only remains to prove the approximability (in the weak* topology) of the pure states $\rho \in \mathcal{P}_{(\xi,\infty)}$, with $\xi \in \dot{\mathfrak{M}}^\infty_\Lambda$, by pure states in $\mathcal{P}_{(\xi,x)}$, with $(\xi, x) \in \dot{\mathfrak{M}}_{arc}$. In that case $M_0 = \dot{\mathfrak{M}}_{arc}$ in condition (A3).

Fix $\xi \in M_t(QC(\mathbb{T}))$ for $t \in \Lambda$. By (8.4) and the proof of Lemma 6.2, every open neighborhood of $\rho^{(1)}_{\xi,\pm\infty}$ and $\rho^{(2)}_{\xi,\pm\infty}$ in the weak* topology contains, respectively, pure states $\rho^{(1)}_{\zeta,\pm\infty}$ and $\rho^{(2)}_{\zeta,\pm\infty}$, where $\zeta \in \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(QC(\mathbb{T}))$ and τ is on the right of t for $\rho^{(1)}_{\xi,\pm\infty}$ and $\rho^{(2)}_{\xi,\pm\infty}$ if, respectively, $\xi \in M^+_t(QC(\mathbb{T}))$ and $\xi \in \widetilde{M}^+_t(QC(\mathbb{T}))$, and τ is on the left of t for $\rho^{(1)}_{\xi,\pm\infty}$ and $\rho^{(2)}_{\xi,\pm\infty}$ if, respectively, $\xi \in \widetilde{M}^-_t(QC(\mathbb{T}))$ and $\xi \in M^-_t(QC(\mathbb{T}))$. Thus, condition (A3) is also fulfilled for the C^*-algebra $\mathfrak{B}_{arc}$.

For each $(\xi, x) \in \mathfrak{M}_{arc}$, we consider the representation

$$\pi_{(\xi,x)} : \mathfrak{B}_{arc} \to \mathcal{B}(l^2(G, \mathbb{C}^2)) \tag{8.5}$$

given on the generators of the C^*-algebra $\mathfrak{B}_{arc}$ by

$$\begin{aligned} \big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi((aI)^\pi)\big)f\big](g) &= \big[(\mathrm{Sym}\,((a\circ g)I))(\xi,x)\big]f(g), \\ \big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(S_{\mathbb{T}}^\pi)\big)f\big](g) &= \big[(\mathrm{Sym}\,S_{\mathbb{T}})(\xi,x)\big]f(g), \\ \big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_h^\pi)\big)f\big](g) &= f(gh), \end{aligned} \tag{8.6}$$

where $a \in PQC(\mathbb{T})$, $g,h \in G$ and $f \in l^2(G,\mathbb{C}^2)$.

Fix a set $\mathcal{O}_{arc} \subset \mathbb{T}\setminus\Lambda$ which contains exactly one point in each orbit defined by the group of shifts G on $\mathbb{T}\setminus\Lambda$, and consider the sets

$$\dot{\mathfrak{N}}_{arc} := \bigcup_{\tau\in\mathcal{O}_{arc}} M_\tau(QC(\mathbb{T}))\times\dot{\mathbb{R}}, \quad \mathfrak{N}_{arc} = \bigcup_{\tau\in\mathcal{O}_{arc}} M_\tau(QC(\mathbb{T}))\times\overline{\mathbb{R}}. \tag{8.7}$$

Theorem 8.1. *For each $B\in\mathfrak{B}$, the operator $B_{arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi)\in\mathfrak{B}_{arc}$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ if and only if for all $(\xi,x)\in\mathfrak{N}_{arc}$ the operators $\pi_{(\xi,x)}(B_{arc})$ are invertible on the space $l^2(G,\mathbb{C}^2)$ and*

$$\sup_{(\xi,x)\in\mathfrak{N}_{arc}} \big\|(\pi_{(\xi,x)}(B_{arc}))^{-1}\big\| < \infty. \tag{8.8}$$

Proof. The set $\dot{\mathfrak{N}}_{arc}$ given by (8.7) contains exactly one point in each G-orbit defined on the set $\dot{\mathfrak{M}}_{arc} \subset \widetilde{\dot{\mathfrak{M}}}_{arc}$ by the group $\{\beta_{g,\,arc} : g\in G\}$ of homeomorphisms given by (8.1). Thus, for each $(\xi,x)\in\dot{\mathfrak{N}}_{arc}$, starting with the representations (8.2)–(8.3) and following (4.3)–(4.5), we obtain the family of representations (8.5) defined by (8.6) and indexed by the points of the set $\mathfrak{N}_{arc}$, where we take into account the fact that the Hilbert spaces $l^2(G,\mathbb{C}^4)$ and $l^2(G,\mathbb{C}^2)\oplus l^2(G,\mathbb{C}^2)$ are isometrically isomorphic. Since assumptions (A1)–(A3) are fulfilled for the C^*-algebra $\mathfrak{B}_{arc}$, Theorem 4.1 implies the assertion of the theorem. □

For each $(\xi,x)\in\mathfrak{N}_{arc}$, we now consider the representation

$$\mathfrak{B}\to\mathcal{B}(l^2(G,\mathbb{C}^2)), \quad B\mapsto\pi_{(\xi,x)}(B_{arc}). \tag{8.9}$$

It is easily seen from (8.5)–(8.6) that representation (8.9) coincides with the representation $\Phi_{\xi,x}$ given by (3.2)–(3.3), that is,

$$\Phi_{\xi,x}(B) = \pi_{(\xi,x)}(B_{arc}) \quad \text{for all} \quad B\in\mathfrak{B} \quad \text{and all} \quad (\xi,x)\in\mathfrak{N}_{arc}.$$

Consequently, Theorem 8.1 is equivalent to part (i) of Theorem 3.1.

9. The C^*-algebra $\mathfrak{B}_\Lambda^\circ$

In this section we establish an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_\Lambda^\circ = P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(\mathfrak{B}^\pi)$ represented in the form (6.15).

Theorem 9.1. *The mapping given by*

$$P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(A^\pi)\mapsto P_\phi(\mathfrak{M}_\Lambda^\circ)\phi(A^\pi) \quad \textit{for all} \quad A\in\mathfrak{A} \tag{9.1}$$

is a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{A}_\Lambda^\circ := P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(\mathfrak{A}^\pi)$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_\Lambda^\circ := P_\phi(\mathfrak{M}_\Lambda^\circ)\phi(\mathfrak{A}^\pi)$.

Proof. Since $\mathfrak{M}^\circ_\Lambda$ is an open subset of $\dot{\mathfrak{M}}$ and the C^*-algebras $\varphi(\mathfrak{A}^\pi)$ and $\phi(\mathfrak{A}^\pi)$ are isometrically *-isomorphic, applying [5, Lemma 3.5], we infer that

$$\|P_\varphi(\mathfrak{M}^\circ_\Lambda)\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\mathfrak{M}^\circ_\Lambda)\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \quad \text{for all} \quad A \in \mathfrak{A} \tag{9.2}$$

by analogy with (7.6). This implies that (9.1) is a well-defined isometric *-isomorphism of the C^*-algebra $\mathfrak{A}^\circ_\Lambda$ onto the C^*-algebra $\widetilde{\mathfrak{A}}^\circ_\Lambda$. □

Let $\overline{\mathfrak{M}^\circ_\Lambda} := \bigcup_{t\in\Lambda} \overline{\mathfrak{M}^\circ_t}$, where $\overline{\mathfrak{M}^\circ_t}$ is given by (6.2). Since the Hilbert space $P_\phi(\mathfrak{M}^\circ_\Lambda)\mathcal{H}_\phi$ is isometrically isomorphic to the Hilbert space

$$\mathcal{H}^\circ_\Lambda := l^2(M^0_\Lambda(QC(\mathbb{T})), L^2_2(\mathbb{R})), \tag{9.3}$$

where $M^0_\Lambda(QC(\mathbb{T}))$ is given by (6.7), we immediately obtain from (6.8) and (6.13) the following.

Lemma 9.2. *The mapping*

$$\widetilde{\mathfrak{A}}^\circ_\Lambda \to \mathcal{B}(\mathcal{H}^\circ_\Lambda), \quad P_\phi(\mathfrak{M}^\circ_\Lambda)\phi(A^\pi) \mapsto \bigoplus_{\xi\in M^0_\Lambda(QC(\mathbb{T}))} (\mathrm{Sym}\, A)(\xi,\cdot)I$$

is an isometric C^*-algebra homomorphism. An operator $P_\phi(\mathfrak{M}^\circ_\Lambda)\phi(A^\pi)$ for $A \in \mathfrak{A}$ is invertible on the space $P_\phi(\mathfrak{M}^\circ_\Lambda)\mathcal{H}_\phi$ if and only if

$$\det((\mathrm{Sym}\, A)(\xi,x)) \neq 0 \quad \textit{for all} \quad (\xi,x) \in \overline{\mathfrak{M}^\circ_\Lambda}. \tag{9.4}$$

Theorem 9.1 and Lemma 9.2 imply the following.

Theorem 9.3. *The map $\mathrm{Sym}^\circ_\Lambda : \mathfrak{A}^\circ_\Lambda \to \mathcal{B}(\mathcal{H}^\circ_\Lambda)$, defined by*

$$P_\varphi(\mathfrak{M}^\circ_\Lambda)\varphi(A^\pi) \mapsto \bigoplus_{\xi\in M^0_\Lambda(QC(\mathbb{T}))} (\mathrm{Sym}\, A)(\xi,\cdot)I \quad \textit{for} \quad A \in \mathfrak{A},$$

is an isometric C^*-algebra homomorphism. An operator $P_\varphi(\mathfrak{M}^\circ_\Lambda)\varphi(A^\pi)$ for $A \in \mathfrak{A}$ is invertible on the space $P_\varphi(\mathfrak{M}^\circ_\Lambda)\mathcal{H}_\varphi$ if and only if (9.4) holds.

Given $t \in \Lambda$, let $\widetilde{\mathfrak{H}}^\pi_t$ be the closed two-sided ideal of the C^*-algebra $\mathcal{Z}^\pi$ generated by all the cosets $H^\pi_{P,t} = H_{P,t} + \mathcal{K}$ with $P \in \mathcal{P}$ (see (5.20) and (5.21)). Since $\mathcal{Z}^\pi$ is a commutative C^*-algebra, we conclude that

$$\widetilde{\mathfrak{H}}^\pi_t = \mathrm{clos}\left\{ \sum_{j=1}^m Z^\pi_j H^\pi_{P_j,t} : \, Z_j \in \mathcal{Z}, \; P_j \in \mathcal{P}, \; m \in \mathbb{N} \right\}, \tag{9.5}$$

where the closure is taken in the essential norm $\|\cdot\|$. Let $\widetilde{\mathfrak{H}}^\pi_\Lambda$ be the closed two-sided ideal of $\mathcal{Z}^\pi$ generated by all ideals $\widetilde{\mathfrak{H}}^\pi_t$ $(t \in \Lambda)$.

Following [5, Lemma 3.5], we define the set

$$\mathcal{Z}(\mathfrak{M}^\circ_\Lambda) := \left\{ Z^\pi \in \mathcal{Z}^\pi : \, \mathrm{supp}\, z(\cdot,\cdot) \subset \dot{\mathfrak{M}}^\circ_\Lambda, \; z(\xi,x) \in [0,1] \text{ for all } (\xi,x) \in \dot{\mathfrak{M}} \right\}, \tag{9.6}$$

where $z(\cdot,\cdot) \in C(\dot{\mathfrak{M}})$ is the Gelfand transform of the coset $\mathcal{Z}^\pi$, $\mathrm{supp}\, z(\cdot,\cdot)$ is the support of $z(\cdot,\cdot)$, and $\dot{\mathfrak{M}}^\circ_\Lambda := \bigcup_{t\in\Lambda} M^0_t(QC(\mathbb{T})) \times \dot{\mathbb{R}}$ is the closure in $\dot{\mathfrak{M}}$ of the set $\mathfrak{M}^\circ_\Lambda$ given by (6.3).

Lemma 9.4. *For the ideal $\widetilde{\mathfrak{H}}^\pi_\Lambda$, the following assertions hold:*

(i) *for every* $B^\pi \in \mathfrak{B}^\pi$ *and every* $H^\pi_\Lambda \in \widetilde{\mathfrak{H}}^\pi_\Lambda$,

$$B^\pi H^\pi_\Lambda = H^\pi_\Lambda B^\pi \in \mathfrak{A}^\pi; \tag{9.7}$$

(ii) $\widetilde{\mathfrak{H}}^\pi_\Lambda = \mathcal{Z}(\mathfrak{M}^\circ_\Lambda)$.

Proof. (i) For $B^\pi \in \mathfrak{A}^\pi$, the property (9.7) follows from the fact that $\mathcal{Z}^\pi$ is a central subalgebra of $\mathfrak{A}^\pi$. Since every coset $H^\pi \in \widetilde{\mathfrak{H}}^\pi_\Lambda$ is of the form $H^\pi = \sum_{t\in\Lambda} H^\pi_t$, where $H^\pi_t \in \widetilde{\mathfrak{H}}^\pi_t$, we can see that relation (9.7) for $B^\pi = U^\pi_g$ follows from Lemma 5.3 and (5.20). Hence, part (i) is proved for every $B^\pi \in \mathfrak{B}^\pi$.

(ii) Fix $t \in \Lambda$. The symbols of operators $H_{P,t}$ ($P \in \mathcal{P}$) are equal to $0_{2\times 2}$ for all $(\xi, x) \in \mathfrak{M} \setminus \mathfrak{M}^\circ_t$ (see Lemma 5.4). Hence, by Theorem 5.6, the Gelfand transform of the coset $H^\pi_{P,t}$ is a continuous function on $\dot{\mathfrak{M}}$ with support contained in the closure $\dot{\mathfrak{M}}^\circ_t = M^0_t(QC(\mathbb{T})) \times \dot{\mathbb{R}}$ of the set $\mathfrak{M}^\circ_t$ in $\dot{\mathfrak{M}}$. By (9.5), this property also holds for the Gelfand transform of any element H^π_t of the ideal $\widetilde{\mathfrak{H}}^\pi_t$. Thus, $\widetilde{\mathfrak{H}}^\pi_t \subset \mathcal{Z}(\mathfrak{M}^\circ_t)$, where

$$\mathcal{Z}(\mathfrak{M}^\circ_t) := \Big\{ Z^\pi \in \mathcal{Z}^\pi : \operatorname{supp} z(\cdot,\cdot) \subset \dot{\mathfrak{M}}^\circ_t,\ z(\xi,x) \in [0,1] \text{ for all } (\xi,x) \in \dot{\mathfrak{M}} \Big\}.$$

On the other hand, we infer from [5, Lemma 6.2] and (9.5) that any element $Z^\pi \in \mathcal{Z}(\mathfrak{M}^\circ_t)$ can be approximated in the C^*-algebra $\mathcal{Z}^\pi$ by cosets $H^\pi_t \in \widetilde{\mathfrak{H}}^\pi_t$, which implies that $\mathcal{Z}(\mathfrak{M}^\circ_t) \subset \widetilde{\mathfrak{H}}^\pi_t$. Moreover, Theorem 5.6 and [5, Lemma 6.2] imply that the ideal $\widetilde{\mathfrak{H}}^\pi_t$ is isomorphic to the ideal of all continuous functions on the compact $\dot{\mathfrak{M}}$ that vanish on $\dot{\mathfrak{M}} \setminus \mathfrak{M}^\circ_t$. Hence, $\widetilde{\mathfrak{H}}^\pi_t = \mathcal{Z}(\mathfrak{M}^\circ_t)$, which together with (9.6) implies part (ii). □

Consider the Hilbert space $\mathcal{H}^\circ_\Lambda$ given by (9.3), with $M^0_\Lambda(QC(\mathbb{T}))$ given by (6.7), and introduce the C^*-algebra

$$\Psi(\mathfrak{B}^\circ_\Lambda) := \operatorname{alg}\big\{\Psi^\circ_\Lambda(A^\pi),\ \Psi^\circ_\Lambda(U^\pi_g) :\ A \in \mathfrak{A},\ g \in G\big\} \subset \mathcal{B}(\mathcal{H}^\circ_\Lambda) \tag{9.8}$$

generated by the operators $\Psi^\circ_\Lambda(A^\pi)$ ($A \in \mathfrak{A}$) and $\Psi^\circ_\Lambda(U^\pi_g)$ ($g \in G$), where

$$\Psi^\circ_\Lambda(A^\pi) := \bigoplus_{\xi \in M^0_\Lambda(QC(\mathbb{T}))} (\operatorname{Sym} A)(\xi,\cdot)I, \quad \Psi^\circ_\Lambda(U^\pi_g) := \bigoplus_{\xi \in M^0_\Lambda(QC(\mathbb{T}))} e_{\ln g'(\xi)}(\cdot)I, \tag{9.9}$$

and $e_{\ln g'(\xi)}(x) := e^{ix \ln g'(\xi)}$ for all $(\xi,x) \in \mathfrak{M}^\circ_\Lambda$.

The mapping $g \mapsto \Psi^\circ_\Lambda(U^\pi_g)$ is a unitary representation of the group G in the Hilbert space $\mathcal{H}^\circ_\Lambda$, the adjoint operator $\Psi^\circ_\Lambda(U^\pi_g)^*$ equals $\Psi^\circ_\Lambda(U^\pi_{g^{-1}})$, and

$$\Psi^\circ_\Lambda(U^\pi_g)\Psi^\circ_\Lambda(A^\pi)\Psi^\circ_\Lambda(U^\pi_g)^* = \Psi^\circ_\Lambda(A^\pi)$$

for all $g \in G$ and all $A \in \mathfrak{A}$ due to (9.9). Consequently, the C^*-algebra $\Psi(\mathfrak{B}^\circ_\Lambda)$ is the closure of the algebra $\Psi(\mathfrak{B}^\circ_\Lambda)^0$ composed by the finite sums of the form $\sum_g \Psi^\circ_\Lambda(A^\pi_g)\Psi^\circ_\Lambda(U^\pi_g)$, where $A_g \in \mathfrak{A}$.

Theorem 9.5. *The mapping*

$$P_\varphi(\mathfrak{M}^\circ_\Lambda)\varphi\Big(\sum_{g\in F} A^\pi_g U^\pi_g\Big) \mapsto \sum_{g\in F} \Psi^\circ_\Lambda(A^\pi_g)\Psi^\circ_\Lambda(U^\pi_g), \tag{9.10}$$

where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$, extends to a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{B}_\Lambda^\circ$ onto the C^*-algebra $\Psi(\mathfrak{B}_\Lambda^\circ)$ given by (9.8).

Proof. Fix an operator $B = \sum_{g\in F} A_g U_g \subset \mathfrak{B}$, where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$, and put $\Psi_\Lambda^\circ(B^\pi) := \sum_{g\in F} \Psi_\Lambda^\circ(A_g^\pi)\Psi_\Lambda^\circ(U_g^\pi)$. Since the set $\mathfrak{M}_\Lambda^\circ$ is open and since $P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi) = \varphi(B^\pi)P_\varphi(\mathfrak{M}_\Lambda^\circ)$, we infer similarly to [5, Lemma 3.5] that

$$\|P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \sup_{Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ)} \|\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \tag{9.11}$$

where the set $\mathcal{Z}(\mathfrak{M}_\Lambda^\circ)$ is given by (9.6).

For each coset $H_\Lambda^\pi \in \widetilde{\mathfrak{H}}_\Lambda^\pi$, it follows from Lemma 9.4(i) that $B^\pi H_\Lambda^\pi \in \mathfrak{A}^\pi$. Hence, by (6.8), (5.3), (9.9) and by Lemmas 5.3–5.4, we obtain $\phi_\Lambda(B^\pi H_\Lambda^\pi) = \Psi_\Lambda^\circ(B^\pi)\phi_\Lambda(H_\Lambda^\pi)$ where $\phi_\Lambda : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_\Lambda^\circ)$ is the restriction of the representation ϕ (see (6.6), (6.8) and (6.11)) to the space $\mathcal{H}_\Lambda^\circ$ considered as an invariant Hilbert subspace of $\mathcal{H}_\phi$. Therefore, applying (9.2) or Theorem 9.3, we get

$$\begin{aligned}\|P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi H_\Lambda^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\phi(\mathfrak{M}_\Lambda^\circ)\phi(B^\pi H_\Lambda^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)}\\ &= \|\Psi_\Lambda^\circ(B^\pi)\phi_\Lambda(H_\Lambda^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)}.\end{aligned} \tag{9.12}$$

Since $\mathcal{Z}(\mathfrak{M}_\Lambda^\circ) = \widetilde{\mathfrak{H}}_\Lambda^\pi$ due to Lemma 9.4(ii) and since

$$P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi Z^\pi) = \varphi(B^\pi Z^\pi) \quad \text{for all} \quad Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ),$$

we deduce from (9.5) and the equalities (9.11)–(9.12) that

$$\begin{aligned}\sup_{Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ)} \|\Psi_\Lambda^\circ(B^\pi)\phi_\Lambda(Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)} &= \sup_{Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ)} \|P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}\\ = \sup_{Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ)} \|\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}.\end{aligned} \tag{9.13}$$

We now consider the identical representation π of the unital C^*-algebra $\Psi(\mathfrak{B}_\Lambda^\circ)$ in the Hilbert space $\mathcal{H}_\Lambda^\circ$. By (9.9), $\phi_\Lambda(\mathcal{Z}^\pi)$ is a central C^*-subalgebra of $\Psi(\mathfrak{B}_\Lambda^\circ)$ with the same unit. Clearly, the maximal ideal space of $\phi_\Lambda(\mathcal{Z}^\pi)$ coincides with $\dot{\mathfrak{M}}_\Lambda^\circ$. Since the set $\mathfrak{M}_\Lambda^\circ$ is an open subset of $\dot{\mathfrak{M}}_\Lambda$ and since the corresponding spectral projection $P_\pi(\mathfrak{M}_\Lambda^\circ)$ is the identity operator on Hilbert space $\mathcal{H}_\Lambda^\circ$, we conclude from [5, Lemma 3.5] that

$$\begin{aligned}\|\Psi_\Lambda^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)} &= \|P_\phi(\mathfrak{M}_\Lambda^\circ)\Psi_\Lambda^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)}\\ &= \sup_{Z^\pi \in \mathcal{Z}(\mathfrak{M}_\Lambda^\circ)} \|\Psi_\Lambda^\circ(B^\pi)\phi_\Lambda(Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)},\end{aligned}$$

which together with (9.13) implies that

$$\|P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|\Psi_\Lambda^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\Lambda^\circ)} \tag{9.14}$$

for all finite sums $B^\pi = \sum_{g\in F} A_g^\pi U_g^\pi \in \mathfrak{B}^\pi$ with $A_g^\pi \in \mathfrak{A}^\pi$. Since the set of such finite sums is dense in $\mathfrak{B}^\pi$ and since (9.14) holds, the mapping (9.10) uniquely extends to a C^*-algebra isomorphism of $\mathfrak{B}_\Lambda^\circ$ onto $\Psi(\mathfrak{B}_\Lambda^\circ)$. □

Every coset B^π of the C^*-algebra $\mathfrak{B}^\pi$ is the limit of a sequence of cosets of the form $B_n^\pi = \sum_{g\in F_n} A_{g,n}^\pi U_g^\pi$ where $A_{g,n}^\pi \in \mathfrak{A}^\pi$ and g runs through finite subsets F_n of G ($n \in \mathbb{N}$). Then, according to Theorem 9.5, the operator $\Psi_\Lambda(B^\pi)$ in the C^*-algebra $\Psi(\mathfrak{B}_\Lambda^\circ)$ has the form

$$\Psi_\Lambda^\circ(B^\pi) = \lim_{n\to\infty} \sum_{g\in F_n} \Psi_\Lambda^\circ(A_{g,n}^\pi)\Psi_\Lambda^\circ(U_g^\pi),$$

where the *-homomorphism $\Psi_\Lambda^\circ : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\Lambda^\circ)$ is an extension of the *-homomorphism $\phi_\Lambda : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_\Lambda^\circ)$ to the C^*-algebra $\mathfrak{B}^\pi$ in view of (6.8) and (9.9). Thus,

$$\begin{aligned} \Psi_\Lambda^\circ(B^\pi) &= \bigoplus_{\xi\in M_\Lambda^0(QC(\mathbb{T}))} B_\Lambda^\circ(\xi,\cdot)I \in \mathcal{B}(\mathcal{H}_\Lambda^\circ), \\ B_\Lambda^\circ(\xi,\cdot) :\ \mathbb{R} \to \mathbb{C}^{2\times 2},\quad & x \mapsto \lim_{n\to\infty} \sum_{g\in F_n} (\operatorname{Sym} A_{g,n})(\xi,x)\, e_{\ln g'(\xi)}(x). \end{aligned} \tag{9.15}$$

Taking into account the inverse closedness of the C^*-algebras $\mathfrak{B}_\Lambda^\circ$ and $\Psi(\mathfrak{B}_\Lambda^\circ)$ in the C^*-algebras $\mathcal{B}(P_\varphi(\mathfrak{M}_\Lambda^\circ)\mathcal{H}_\varphi)$ and $\mathcal{B}(\mathcal{H}_\Lambda^\circ)$, respectively, we immediately obtain an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_\Lambda^\circ$ from Theorem 9.5 and (9.15).

Theorem 9.6. *For each $B \in \mathfrak{B}$, the operator $B_\Lambda^\circ = P_\varphi(\mathfrak{M}_\Lambda^\circ)\varphi(B^\pi) \in \mathfrak{B}_\Lambda^\circ$ is invertible on the space $P_\varphi(\mathfrak{M}_\Lambda^\circ)\mathcal{H}_\varphi$ if and only if the operator $\Psi_\Lambda^\circ(B^\pi) \in \Psi(\mathfrak{B}_\Lambda^\circ)$ is invertible on the space $\mathcal{H}_\Lambda^\circ = l^2(M_\Lambda^0(QC(\mathbb{T})), L_2^2(\mathbb{R}))$, that is, if*

$$\min_{\xi\in M_\Lambda^0(QC(\mathbb{T}))} \inf_{x\in\mathbb{R}} \big|\det(B_\Lambda^\circ(\xi,x))\big| > 0.$$

For each $(\xi,x) \in \mathfrak{N}_\Lambda$, where $\mathfrak{N}_\Lambda$ is given by (3.1), we introduce the representation

$$\mathfrak{B} \to \mathcal{B}(\mathbb{C}^2), \quad B \mapsto B_\Lambda^\circ(\xi,x)I, \tag{9.16}$$

given on the generators of the C^*-algebra $\mathfrak{B}$ by the following 2×2 matrix functions $B_\Lambda^\circ(\cdot,\cdot)$ according to (5.3), (5.4) and (9.15):

$$\begin{aligned} (aI)_\Lambda^\circ(\xi,x) &= \operatorname{diag}\big\{a(\xi,1), a(\xi,0)\big\}, \\ (S_{\mathbb{T}})_\Lambda^\circ(\xi,x) &= \begin{bmatrix} \tanh(\pi x) & 1/\cosh(\pi x) \\ 1/\cosh(\pi x) & -\tanh(\pi x) \end{bmatrix}, \\ (U_h)_\Lambda^\circ(\xi,x) &= \operatorname{diag}\big\{e^{ix\ln h'(\xi)}, e^{ix\ln h'(\xi)}\big\}, \end{aligned} \tag{9.17}$$

where $a \in PQC(\mathbb{T})$, $a(\xi,\mu)$ is the value of the Gelfand transform of a at the point $(\xi,\mu) \in M(PQC(\mathbb{T}))$ and $h \in G$. It is easily seen from (9.17) that for each $(\xi,x) \in \mathfrak{N}_\Lambda$ representation (9.16) coincides with the representation $\Phi_{\xi,x}$ defined by (3.4)–(3.5), that is,

$$\Phi_{\xi,x}(B) = B_\Lambda^\circ(\xi,x)I \quad \text{for all} \quad B \in \mathfrak{B} \quad \text{and all} \quad (\xi,x) \in \mathfrak{N}_\Lambda.$$

Hence, identifying the operators $\Phi_{\xi,x}(B)$ and the matrices $B_\Lambda^\circ(\xi,x)$, we conclude that Theorem 9.6 is equivalent to part (ii) of Theorem 3.1.

10. Invertibility of functional operators

Let us study the invertibility of functional operators in the C^*-algebra

$$\mathcal{A} := \text{alg}(PQC(\mathbb{T}), U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$$

generated by the multiplication operators by piecewise quasicontinuous functions on $\mathbb{T}$ and by the isometric shift operators U_g $(g \in G)$.

The C^*-algebra $\mathcal{A}$ is the closure of the algebra $\mathcal{A}^0 \subset \mathcal{A}$ consisting of the functional operators $A = \sum_{g\in F} a_g U_g$, where $a_g \in PQC^0(\mathbb{T})$ and F runs through the finite subsets of G. In order to obtain an invertibility criterion for the operators $A \in \mathcal{A}$, we will apply the local-trajectory method.

Let $\widetilde{\mathfrak{A}} := \widetilde{\mathcal{Z}} := \{aI : a \in PQC(\mathbb{T})\}$. As $\widetilde{\mathcal{Z}} \cong PQC(\mathbb{T})$, we get $M(\widetilde{\mathcal{Z}}) = M(PQC(\mathbb{T}))$. Let us check the fulfillment of assumptions made in Section 4.

Lemma 2.3 implies that for every $g \in G$ and all $a \in PQC(\mathbb{T})$ the mapping $\widetilde{\alpha}_g : aI \mapsto U_g a U_g^{-1} = (a \circ g)I$ is a *-automorphism of the commutative C^*-algebras $\widetilde{\mathfrak{A}} = \widetilde{\mathcal{Z}} \subset \mathcal{B}(L^2(\mathbb{T}))$. Since G is a commutative and therefore an amenable group, we see that conditions (A1)–(A2) for the C^*-algebra $\mathcal{A}$ are satisfied.

For every $g \in G$, the *-automorphism $\widetilde{\alpha}_g$ induces the homeomorphism

$$\widetilde{\beta}_g : M(PQC(\mathbb{T})) \to M(PQC(\mathbb{T})), \quad (\xi, \mu) \mapsto (g(\xi), \mu), \tag{10.1}$$

where $g(\xi) \in M(QC(\mathbb{T}))$ is given by (5.30). If the ideal $\xi \in M(QC(\mathbb{T}))$ belongs to the fiber $M_t(SO(\mathbb{T}))$, then $g(\xi) \in M_{g(t)}(QC(\mathbb{T}))$. Moreover, if ξ belongs to $M^0(QC(\mathbb{T}))$ or $\widetilde{M}^{\pm}(QC(\mathbb{T}))$, respectively, then so is $g(\xi)$. Hence, taking into account the topologically free action of the group G on $M(QC(\mathbb{T}))$ (see [11, Lemma 4.2]) and the Gelfand topology (2.5) on $M(PQC(\mathbb{T}))$, we easily conclude that condition (A3) for the C^*-algebra $\mathcal{A}$ also holds, with $M_0 := M_{\mathbb{T}\setminus\Lambda}(PQC(\mathbb{T}))$, where $M_{\mathbb{T}\setminus\Lambda}(PQC(\mathbb{T})) := \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(PQC(\mathbb{T}))$.

With each maximal ideal $(\xi, \mu) \in M(PQC(\mathbb{T}))$ we associate the representation

$$\Pi_{(\xi,\mu)} : \mathcal{A} \to \mathcal{B}(l^2(G)), \quad A \mapsto A_{(\xi,\mu)} \tag{10.2}$$

given for the operators $A = \sum_{g\in F} a_g U_g \in \mathcal{A}^0$ by

$$(A_{(\xi,\mu)} f)(h) = \sum_{g\in F} [(a_g \circ h)(\xi, \mu)] f(hg) \quad (h \in G,\ f \in l^2(G)). \tag{10.3}$$

Following Section 8, we fix a set $\mathcal{O}_{arc} \subset \mathbb{T} \setminus \Lambda$ containing exactly one point in each orbit defined by the group of shifts G on $\mathbb{T} \setminus \Lambda$ and consider the set

$$\mathfrak{R}_{arc} := \bigcup_{\tau\in\mathcal{O}_{arc}} M_\tau(PQC(\mathbb{T})). \tag{10.4}$$

The set $\mathfrak{R}_{arc}$ contains exactly one point in each G-orbit defined by the action of the group G on $M_{\mathbb{T}\setminus\Lambda}(PQC(\mathbb{T}))$ by means of the homeomorphisms $\widetilde{\beta}_g$ $(g \in G)$ given by (10.1).

Since conditions (A1)–(A3) are fulfilled, we get the following.

Theorem 10.1. *A functional operator $A \in \mathcal{A}$ is invertible on the space $L^2(\mathbb{T})$ if and only if for all $(\xi, \mu) \in \mathfrak{R}_{arc}$ the operators $A_{(\xi,\mu)}$ are invertible on the space $l^2(G)$ and*

$$\sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \left\|(A_{(\xi,\mu)})^{-1}\right\| < \infty. \tag{10.5}$$

Proof. Take the maximal ideal $J_{(\xi,\mu)} := \{aI : a \in PQC(\mathbb{T}),\, a(\xi,\mu) = 0\}$ of $\widetilde{\mathcal{Z}}$ associated to each functional $(\xi,\mu) \in M(PQC(\mathbb{T}))$. As $\widetilde{\mathfrak{A}} = \widetilde{\mathcal{Z}}$, the mapping

$$\widetilde{\Pi}_{(\xi,\mu)} : \widetilde{\mathfrak{A}}/J_{(\xi,\mu)} \to \mathbb{C}, \quad aI + J_{(\xi,\mu)} \mapsto a(\xi,\mu),$$

is an isometric representation of the C^*-algebra $\widetilde{\mathfrak{A}}/J_{(\xi,\mu)}$ in $\mathbb{C}$. Following (4.3)–(4.5), we construct representations of the C^*-algebra $\mathcal{A}$ in the Hilbert space $l^2(G)$ by formulas (10.2) and (10.3). Since $\mathcal{A}$ satisfies conditions (A1)–(A3) of the local-trajectory method, Theorem 4.1 immediately implies the statement of the theorem. □

Applying Theorems 10.1 and 8.1 we can prove the following result.

Theorem 10.2. *For any functional operator $A \in \mathcal{A}$, its Fredholmness on the space $L^2(\mathbb{T})$ is equivalent to its invertibility on this space.*

Proof. Obviously, we only need to prove that the Fredholmness of each functional operator A implies its invertibility. Suppose that an operator $A \in \mathcal{A}$ is Fredholm on the space $L^2(\mathbb{T})$. Then the operator $A_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$. Consequently, by Theorem 8.1, for all $(\xi, x) \in \mathfrak{N}_{arc}$ the operators $\pi_{(\xi,x)}(A_{arc})$ are invertible on the space $l^2(G, \mathbb{C}^2)$ and

$$\sup_{(\xi,x)\in\mathfrak{N}_{arc}} \left\|(\pi_{(\xi,x)}(A_{arc}))^{-1}\right\| < \infty. \tag{10.6}$$

By (8.5)–(8.6), (5.3) and (10.2)–(10.3), we infer that, for all $x \in \overline{\mathbb{R}}$,

$$\pi_{(\xi,x)}(A_{arc}) = \begin{cases} \operatorname{diag}\{A_{(\xi,0)}, A_{(\xi,0)}\} & \text{if } \xi \in \widetilde{M}^{-}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \operatorname{diag}\{A_{(\xi,1)}, A_{(\xi,0)}\} & \text{if } \xi \in M^{0}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \operatorname{diag}\{A_{(\xi,1)}, A_{(\xi,1)}\} & \text{if } \xi \in \widetilde{M}^{+}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \end{cases} \tag{10.7}$$

where

$$\widetilde{M}^{\pm}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})) := \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} \widetilde{M}^{\pm}_{\tau}(QC(\mathbb{T})), \quad M^{0}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})) := \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M^{0}_{\tau}(QC(\mathbb{T})). \tag{10.8}$$

Comparing the sets $\mathfrak{N}_{arc}$ and $\mathfrak{R}_{arc}$ given by (3.1) and (10.4), respectively, we conclude from (10.7) and (10.6) that for every $(\xi,\mu) \in \mathfrak{R}_{arc}$ the operator $A_{(\xi,\mu)}$ is invertible on the space $l^2(G)$ and condition (10.5) is fulfilled. Hence, by Theorem 10.1, the operator A is invertible on the space $L^2(\mathbb{T})$. □

11. The C^*-algebra $\mathfrak{B}_\Lambda^\diamond$

In this section we will show that for every $B \in \mathfrak{B}$ the invertibility of the operator $B_{arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi)$ on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ implies the invertibility of the operators $B_\Lambda^\diamond = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B^\pi)$ on the Hilbert space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$. This means that condition (iii) in Theorem 6.3 is superfluous.

Observe that the C^*-algebra $\mathfrak{B} = \mathrm{alg}(PQC(\mathbb{T}), S_\mathbb{T}, U_G)$ can be viewed as the C^*-algebra $\mathfrak{B} = \mathrm{alg}(\mathcal{A}, S_\mathbb{T})$ generated by the C^*-algebra $\mathcal{A}$ studied in Section 10 and by the Cauchy singular integral operator $S_\mathbb{T}$. By analogy with Theorem 5.5, we can get another general form of operators $B \in \mathfrak{B}$.

Let $\mathfrak{B}^0$ denote the dense non-closed subalgebra of the C^*-algebra $\mathfrak{B}$ consisting of all operators of the form $\sum_{i=1}^n T_{i1}T_{i2}\dots T_{ij_i}$ where $n, j_i \in \mathbb{N}$ and $T_{i,k} \in \mathcal{A}^0 \cup \{S_\mathbb{T}\}$. We denote by $\mathfrak{H}$ the closed two-sided ideal of $\mathfrak{B}$ generated by all the commutators $[aI, S_\mathbb{T}] := aS_\mathbb{T} - S_\mathbb{T}aI$ with $a \in PC(\mathbb{T})$, that is, the closure of the set

$$\mathfrak{H}^0 := \left\{ \sum_{i=1}^n B_i H_i C_i : \, B_i, C_i \in \mathfrak{B}^0, \, H_i = [a_i I, S_\mathbb{T}], \, a_i \in PC^0(\mathbb{T}), \, n \in \mathbb{N} \right\}.$$

The ideal $\mathcal{K}$ of all compact operators on $L^2(\mathbb{T})$ is contained in $\mathfrak{H}$ (see, e.g., [16]). Hence, the commutators $[aI, S_\mathbb{T}]$ $(a \in QC(\mathbb{T}))$ and $[U_g, S_\mathbb{T}]$ $(g \in G)$ belong to the ideal $\mathfrak{H}$ because these operators are compact on the space $L^2(\mathbb{T})$ according to (2.6) and (2.7). As a result, for every $A \in \mathcal{A}$ the commutators $[A, S_\mathbb{T}]$ also are in $\mathfrak{H}$.

Let $\widetilde{\mathcal{A}}$ be the C^*-algebra of the 2×2 diagonal matrices with $\mathcal{A}$-valued entries. We now obtain an analogue of Theorem 5.5 for the C^*-algebra $\mathfrak{B}$ (cf. [20], [21]).

Theorem 11.1. *Every operator $B \in \mathfrak{B}$ is uniquely represented in the form*

$$B = A^+P_\mathbb{T}^+ + A^-P_\mathbb{T}^- + H_B, \tag{11.1}$$

where $A^\pm$ are functional operators in the C^*-algebra $\mathcal{A}$, $P_\mathbb{T}^\pm = (I \pm S_\mathbb{T})/2$ are the orthogonal projections associated with the Cauchy singular integral operator $S_\mathbb{T}$, $H_B \in \mathfrak{H}$, the map $B \mapsto \mathrm{diag}\{A^+, A^-\}$ is a C^*-algebra homomorphism of the C^*-algebra $\mathfrak{B}$ onto the C^*-algebra $\widetilde{\mathcal{A}}$ with kernel $\mathfrak{H}$, and

$$\|A^\pm\| \le \inf_{H \in \mathfrak{H}} \|B + H\| \le |B| = \inf_{K \in \mathcal{K}} \|B + K\|. \tag{11.2}$$

Proof. Obviously, every operator $B \in \mathfrak{B}^0$ is represented in the form (11.1), and the mapping $B \mapsto \mathrm{diag}\{A^+, A^-\}$ is an algebraic homomorphism of the non-closed algebra $\mathfrak{B}^0$ into $\widetilde{\mathcal{A}}$ with the kernel contained in $\mathfrak{H}^0$. This mapping is defined for the generators of the algebra $\mathfrak{B}$ by

$$aI \mapsto \mathrm{diag}\{aI, aI\}, \quad U_g \mapsto \mathrm{diag}\{U_g, U_g\}, \quad S_\mathbb{T} \mapsto \mathrm{diag}\{I, -I\}.$$

Obviously, it only remains to prove (11.2) for all $B \in \mathfrak{B}^0$.

Taking into account the facts that the ideal $\mathfrak{H} \subset \mathfrak{B}$ is generated by all the commutators $[aI, S_\mathbb{T}]$ $(a \in PC(\mathbb{T}))$ and that

$$(\mathrm{Sym}([aI, S_\mathbb{T}]))(\xi, \pm\infty) = 0_{2\times 2} \quad \text{for all } \; a \in PC(\mathbb{T}) \; \text{ and all } \; \xi \in M(QC(\mathbb{T}))$$

according to (5.3), we infer from (8.6) that for any operator $H \in \mathfrak{H}$,

$$\pi_{(\xi,\pm\infty)}(H_{arc}) = 0 \quad \text{for all } \ \xi \in M(QC(\mathbb{T})), \tag{11.3}$$

where $H_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(H^\pi)$. Analogously, by (5.3), for all $\xi \in M(QC(\mathbb{T}))$,

$$(\mathrm{Sym}\, P_{\mathbb{T}}^{\pm})(\xi, \pm\infty) = \mathrm{diag}\{1, 0\}, \quad (\mathrm{Sym}\, P_{\mathbb{T}}^{\pm})(\xi, \mp\infty) = \mathrm{diag}\{0, 1\}. \tag{11.4}$$

Further, from (10.7) it follows that for $x \in \{+\infty, -\infty\}$,

$$\pi_{(\xi,x)}(A^{\pm}_{arc}) = \begin{cases} \mathrm{diag}\{A^{\pm}_{(\xi,0)}, A^{\pm}_{(\xi,0)}\} & \text{if } \ \xi \in \widetilde{M}^{-}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \mathrm{diag}\{A^{\pm}_{(\xi,1)}, A^{\pm}_{(\xi,0)}\} & \text{if } \ \xi \in M^{0}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \mathrm{diag}\{A^{\pm}_{(\xi,1)}, A^{\pm}_{(\xi,1)}\} & \text{if } \ \xi \in \widetilde{M}^{+}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \end{cases} \tag{11.5}$$

where the sets on the right of (11.5) are given by (10.8). Hence, for every operator $B = A^+P^+_{\mathbb{T}} + A^-P^-_{\mathbb{T}} + H_B \in \mathfrak{B}^0$ we deduce from (11.3)–(11.5) that

$$\pi_{(\xi,\pm\infty)}(B_{arc}) = \mathrm{diag} \begin{cases} \{A^{\pm}_{(\xi,0)}, A^{\mp}_{(\xi,0)}\} & \text{if } \ \xi \in \widetilde{M}^{-}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \{A^{\pm}_{(\xi,1)}, A^{\mp}_{(\xi,0)}\} & \text{if } \ \xi \in M^{0}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})), \\ \{A^{\pm}_{(\xi,1)}, A^{\mp}_{(\xi,1)}\} & \text{if } \ \xi \in \widetilde{M}^{+}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})). \end{cases} \tag{11.6}$$

Therefore, letting $M^{\pm}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T})) := \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M^{\pm}_{\tau}(QC(\mathbb{T}))$, we infer that

$$\begin{aligned} &\max\Big\{ \sup_{\xi\in M^{-}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T}))} \|A^{\pm}_{(\xi,0)}\|, \sup_{\xi\in M^{+}_{\mathbb{T}\setminus\Lambda}(QC(\mathbb{T}))} \|A^{\pm}_{(\xi,1)}\| \Big\} \\ &\quad \le \|P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi + H^\pi)\| \le \|B + H\| \quad \text{for all } \ H \in \mathfrak{H}. \end{aligned} \tag{11.7}$$

By Theorem 10.1, the C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to the C^*-algebra

$$\Big\{ \bigoplus_{(\xi,\mu)\in\mathfrak{R}_{arc}} A_{(\xi,\mu)} : \ A \in \mathcal{A} \Big\} \subset \bigoplus_{(\xi,\mu)\in\mathfrak{R}_{arc}} \mathcal{B}(l^2(G))$$

equipped with the norm $\sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \|A_{(\xi,\mu)}\|$. Therefore, for every $A \in \mathcal{A}$,

$$\|A\| = \sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \|A_{(\xi,\mu)}\|. \tag{11.8}$$

Finally, from (11.7) and (11.8) it follows that $\|A^{\pm}\| \le \|B+H\|$ for all $B \in \mathfrak{B}^0$ and all $H \in \mathfrak{H}$, which immediately implies (11.2). □

Let $G(\mathbb{T})$ denote the set of the G-orbits $G(t) := \{g(t) : g \in G\} \subset \mathbb{T}$ for all $t \in \mathbb{T}$. For each $w \in G(\mathbb{T})$ we denote by $\mathfrak{H}_w$ the closed two-sided ideal of $\mathfrak{B}$ generated by the operator V_t given by (5.6), where t is an arbitrary point of the G-orbit w. Observe that $V_tV_\tau \simeq 0$ for $t, \tau \in \mathbb{T}$ and $t \ne \tau$. Hence, $H_wH_v \simeq 0$ if $w, v \in G(\mathbb{T})$ and $w \ne v$. Similarly to [21, Lemma 5.4] we get the following result.

Lemma 11.2. *Every operator $H \in \mathfrak{H}^0$ is of the form $H \simeq \sum_{w\in G_H} H_w$, where G_H is an at most finite subset of $G(\mathbb{T})$ and $H_w \in \mathfrak{H}_w$.*

Consider now the C^*-algebra $\mathfrak{B}_\Lambda^\diamond = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(\mathfrak{B}^\pi)$ (see (6.16)). Given $t \in \Lambda$, let $\mathfrak{H}_t$ be the closed two-sided ideal of $\mathfrak{B}$ generated by the operator V_t given by (5.6). By Lemma 9.4(i), the ideal $\mathfrak{H}_t$ is contained in the C*-algebra $\mathfrak{A}$. Clearly, the ideal $\mathfrak{H}$ is generated by all ideals $\mathfrak{H}_t$ $(t \in \mathbb{T})$.

Lemma 11.3. *If $H \in \mathfrak{H}$, then*

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(H^\pi) = 0. \tag{11.9}$$

Proof. First, let us prove that, for the set $\mathfrak{M}_\Lambda^\diamond$ closed in $\dot{\mathfrak{M}}$,

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(V_\tau^\pi) = 0 \quad \text{for all} \ \ \tau \in \mathbb{T}. \tag{11.10}$$

Fix $\tau \in \mathbb{T}$. Let Δ_n be a sequence of open sets of $\dot{\mathfrak{M}}$ such that $\bigcap_n \Delta_n = \mathfrak{M}_\Lambda^\diamond$. Analogously to Theorem 7.1 one can establish that

$$\|P_\varphi(\Delta_n)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\Delta_n)\phi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \quad \text{for all } n \in \mathbb{N}. \tag{11.11}$$

If $\tau \in \mathbb{T} \setminus \Lambda$, then for the Hilbert space $\mathcal{H}_\phi$ we have the equality

$$\lim_{n\to\infty} \|P_\phi(\Delta_n)\phi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} = 0 \tag{11.12}$$

because $\tau \notin \overline{\Delta}_n$ for all sufficiently large n, where $\overline{\Delta}_n$ is the closure of Δ_n in $\dot{\mathfrak{M}}$. Hence, we derive from (11.11) that, for the abstract Hilbert space $\mathcal{H}_\varphi$,

$$\lim_{n\to\infty} \|P_\varphi(\Delta_n)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = 0 \tag{11.13}$$

for all $\tau \in \mathbb{T} \setminus \Lambda$. Let now $\tau \in \Lambda$. Then we conclude from Lemma 5.2 that again (11.12) holds, which in view of (11.11) gives (11.13) for all $\tau \in \Lambda$. Thus,

$$\lim_{n\to\infty} \|P_\varphi(\Delta_n)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = 0 \quad \text{for all} \ \ \tau \in \mathbb{T}. \tag{11.14}$$

Let $\varphi|_{\mathcal{Z}^\pi}$ be the restriction of the representation φ to the central C^*-subalgebra $\mathcal{Z}^\pi$. Consider the decomposition of the representation $\varphi|_{\mathcal{Z}^\pi}$ into the direct sum of cyclic representations φ_α in the mutually orthogonal subspaces $\mathcal{H}_{\varphi,\alpha}$ of the space $\mathcal{H}_\varphi$. Then, by [5, Lemma 3.4], for each subspace $\mathcal{H}'_\varphi = \bigoplus_{\alpha\in Q} \mathcal{H}_{\varphi,\alpha} \subset \mathcal{H}_\varphi$ with a finite set Q, we have

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(V_\tau^\pi) = \operatorname*{s-lim}_{n\to\infty} P_\varphi(\Delta_n)\varphi(V_\tau^\pi).$$

Hence,

$$\begin{aligned} 0 \le \|P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}'_\varphi)} &\le \liminf_{n\to\infty} \|P_\varphi(\Delta_n)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}'_\varphi)} \\ &\le \liminf_{n\to\infty} \|P_\varphi(\Delta_n)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \end{aligned}$$

which implies according to (11.14) that for all considered subspaces $\mathcal{H}'_\varphi \subset \mathcal{H}_\varphi$,

$$\|P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(V_\tau^\pi)\|_{\mathcal{B}(\mathcal{H}'_\varphi)} = 0. \tag{11.15}$$

Since the set of vectors $\xi = \{\xi_\alpha\}_\alpha \in \mathcal{H}_\varphi$ with a finite number of non-zero entries $\xi_\alpha \in \mathcal{H}_{\varphi,\alpha}$ is dense in $\mathcal{H}_\varphi$, the relations (11.15) imply (11.10).

Consider now an operator $H_w \in \mathfrak{H}_w$ with $w \in G(\mathbb{T})$. It follows from (5.16), (5.18) and (5.28) that

$$H_w^\pi = \lim_{n\to\infty} \sum_{\tau\in G_{n,w}} B_{n,\tau}^\pi V_\tau^\pi, \tag{11.16}$$

where $G_{n,w}$ are finite subsets of the G-orbit w and $B_{n,\tau}$ are operators in the C^*-algebra $\mathfrak{B}$. Since the set $\mathfrak{M}_\Lambda^\diamond$ belongs to the family $\mathfrak{R}_G(\dot{\mathfrak{M}})$ defined in (6.5), we infer from (4.8) that for all $B \in \mathfrak{B}$,

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B^\pi) = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B^\pi)P_\varphi(\mathfrak{M}_\Lambda^\diamond).$$

Therefore, from (11.16) and (11.10) it follows that

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(H_w^\pi) = \lim_{n\to\infty} \sum_{\tau\in G_{n,w}(\mathbb{T})} P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B_{n,\tau}^\pi)P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(V_\tau^\pi) = 0. \tag{11.17}$$

Finally, by Lemma 11.2, for any operator $H \in \mathfrak{H}^0$ there are a finite subset G_H of $G(\mathbb{T})$ and operators $H_w \in \mathfrak{H}_w$ such that $H^\pi = \sum_{w\in G_H} H_w^\pi$. Hence, by (11.17), for $H \in \mathfrak{H}^0$ we obtain

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(H^\pi) = \sum_{w\in G_H} P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(H_w^\pi) = 0,$$

which implies (11.9) because the set $\mathfrak{H}^0$ is dense in $\mathfrak{H}$. □

Lemma 11.3 shows that the operators $H \in \mathfrak{H}$ do not have influence on the operators in the C^*-algebra $\mathfrak{B}_\Lambda^\diamond$ and, in particular, implies the following.

Corollary 11.4. *If A is a functional operator in $\mathcal{A}$, then*

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi([A, S_\mathbb{T}]^\pi) = 0.$$

Theorem 11.5. *If $B \in \mathfrak{B}$ and the operator $B_{arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$, then the operator $B_\Lambda^\diamond := P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$.*

Proof. Fix an operator $B \in \mathfrak{B}$. By Theorem 11.1, the operator B has the form $B = A^+P_\mathbb{T}^+ + A^-P_\mathbb{T}^- + H_B$, with $A^\pm \in \mathcal{A}$ and $H_B \in \mathfrak{H}$. We then infer from Lemma 11.3 that

$$B_\Lambda^\diamond = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((A^+P_\mathbb{T}^+ + A^-P_\mathbb{T}^-)^\pi). \tag{11.18}$$

If the operator B_{arc} is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$, then the operators $(A^\pm)_\Lambda^\diamond := P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((A^\pm)^\pi)$ are invertible on the space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$. Indeed, if the operator B_{arc} is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$, then, by Theorem 8.1, the operators $\pi_{(\xi,x)}(B_{arc})$ for all $(\xi, x) \in \mathfrak{N}_{arc}$ are invertible on the Hilbert space $l^2(G, \mathbb{C}^2)$ and condition (8.8) is fulfilled. In particular, the operators $\pi_{(\xi,\pm\infty)}(B_{arc})$ are invertible on the space $l^2(G, \mathbb{C}^2)$ for all $(\xi, \pm\infty) \in \mathfrak{R}_{arc}$, where $\mathfrak{R}_{arc}$ is defined by (10.4). Then it follows from (11.6) that all the operators $A_{(\xi,\mu)}^\pm$ for $(\xi, \mu) \in \mathfrak{R}_{arc}$ are invertible on the space $l^2(G)$ and

$$\sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \left\| \left(A_{(\xi,\mu)}^\pm\right)^{-1} \right\| < \infty.$$

Hence, by Theorem 10.1, the functional operators $A^\pm$ are invertible on the space $L^2(\mathbb{T})$, which implies the invertibility of both the operators $(A^\pm)_\Lambda^\diamond$ on the Hilbert space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$.

Corollary 11.4 implies that for every functional operator $A \in \mathcal{A}$,

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(A^\pi)P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^\pm)^\pi) = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^\pm)^\pi)P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi(A^\pi).$$

Therefore, taking into account the invertibility of the operators $(A^\pm)_\Lambda^\diamond$ and the equalities

$$P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^+)^\pi)P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^-)^\pi) = P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^+P_{\mathbb{T}}^-)^\pi) = 0,$$

we conclude from (11.18) that the operator

$$(B_\Lambda^\diamond)^{-1} := ((A^+)_\Lambda^\diamond)^{-1}P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^+)^\pi) + ((A^-)_\Lambda^\diamond)^{-1}P_\varphi(\mathfrak{M}_\Lambda^\diamond)\varphi((P_{\mathbb{T}}^-)^\pi),$$

where $((A^\pm)_\Lambda^\diamond)^{-1}$ are the inverses for the operators $(A^\pm)_\Lambda^\diamond$, is the inverse of the operator $B_\Lambda^\diamond$ on the space $P_\varphi(\mathfrak{M}_\Lambda^\diamond)\mathcal{H}_\varphi$. □

Finally, to complete the proof of Theorem 3.1, which gives the Fredholm criterion for the operators B in the C^*-algebra $\mathfrak{B}$, it remains to combine Theorems 6.3, 8.1, 9.6 and 11.5. This also gives the Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ (see (3.7)).

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