Density of Analytic Polynomials in Abstract Hardy Spaces

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Abstract. Let X be a separable Banach function space on the unit circle $\mathbb{T}$ and $H[X]$ be the abstract Hardy space built upon X. We show that the set of analytic polynomials is dense in $H[X]$ if the Hardy-Littlewood maximal operator is bounded on the associate space X'. This result is specified to the case of variable Lebesgue spaces.

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1. Introduction

For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane $\mathbb{C}$. For $f \in L^1$, let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi}\, d\varphi, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. The classical Hardy spaces H^p are given by

$$H^p := \left\{ f \in L^p \ : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

A function of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T}, \quad \alpha_0, \dots, \alpha_n \in \mathbb{C},$$

is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by $\mathcal{P}_A$. It is well known that that the set $\mathcal{P}_A$ is dense in H^p whenever $1 \le p < \infty$ (see, e.g., [3, Chap. III, Corollary 1.7(a)]).

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Let X be a Banach space continuously embedded in L^1. Following [17, p. 877], we will consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \{f \in X : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0\}.$$

It is clear that if $1 \le p \le \infty$, then $H[L^p]$ is the classical Hardy space H^p. The aim of this note is to find sufficient conditions for the density of the set $\mathcal{P}_A$ in the space $H[X]$ when X falls into the class of so-called Banach function spaces.

We equip $\mathbb{T}$ with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. Let L^0 be the space of all measurable complex-valued functions on $\mathbb{T}$. As usual, we do not distinguish functions, which are equal almost everywhere (for the latter we use the standard abbreviation a.e.). Let L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E.

Following [1, Chap. 1, Definition 1.1], a mapping $\rho : L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \ge 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

$$\begin{aligned}
&\text{(A1)} && \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \ \rho(af) = a\rho(f), \ \rho(f+g) \le \rho(f) + \rho(g),\\
&\text{(A2)} && 0 \le g \le f \ \mu - \text{a.e.} \ \Rightarrow \ \rho(g) \le \rho(f) \quad \text{(the lattice property)},\\
&\text{(A3)} && 0 \le f_n \uparrow f \text{ a.e.} \ \Rightarrow \ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)},\\
&\text{(A4)} && m(E) < \infty \ \Rightarrow \ \rho(\chi_E) < \infty,\\
&\text{(A5)} && \int_E f(t)\, dm(t) \le C_E \rho(f)
\end{aligned}$$

with the constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X under the natural linear space operations and under this norm becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(t)g(t)\, d\mu(t) \ : \ f \in L^0_+, \ \rho(f) \le 1\right\}, \ g \in L^0_+.$$

It is a Banach function norm itself [1, Chap. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X. The associate space X' can be viewed a subspace of the (Banach) dual space X^*.

The distribution function m_f of an a.e. finite function $f \in L^0$ is defined by

$$m_f(\lambda) := m\{t \in \mathbb{T} : |f(t)| > \lambda\}, \quad \lambda \ge 0.$$

Two a.e. finite functions $f, g \in L^0$ are said to be equimeasurable if

$$m_f(\lambda) = m_g(\lambda) \quad \text{for all} \quad \lambda \ge 0.$$

The non-increasing rearrangement of an a.e. finite function $f \in L^0$ is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \le x\}, \quad x \ge 0.$$

We refer to [1, Chap. 2, Section 1] and [11, Chap. II, Section 2] for properties of distribution functions and non-increasing rearrangements. A Banach function space X is called rearrangement-invariant if for every pair of a.e. finite equimeasurable functions $f, g \in L^0$, one has the following property: if $f \in X$, then $g \in X$ and the equality $\|f\|_X = \|g\|_X$ holds. Lebesgue spaces L^p, $1 \le p \le \infty$, as well as, more general Orlicz spaces, Lorentz spaces, and Marcinkiewicz spaces are classical examples of rearrangement-invariant Banach function spaces (see [1, 11]). For more recent examples of rearrangement-invariant spaces, like Cesàro, Copson, and Tandori spaces, we refer to the paper of Maligranda and Leśnik [13].

One of our motivations for this work is the recent progress in the study of Harmonic Analysis in the setting of variable Lebesgue spaces [4, 6, 10]. Let $\mathfrak{P}(\mathbb{T})$ be the set of all measurable functions $p : \mathbb{T} \to [1, \infty]$. For $p \in \mathfrak{P}(\mathbb{T})$, put

$$\mathbb{T}_\infty^{p(\cdot)} := \{t \in \mathbb{T} \ : \ p(t) = \infty\}.$$

For a measurable function $f : \mathbb{T} \to \mathbb{C}$, consider

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{T}\setminus\mathbb{T}_\infty^{p(\cdot)}} |f(t)|^{p(t)} dm(t) + \|f\|_{L^\infty(\mathbb{T}_\infty^{p(\cdot)})}.$$

According to [4, Definition 2.9], the variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\}$$

(see, e.g., [4, Theorems 2.17, 2.71 and Section 2.10.3]). If $p \in \mathfrak{P}(\mathbb{T})$ is constant, then $L^{p(\cdot)}$ is nothing but the standard Lebesgue space L^p. If $\in \mathfrak{P}(\mathbb{T})$ is not constant, then $L^{p(\cdot)}$ is not rearrangement-invariant [4, Example 3.14]. Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda's paper [14] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces. The associate space of $L^{p(\cdot)}$ is isomorphic to the space $L^{p'(\cdot)}$, where $p' \in \mathfrak{P}(\mathbb{T})$ is defined so that $1/p(t) + 1/p'(t) = 1$ for a.e. $t \in \mathbb{T}$ with the usual convention $1/\infty := 0$ [6, Theorem 3.2.13]. For $p \in \mathfrak{P}(\mathbb{T})$, put

$$p_- := \operatorname*{ess\,inf}_{t\in\mathbb{T}} p(t), \quad p_+ := \operatorname*{ess\,sup}_{t\in\mathbb{T}} p(t).$$

The space variable Lebesgue space $L^{p(\cdot)}$ is separable if and only if $p_+ < \infty$ (see, e.g., [4, Theorem 2.78]).

The following result is a kind of folklore.

Theorem 1.1. *Let X be a separable rearrangement-invariant Banach function space on $\mathbb{T}$. Then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[X]$. Moreover, for every $f \in H[X]$, there is a sequence of* *analytic polynomials* $\{p_n\}$ *such that* $\|p_n\|_X \le \|f\|_X$ *for all* $n \in \mathbb{N}$ *and* $p_n \to f$ *in the norm of* X *as* $n \to \infty$.

Surprisingly enough, we could not find in the literature neither Theorem 1.1 explicitly stated nor any result on the density of $\mathcal{P}_A$ in abstract Hardy spaces $H[X]$ in the case when X is an arbitrary Banach function space beyond the class of rearrangement-invariant spaces. The aim of this note is to fill in this gap.

Given $f \in L^1$, the Hardy-Littlewood maximal function is defined by

$$(Mf)(t) := \sup_{I \ni t} \frac{1}{m(I)} \int_I |f(\tau)|\, dm(\tau), \quad t \in \mathbb{T},$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ containing $t \in \mathbb{T}$. The operator $f \mapsto Mf$ is called the Hardy-Littlewood maximal operator.

Theorem 1.2 (Main result). *Suppose* X *is a separable Banach function space on* $\mathbb{T}$. *If the Hardy-Littlewood maximal operator* M *is bounded on the associate space* X', *then the set of analytic polynomials* $\mathcal{P}_A$ *is dense in the abstract Hardy space* $H[X]$.

To illustrate this result in the case of variable Lebesgue spaces, we will need the following classes of variable exponents. Following [4, Definition 2.2], one says that $r : \mathbb{T} \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $C_0 > 0$ such that

$$|r(x) - r(y)| = C_0/(-\log|x-y|)) \quad \text{for all} \quad x, y \in \mathbb{T}, \quad |x-y| < 1/2.$$

The class of all locally log-Hölder continuous functions is denoted by $LH_0(\mathbb{T})$. If $p_+ < \infty$, then $p \in LH_0(\mathbb{T})$ if and only if $1/p \in LH_0(\mathbb{T})$. By [4, Theorem 4.7], if $p \in \mathfrak{P}(\mathbb{T})$ is such that $1 < p_-$ and $1/p \in LH_0(\mathbb{T})$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}$. This condition was initially referred to as "almost necessary" (see [4, Section 4.6.1] for further details). However, Lerner [12] constructed an example of discontinuous variable exponent such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

Kapanadze and Kopaliani [7] developed further Lerner's ideas. They considered the following class of variable exponents. Recall that a function $f \in L^1$ belongs to the space BMO if

$$\|f\|_* := \sup_{I \subset \mathbb{T}} \frac{1}{m(I)} \int_I |f(t) - f_I|\, dm(t) < \infty,$$

where f_I is the integral average of f on the arc I and the supremum is taken over all arcs $I \subset \mathbb{T}$. For $f \in BMO$, put

$$\gamma(f, r) := \sup_{m(I) \le r} \frac{1}{m(I)} \int_I |f(t) - f_I|\, dm(t).$$

Let $VMO^{1/|\log|}$ be the set of functions $f \in BMO$ such that

$$\gamma(f, r) = o(1/|\log r|) \quad \text{as} \quad r \to 0.$$

Note that $VMO^{1/|\log|}$ contains discontinuous functions. We will say that $p \in \mathfrak{P}(\mathbb{T})$ belongs to the Kapanadze-Kopaliani class $\mathfrak{K}(\mathbb{T})$ if $1 < p_- \le p_+ < \infty$ and $p \in VMO^{1/|\log|}$. It is shown in [7, Theorem 2.1] that if $p \in \mathfrak{K}(\mathbb{T})$, then the Hardy-Littelwood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}$.

Corollary 1.3. *Suppose $p \in \mathfrak{P}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[L^{p(\cdot)}]$ built upon the variable Lebesgue space $L^{p(\cdot)}$.*

The paper is organized as follows. In Section 2, we prove that the separability of a Banach function space X is equivalent to the density of the set of trigonometric polynomials $\mathcal{P}$ in X and to the density of the set of all continuous functions C in X. Further, we recall a pointwise estimate of the Fejér means $f * K_n$, where K_n is the n-th Fejér kernel, by the Hardy-Littlewood maximal function Mf. In Section 3 we show that the norms of the operators $F_n f = f * K_n$ are uniformly bounded on a Banach function space X if X is rearrangement-invariant or if the Hardy-Littlewood maximal operator is bounded on X'. Moreover, if X is rearrangement-invariant, then $\|F_n\|_{\mathcal{B}(X)} \le 1$ for all $n \in \mathbb{N}$. Further, we prove that under the assumptions of Theorem 1.1 or 1.2, $\|f * K_n - f\|_X \to 0$ as $n \to \infty$. It remains to observe that $f * K_n \in \mathcal{P}_A$ if $f \in H[X]$, which will complete the proof of Theorems 1.1 and 1.2.

2. Preliminaries

2.1. Elementary lemma

We start with the following elementary lemma, whose proof can be found, e.g., in [3, Chap. III, Proposition 1.6(a)]. Here and in what follows, the space of all bounded linear operators on a Banach space E will be denoted by $\mathcal{B}(E)$.

Lemma 2.1. *Le E be a Banach space and $\{T_n\}$ be a sequence of bounded operators on E such that*

$$\sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{B}(E)} < \infty.$$

If D is a dense subset of E and for all $x \in D$,

$$\|T_n x - x\|_E \to 0 \quad \text{as} \quad n \to \infty, \tag{2.1}$$

then (2.1) holds for all $x \in E$.

2.2. Density of continuous function and trigonometric polynomials in Banach function spaces

A function of the form

$$q(t) = \sum_{k=-n}^{n} \alpha_k t^k, \quad t \in \mathbb{T}, \quad \alpha_{-n}, \dots, \alpha_n \in \mathbb{C},$$

is said to be a trigonometric (or Laurent) polynomial on $\mathbb{T}$. The set of all trigonometric polynomials is denoted by $\mathcal{P}$.

Lemma 2.2. *Let X be a Banach function space on $\mathbb{T}$. The following statements are equivalent:*

(a) *the set $\mathcal{P}$ of all trigonometric polynomials is dense in X;*
(b) *the space C of all continuous functions on $\mathbb{T}$ is dense in X;*
(c) *the Banach function space X is separable.*

Proof. The proof is developed by analogy with [8, Lemma 1.3].

(a) $\Rightarrow$ (b) is trivial because $\mathcal{P} \subset C \subset X$.

(b) $\Rightarrow$ (c). Since C is separable and $C \subset X$ is dense in X, we conclude that X is separable.

(c) $\Rightarrow$ (a). Assume that X is separable and $\mathcal{P}$ is not dense in X. Then by the corollary of the Hahn-Banach theorem (see, e.g., [2, Chap. 7, Theorem 4.2]), there exists a nonzero functional $\Lambda \in X^*$ such that $\Lambda(p) = 0$ for all $p \in \mathcal{P}$. Since X is separable, from [1, Chap. 1, Corollaries 4.3 and 5.6] it follows that the Banach dual X^* of X is canonically isometrically isomorphic to the associate space X'. Hence there exists a nonzero function $h \in X' \subset L^1$ such that

$$\int_{\mathbb{T}} p(t)h(t)\,dm(t) = 0 \quad \text{for all} \quad p \in \mathcal{P}.$$

Taking $p(t) = t^n$ for $n \in \mathbb{Z}$, we obtain that all Fourier coefficients of $h \in L^1$ vanish, which implies that $h = 0$ a.e. on $\mathbb{T}$ by the uniqueness theorem of the Fourier series (see, e.g., [9, Chap. I, Theorem 2.7]). This contradiction proves that $\mathcal{P}$ is dense in X. □

2.3. Pointwise estimate for the Fejér means

Recall that L^1 is a commutative Banach algebra under the convolution multiplication defined for $f, g \in L^1$ by

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta - i\varphi})g(e^{i\varphi})\,d\varphi, \quad e^{i\theta} \in \mathbb{T}.$$

For $n \in \mathbb{N}$, let

$$K_n(e^{i\theta}) := \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{i\theta k} = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^2, \quad e^{i\theta} \in \mathbb{T},$$

be the n-th Fejér kernel. It is well-known that $\|K_n\|_{L^1} \le 1$. For $f \in L^1$, the n-th Fejér mean of f is defined as the convolution $f * K_n$. Then

$$(f * K_n)(e^{i\theta}) = \sum_{k=-n}^{n} \widehat{f}(k)\left(1 - \frac{|k|}{n+1}\right)e^{i\theta k}, \quad e^{i\theta} \in \mathbb{T} \tag{2.2}$$

(see, e.g., [9, Chap. I]). This means that if $f \in L^1$, then $f * K_n \in \mathcal{P}$. Moreover, if $f \in H^1 = H[L^1]$, then $f * K_n \in \mathcal{P}_A$.

Lemma 2.3. *For every $f \in L^1$ and $t \in \mathbb{T}$,*

$$\sup_{n\in\mathbb{N}} |(f * K_n)(t)| \le \frac{\pi^2}{2}(Mf)(t). \tag{2.3}$$

Proof. Since $|\sin\varphi| \ge 2|\varphi|/\pi$ for $|\varphi| \le \pi/2$, we have for $\theta \in [-\pi, \pi]$,

$$\begin{aligned} K_n(e^{i\theta}) &\le \frac{\pi^2}{n+1}\frac{\sin^2\left(\frac{n+1}{2}\theta\right)}{\theta^2} \\ &= \frac{\pi^2}{4}(n+1)\frac{\sin^2\left(\frac{n+1}{2}\theta\right)}{\left(\frac{n+1}{2}\theta\right)^2} \\ &\le \frac{\pi^2}{4}(n+1)\min\left\{1, \left(\frac{n+1}{2}\theta\right)^{-2}\right\} \\ &\le \frac{\pi^2}{2}\frac{n+1}{1+\left(\frac{n+1}{2}\theta\right)^2} =: \Psi_n(\theta). \end{aligned} \tag{2.4}$$

It is easy to see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_n(\theta)\,d\theta \le \frac{\pi^2}{2} \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.5}$$

From [15, Lemma 2.11] and estimates (2.4)–(2.5) we immediately get estimate (2.3). □

3. Proofs of the main results

3.1. Norm estimates for the Fejér means

First we consider the case of rearrangement-invariant Banach function spaces.

Lemma 3.1. *Let X be a rearrangement-invariant Banach function space on $\mathbb{T}$. Then for each $n \in \mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and*

$$\sup_{n\in\mathbb{N}} \|F_n\|_{\mathcal{B}(X)} \le 1.$$

Proof. By [1, Chap. 3, Lemma 6.1], for every $f \in X$ and every $n \in \mathbb{N}$,

$$\|f * K_n\|_X \le \|K_n\|_{L^1}\|f\|_X.$$

It remains to recall that $\|K_n\|_{L_1} \le 1$ for all $n \in \mathbb{N}$. □

Now we will show the corresponding results for Banach function spaces such that the Hardy-Littlewood maximal operator is bounded on X'.

Theorem 3.2. *Let X be a Banach function space on $\mathbb{T}$ such that the Hardy-Littlewood maximal operator M is bounded on its associate space X'. Then for each $n \in \mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and*

$$\sup_{n\in\mathbb{N}} \|F_n\|_{\mathcal{B}(X)} \le \pi^2\|M\|_{X'\to X'}.$$

Proof. The idea of the proof is borrowed from the proof of [4, Theorem 5.1]. Fix $f \in X$ and $n \in \mathbb{N}$. Since $K_n \ge 0$, we have $|f * K_n| \le |f| * K_n$. Then from the Lorentz-Luxemburg theorem (see, e.g., [1, Chap. 1, Theorem 2.7]) we deduce that

$$\begin{aligned}\|f * K_n\|_X &\le \| \, |f| * K_n\|_X = \| \, |f| * K_n\|_{X''} \\ &= \sup\left\{\int_{\mathbb{T}} (|f| * K_n)(t)|g(t)| \, dm(t) \ : \ g \in X', \ \|g\|_{X'} \le 1\right\}.\end{aligned}$$

Hence there exists a function $h \in X'$ such that $h \ge 0$, $\|h\|_{X'} \le 1$, and

$$\|f * K_n\|_X \le 2 \int_{\mathbb{T}} (|f| * K_n)(t) h(t) \, dm(t). \tag{3.1}$$

Taking into account that $K_n(e^{i\theta}) = K_n(e^{-i\theta})$ for all $\theta \in \mathbb{R}$, by Fubini's theorem, we get

$$\int_{\mathbb{T}} (|f| * K_n)(t) h(t) \, dm(t) = \int_{\mathbb{T}} (h * K_n)(t) |f(t)| \, dm(t).$$

From this identity and Hölder's inequality for X (see, e.g., [1, Chap. 1, Theorem 2.4]), we obtain

$$\int_{\mathbb{T}} (|f| * K_n)(t) h(t) \, dm(t) \le \|f\|_X \|h * K_n\|_{X'}. \tag{3.2}$$

Applying Lemma 2.3 to $h \in X' \subset L^1$, by the lattice property, we see that

$$\|h * K_n\|_{X'} \le \frac{\pi^2}{2} \|Mh\|_{X'}. \tag{3.3}$$

Combining estimates (3.1)–(3.3) and taking into account that M is bounded on X' and that $\|h\|_{X'} \le 1$, we arrive at

$$\|f * K_n\|_X \le \pi^2 \|M\|_{X' \to X'} \|f\|_X.$$

Hence

$$\sup_{n \in \mathbb{N}} \|F_n\|_{\mathcal{B}(X)} = \sup_{n \in \mathbb{N}} \sup_{f \in X \setminus \{0\}} \frac{\|f * K_n\|_X}{\|f\|_X} \le \pi^2 \|M\|_{X' \to X'} < \infty,$$

which completes the proof. □

3.2. Convergence of the Fejér means in the norm

The following statement is the heart of the proof of the main results.

Theorem 3.3. *Suppose X is a separable Banach function space on $\mathbb{T}$. If X is rearrangement-invariant or the Hardy-Littlewood maximal operator is bounded on the associate space X', then for every $f \in X$,*

$$\lim_{n \to \infty} \|f * K_n - f\|_X = 0. \tag{3.4}$$

Proof. It is well-known that for every $f \in C$,

$$\lim_{n \to \infty} \|f * K_n - f\|_C = 0$$

(see, e.g., [3, Chap. III, Theorem 1.1(a)] or [9, Theorem 2.11]). From the definition of the Banach function space X it follows that $C \subset X \subset L^1$, where both embeddings are continuous. Then, for every $f \in C$, (3.4) is fulfilled. From Lemma 2.2 we know that the set C is dense in the space X. By Lemma 3.1 and Theorem 3.2,

$$\sup_{n\in\mathbb{N}} \|F_n\|_{\mathcal{B}(X)} < \infty,$$

where $F_n f = f * K_n$. It remains to apply Lemma 2.1. □

This statement for rearrangement-invariant Banach function spaces is contained, e.g., in [5, p. 268]. Notice that the assumption of the separability of X is hidden there.

Now we formulate the corollary of the above theorem in the case of variable Lebesgue spaces.

Corollary 3.4. *Suppose $p \in \mathfrak{P}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then for every $f \in L^{p(\cdot)}$,*

$$\lim_{n\to\infty} \|f * K_n - f\|_{L^{p(\cdot)}} = 0.$$

For variable exponents $p \in \mathfrak{P}(\mathbb{T})$ satisfying $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$, this result was obtained by Sharapudinov [16, Section 3.1]. For $p \in \mathfrak{K}(\mathbb{T})$, the above corollary is new.

3.3. Proofs of Theorems 1.1 and 1.2

If $f \in H[X]$, then $p_n = f * K_n \in \mathcal{P}_A$ for all $n \in \mathbb{N}$ in view of (2.2). By Theorem 3.3, $\|p_n - f\|_X \to 0$ as $n \to \infty$. Thus the set $\mathcal{P}_A$ is dense in in the abstract Hardy space $H[X]$ built upon X.

Moreover, if X is a rearrangement-invariant Banach function space, then from Lemma 3.1 it follows that $\|p_n\|_X \le \|f\|_X$ for all $n \in \mathbb{N}$. □

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