On the Bruhat Order of Labeled Graphs

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Abstract

We investigate two Bruhat (partial) orders on graphs with vertices labeled $1, 2, \ldots, n$ and with a specified degree sequence R, equivalently, symmetric $(0, 1)$-matrices with zero trace and a specified row sum vector R (adjacency matrices of such graphs). One is motivated by the classical Bruhat order on permutations while the other, more restrictive, one is defined by a switch of a pair of disjoint edges. In the Bruhat order, one seeks to concentrate the edges of a graph with a given degree sequence among the vertices with smallest labels, thereby

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producing a minimal graph in this order. We begin with a discussion of graphs whose isomorphism class does not change under a switch. Then we are interested in when the two Bruhat orders are identical. For labeled graphs of regular degree k, we show that the two orders are identical for $k \leq 2$ but not for $k = 3$.

Keywords: Bruhat order, labeled graph, switching, degree sequence, adjacency matrix, symmetric matrix.

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1 Introduction

To put our investigations in perspective, we begin with the following general description. Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then $\Sigma(A)$, is the $m \times n$ matrix whose (k, l)-entry is

$$\sigma_{kl}(A) = \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} \quad (1 \leq k \leq m, 1 \leq l \leq n),$$

the sum of the entries the leading $k \times l$ submatrix $A[\{1, 2, \ldots, k\}, \{1, 2, \ldots, l\}]$ of A.

Let $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots, s_m)$ be two arbitrary nonnegative integral vectors such that $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$. Then $\mathcal{A}(R, S)$ denotes the class of all $(0, 1)$-matrices with row sum vector R and column sum vector S. When $R = S = (k, k, \ldots, k)$ then we write $\mathcal{A}(n, k)$ instead of $\mathcal{A}(R, S)$; in this case m must equal n. By the Gale-Ryser theorem (see [1, 2]), the class $\mathcal{A}(R, S)$ is nonempty if and only if, after permuting S so that S is nonincreasing, S is majorized by R^* where R^* denotes the conjugate vector of R.

As a generalization of the Bruhat order on permutations of $\{1, 2, \ldots, n\}$, equivalently, $n \times n$ permutation matrices, the *Bruhat order* on $\mathcal{A}(R, S)$ has been defined by

$$A_1 \preceq_B A_2 \quad (A_1, A_2 \in \mathcal{A}(R, S))$$

provided that

$$\Sigma(A_1) \geq \Sigma(A_2) \text{ (entrywise).} \tag{1}$$

Note that for $A \in \mathcal{A}(R,S)$, the last row of $\Sigma(A)$ equals S and the last column equals R. The Bruhat order on a nonempty class $\mathcal{A}(R,S)$ is a partial order that has been previously investigated [1, 2, 3, 5, 6, 8, 9]. We note that the matrix $\Sigma(A)$ is not invariant under permutations of the rows and columns of A. Thus the Bruhat order between matrices is not what is usually thought of as a combinatorial property.

Let $A_1 \in \mathcal{A}(R,S)$ contain a 2×2 submatrix equal to

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then replacing this submatrix by the 2×2 identity I_2, called an $L_2 \to I_2$ *interchange*, gives a matrix A_2 such that $\Sigma(A_2) \geq \Sigma(A_1)$ and hence $A_2 \preceq_B A_1$. As shown in [3] it is possible that there exist classes $\mathcal{A}(R,S)$ and matrices $A_1, A_2 \in \mathcal{A}(R,S)$ such that $A_2 \preceq_B A_1$ but A_1 does not contain a 2×2 submatrix equal to L_2 so that no $L_2 \to I_2$ interchange or sequence thereof can bring A_1 to A_2. This motivated the introduction of a related partial order.

In [3] a *strong Bruhat order* (called there the secondary Bruhat order) was defined by using $L_2 \to I_2$ interchanges:

$$A_1 \preceq_{\widehat{B}} A_2 \quad (A_1, A_2 \in \mathcal{A}(R,S))$$

provided A_1 can be obtained from A_2 by a sequence of $L_2 \to I_2$ interchanges It follows that

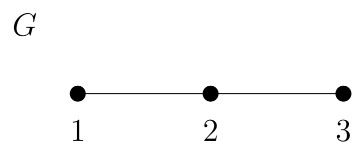
$$A_1 \preceq_{\widehat{B}} A_2 \text{ implies } A_1 \preceq_B A_2,$$

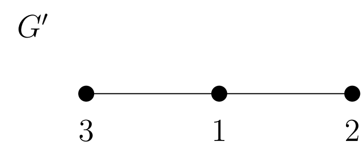
but as already mentioned, the converse does not hold in general. Thus, in general, a stronger property than (1) has to hold for two matrices to be comparable by the partial order $\preceq_{\widehat{B}}$.

Now consider just one nonnegative integral vector $R = (r_1, r_2, \ldots, r_n)$, and let $\mathcal{A}^0_{\text{sym}}(R)$ denote the class of all $n \times n$ symmetric $(0,1)$-matrices with zero trace and row sum vector R. Thus $\mathcal{A}^0_{\text{sym}}(R)$ represents the class of all graphs on n vertices where the vertices of each graph have been given the labels $1, 2, \ldots, n$. Thus, via the adjacency matrix, $\mathcal{A}^0_{\text{sym}}(R)$ can be identified with the class $\mathcal{G}(R)$ of all *labeled graphs* with vertex labels $1, 2, \ldots, n$ whose degree sequence equals R. We freely move between these two interpretations of symmetric $(0,1)$-matrices with zero trace and labeled graphs.

Our purpose in this paper is to investigate these Bruhat orders on the matrix classes $\mathcal{A}^0_{\text{sym}}(R)$ equivalently on the labeled graph classes $\mathcal{G}(R)$. The minimal elements in these orders reflect a heavier concentration of edges among the vertices with the smallest labels.

Example 1.1 Consider the following labeled graphs G and G':





where

$$A_1 = A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } A_2 = A(G') = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

While G and G' are isomorphic, they are different as labeled graphs, and indeed their degree sequence, $R = (1, 2, 1)$ and $R' = (2, 1, 1)$ are different. Thus G and G' are incomparable in the two Bruhat orders; indeed we have

$$\Sigma(A_1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \text{ and } \Sigma(A_2) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

$\square$

Several aspects of the class $\mathcal{A}^0_{\rm sym}(R)$, have been studied in [2]. Since a matrix in a class $\mathcal{A}^0_{\rm sym}(R)$ has zero trace, it contains an even number of 1's. If $R = (k, \ldots, k)$ is a constant n-vector, we write $\mathcal{A}^0_{\rm sym}(n, k)$ in place of $\mathcal{A}^0_{\rm sym}(R)$. Note that if k is odd and $\mathcal{A}_{\rm sym}(n, k)$ is nonempty, then n is even.

If G_1 and G_2 are two graphs with the same degree sequence, then it is well known [4, 7] (see also Corollary 7.2.6 of [2]) that G_1 and G_2 can be obtained from one another by *switching* two edges as follows: Replace an induced subgraph consisting of four vertices and a matching of size 2 (two vertex disjoint edges on four vertices and no other edges) with one of the two other possible matchings of size 2 on the four vertices). In case of labeled graphs and their adjacenvy matrices and in consideration of the Bruhat order, we need to distinguish three types of switching which we now discuss.

Let $A_1, A_2 \in \mathcal{A}^0_{\text{sym}}(R)$, and let $i,\ j,\ k,\ l$ be integers such that $1 \le i < j < k < l \le n$. We say that A_1 is obtained from A_2 by an (i,j,k,l)-*switch* (shortened to *switch* if the rows and columns are not specified), if A_1 and A_2 are equal except in their 4×4 principal submatrices F and B, respectively, determined by rows and columns $i,\ j,\ k,\ l$, which are related in one of three possible ways as indicated below:

$$F_{S_1} := \begin{bmatrix} 0 & * & 0 & 1 \\ * & 0 & 1 & 0 \\ 0 & 1 & 0 & * \\ 1 & 0 & * & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & * & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & * \\ 0 & 1 & * & 0 \end{bmatrix} =: B_{S_1} \tag{2}$$

$$F_{S_2} := \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & * & 1 \\ 1 & * & 0 & 0 \\ * & 1 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 & 0 & * \\ 1 & 0 & * & 0 \\ 0 & * & 0 & 1 \\ * & 0 & 1 & 0 \end{bmatrix} =: B_{S_2} \tag{3}$$

$$F_{S_3} := \begin{bmatrix} 0 & 0 & * & 1 \\ 0 & 0 & 1 & * \\ * & 1 & 0 & 0 \\ 1 & * & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 & * & 0 \\ 1 & 0 & 0 & * \\ * & 0 & 0 & 1 \\ 0 & * & 1 & 0 \end{bmatrix} =: B_{S_3} \tag{4}$$

where $*$ denotes an unspecified entry. In each of (2), (3) and (4), if the submatrix F in A_1 is replaced by the corresponding submatrix B in A_2, we say that A_2 is obtained from A_1 by a *forward switch*, and A_1 is obtained from A_2 by a *backward switch*.

In terms of graphs, let $x_1,\ x_2,\ x_3,\ x_4$ be a sequence of four distinct vertices of a graph G, which have been identified with their labels from $1, \dots, n$, such that $1 \le x_1 < x_2 < x_3 < x_4 \le n$.

(a) If $\{x_1, x_4\}$ and $\{x_2, x_3\}$ are edges of G but $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are not and if G' is the graph obtained from G by deleting the edges $\{x_1, x_4\}$ and $\{x_2, x_3\}$ and including the edges $\{x_1, x_3\}$ and $\{x_2, x_4\}$, then $A(G')$ is obtained from $A(G)$ by a forward (x_1, x_2, x_3, x_4)-switch of type specified by (2).

(b) If $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are edges of G but $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are not and if G' is the graph obtained from G by deleting the edges $\{x_1, x_3\}$

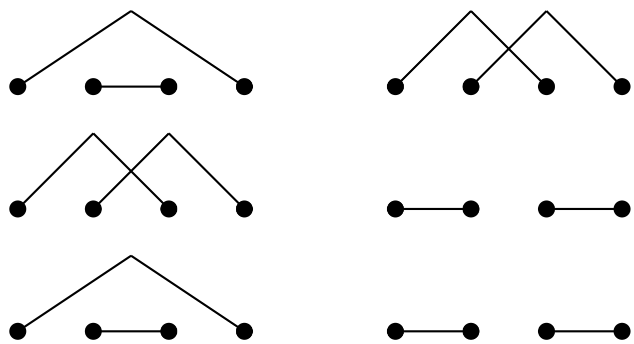


Figure 1: Possible switches on four labeled vertices.

and $\{x_2, x_4\}$ and including the edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$, then $A(G')$ is obtained from $A(G)$ by a forward (x_1, x_2, x_3, x_4)-switch of type specified by (3).

(c) If $\{x_1, x_4\}$ and $\{x_2, x_3\}$ are edges of G but $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are not and if G' is the graph obtained from G by deleting the edges $\{x_1, x_4\}$ and $\{x_2, x_3\}$ and including the edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$, then $A(G')$ is obtained from $A(G)$ by a forward (x_1, x_2, x_3, x_4)-switch of type specified by (4).

In each of the three cases (a), (b), and (c), G' is obtained from G by a *forward switch* and G is obtained from G' by a *backward switch*; less specifically, we say that G and G' are obtained from one another by a *switch*.

In Figure 1 we show the three possibilities of switches (forward switches from left to right) that are possible with four labeled vertices.

We now define a partial order on $\mathcal{A}^0_{sym}(R)$ which is the symmetric analogue of the strong Bruhat order. Let $A_1, A_2 \in \mathcal{A}^0_{\rm sym}(R)$. We define the *Bruhat-graph order*, $A_1 \preceq_{\rm BG} A_2$, provided A_1 can be obtained from A_2 by a sequence of forward switches, and thus A_1 can be obtained from A_2 by a sequence of backward switches. If G_1 and G_2 are the labeled graphs such that $A_1 = A(G_1)$ and $A_2 = A(G_2)$, we also say that G_1 precedes G_2 in the Bruhat-graph order and also write $G_1 \preceq_{\rm BG} G_2$.

We now briefly summarize the contents of this paper. In Section 2 we describe graphs that are isomorphic to the graph obtained when any switch is applied, and hence when any sequence of switches is applied. In particular, we give a complete description of such graphs in the case of trees, more generally forests.

Suppose that two matrices A_1 and A_2 in $\mathcal{A}^0_{\rm sym}(R)$ are comparable in the strong Bruhat order. Then there is a sequence of forward interchanges $L_2 \longrightarrow I_2$ that transforms A_1, say, into A_2. Now because of symmetry, a forward interchange $L_2 \longrightarrow I_2$ in the submatrix indexed by rows s,t and columns f,g implies a forward interchange $L_2 \longrightarrow I_2$ in the submatrix indexed by rows f,g and columns s,t where $|\{s,t,f,g\}| = 4$. Combining these two forward interchanges, we have a forward switch. Consequently, if two matrices are comparable in the strong Bruhat order on $\mathcal{A}^0_{\rm sym}(R)$, then they are comparable in the Bruhat-graph order on $\mathcal{A}^0_{\rm sym}(R)$. Since a forward switch consists of two forward interchanges, if $A_1 \preceq_{BG} A_2$, then $A_1 \preceq_B A_2$. The question arises whether or not the converse is true, that is whether or not "$\preceq_B$" is identical to "$\preceq_{BG}$"on $\mathcal{A}^0_{\rm sym}(R)$. This question will be addressed in Section 3.

The paper is organized as follows. In Section 2 we characterize graphs G such that switching always produces a graph isomorphic to G. In Section 3 we consider the question of whether or not the Bruhat and Bruhat-graph order coincide on some subclasses of $\mathcal{A}^0_{\rm sym}(R)$. In the final section we include some remarks concerning a subsequent paper.

2 Switch invariant graphs

As a warm-up to our primary motivation, we consider graphs that whose isimorphism class does not change under switching. A graph G is a *switch invariant graph* provided, ignoring any labels, all graphs obtained from G by a switch are isomorphic to G. In this section we consider graphs G that are switch invariant. If G is a switch invariant graph. We start by making some simple observations.

Remark 2.1

(i) *If G does not have a matching of two edges in G, then G is switch invariant.*

For instance, the complete graphs K_n, the null graph N_n, the star $K_{1,n-1}$, the graph $K_2 \cup N_{n-2}$ are switch invariant.

(ii) *If G is a switch invariant graph, so is its complement $\overline{G}$.*

For instance, the complement of a star is switch invariant.

(*iii*) *Since the complement of a star* $K_{1,n-1}$ $(n \geq 3)$ *is disconnected, a switch invariant, connected graph can have a disconnected complement that is also switch invariant.*

Recall that the complement of a disconnected graph is connected. In this section we show that a graph G and its complement $\overline{G}$ can both be connected and switch invariant.

(*iv*) *If* G *is a switch invariant, disconnected graph, then each component of* G *is a switch invariant graph.*

The converse is not true. The graphs K_3 and K_2 are switch invariant. However, there union $K_3 \cup K_2$ is easily seen as not switch invariant.

First we show that tree are rarely switch invariant. Recall that $R = (r_1, r_2, \ldots, r_n)$ is the degree sequence of a tree on n vertices if and only if $r_1, r_2, \ldots, r_n$ are positive integers with $r_1 + r_2 + \cdots + r_n = 2(n-1)$. We also note that if R is the degree sequence of a tree then it could also be the degree sequence of a graph which is not a tree. For instance $R = (2, 2, 2, 1, 1)$ is the degree sequence of a tree (a path on five vertices) but it is also the degree sequence of the disjoint union of a 3-cycle and an edge. The degree sequence R of a star has only one $r_i \geq 2$. A *double star*, that is, a union of two stars with an edge between their central vertices, has two $r_i \geq 2$.

Proposition 2.2 *Let* $T = (X, U)$ *be a tree. Then* T *is switch invariant if and only if* T *is a star or a double star.*

Proof. As noted in Remark 2.1, a star is switch invariant. Suppose that T is a double star. A matching of two edges in T has each edge from a different star. Let $u = \{x, y\}$ and $v = \{a, b\}$ be these edges. Since T is a double star, we can assume that the degree of x and a are one and y and b are the central vertices of the stars. Then $\{y, b\} \in U$ and the graph obtained from T by the unique switch that uses the edges u and v is obtained by deleting the edges u and v and inserting the edges $\{a, y\}$ and $\{x, b\}$. Therefore, these two graphs are isomorphic and T is switch invariant.

Conversely, suppose that T is a tree different from a star and a double star. Then there are two vertices c, s of degree one with distance $d(c, s) \geq 4$. Let $g = \{c, h\}$ and $f = \{s, k\}$ be the edges incident in c and s. Let H be the graph obtained from T by deleting the edges g and f and inserting the edges

$\{c, s\}$ and $\{h, k\}$ (this is not an edge of T because $d(c, s) \geq 4$). Then H is a disconnected graph and, therefore, is not isomorphic to T. Consequently, T is not switch invariant. ∎

Since any two graphs with the same degree sequence can be switched into each other, we obtain the following corollary.

Corollary 2.3 *A graph with the degree sequence R of a tree is switch-invariant if and only if R is the degree sequence of a star or double star.* □

We now describe switch invariant, disconnected graphs.

Proposition 2.4 *Let G be a disconnected graph with at least two components with edges. If G has a cycle, then G is not switch invariant.*

Proof. Let G_1 and G_2 be two components of G with edges such that G_1 has a cycle. Let $u = \{a, b\}$ be an edge of G_1 that belongs to a cycle and let $v = \{c, e\}$ be an edge of G_2. Let H be the graph obtained from G by deleting the edges u and v and inserting the edges $z = \{a, c\}$ and $w = \{b, e\}$. We show that this switch changes the two components G_1 and G_2 of G into one component of H and so G is not switch invariant.

Let x, y be two vertices of $G_1 \cup G_2$. If x, y are vertices of G_1, then there is a path between x and y not containing the edge u. Thus there is a path between x and y in H.

If x is a vertex of G_1 and y is a vertex of G_2, then either there is a path in G between y and c without the edge v or there is a path between y and e without the edge v. Since there is a path in G between x and a without the edge u and there is a path in G between x and b without the edge u, we have a path in H between x and y.

If x and y are vertices in G_2, then there is a path in G between x and y. If v is not an edge of this path, then this is a path in H. If v is an edge of this path, arguing as in the last case, there are in H a path between x and a vertex of G_1 and a path between this vertex of G_1 and y. ∎

Using Proposition 2.2 and Remark 2.1 we know that in a switch invariant forest each component is a star or a double star.

Lemma 2.5 *Let G be a forest containing at least two trees each with three or more vertices. Then G is not a switch invariant graph.*

Proof. Let G_1 and G_2 be two components of G each with at least three vertices. Let $u = \{a, b\}$ be an edge of G_1 and let $v = \{c, e\}$ be an edge of G_2 such that the degree of a and c are one. Let H be the graph obtained from G by deleting the edges u and v and inserting the edges $z = \{a, c\}$ and $w = \{b, e\}$. This switch changes the two components G_1 and G_2 of G into two components of H but one of them has two vertices. It follows that G is not switch invariant. ■

Lemma 2.6 *Let G be a forest with at least one component with exactly two vertices of degree greater than or equal to 2 and another component with at least one edge. Then G is not switch invariant.*

Proof. Let G_1 be a component of G with exactly two vertices, x and y, of degree greater than or equal to 2 and let G_2 be another component with at least one edge. Since G_1 is a double star, then $u = \{x, y\}$ is an edge of G_1 and G_1 has at least 4 vertices. By Lemma 2.5, for G to be switch invariant, G_2 has only one edge $v = \{a, b\}$. Let H be the graph obtained from G by deleting the edges u and v and inserting the edges $z = \{a, x\}$ and $w = \{b, y\}$. This switch replaces the two components G_1 and G_2 with two components of H where each one is a star with at least 3 vertices. Thus G is not switch invariant. ■

We now characterize switch invariant forests.

Theorem 2.7 *A forest G is switch invariant if and only if $G = G_1 \cup G_2$, where the following hold:*

(a) *G_1 is a star or a double star.*

(b) *G_2, if non-vacuous, consists of pairwise vertex-disjoint edges and isolated vertices.*

(c) *If G_1 has exactly two vertices of degree greater than or equal to 2, then G_2 consists only of isolated vertices.*

Proof. From Remark 2.1, Proposition 2.2, and Lemmas 2.5 and 2.6, we conclude that, if G is a switch invariant forest, then $G = G_1 \cup G_2$, where G_1 and G_2 satisfy conditions (a), (b), and (c).

Conversely, suppose that $G = G_1 \cup G_2$, where G_1 and G_2 satisfy conditions (a), (b), and (c). If G_1 has exactly two vertices of degree greater than or equal to 2, then, by Proposition 2.2, G is switch invariant. Now suppose that G_1 has at most one vertex of degree greater than 2. Then, if $u = \{x, y\}$ and $v = \{a, b\}$ are edges of G, with $|\{a, b, x, y\}| = 4$, but $z = \{a, x\}$ and $w = \{b, y\}$ are not edges of G, then either u and v are in the same component, in which case it is a double star, or u and v are in different components, in which case one of these components is K_2. In any case, it is easy to see that the graph obtained from G by the switch determined by vertices a, b, x, y is isomorphic to G. ■

3 Bruhat order and Bruhat-graph order

In this section, we consider certain forests with degree sequences R for which the Bruhat order and Bruhat-graph order coincide on the corresponding matrix classes $\mathcal{A}^0_{\text{sym}}(R)$.

Given an $n \times n$ matrix A, the submatrix of A indexed by rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_s$ is denoted by $A[\{i_1, \ldots, i_p\}; \{j_1, \ldots, j_s\}]$. If $\{i_1, \ldots, i_p\} = \{j_1, \ldots, j_s\}$, we simply write $A[\{i_1, \ldots, i_p\}]$.

Lemma 3.1 *Let G be a labeled graph $T \cup N_t$, where T is a double star and N_t is the null graph with t vertices. Let R be the degree sequence of G. Then the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{\text{sym}}(R)$ coincide.*

Proof. Let A and C be matrices of $\mathcal{A}^0_{\text{sym}}(R)$. We know that $A \prec_{BG} C$ implies that $A \prec_B C$. Now we show the converse.

Suppose that $A \prec_B C$ and there is no matrix D in $\mathcal{A}_{\text{sym}}(R)$ with $A \prec_B D \prec_B C$. Let k and l, with $k < l$, be the rows of A that correspond to the central vertices of T. So, if the entry (i, j) of A (respectively C) is nonzero then $\{i, j\} \cap \{k, l\} \neq \emptyset$. Moreover, $a_{kl} = a_{lk} = c_{kl} = c_{lk} = 1$ and if h is an integer such that $1 \leq h \leq n$ and h is different from k and l then the h-row sum and the h-column sum of A (respectively C) is less than or equal to 1.

Since $A \prec_B C$, let j be the minimal such that $a_{kj} = 1$ and $c_{kj} = 0$. Then $\sigma_{kj}(A) > \sigma_{kj}(C)$. Let r be the minimal such that $j < r$, $a_{kr} = 0$ and $c_{kr} = 1$. Since $|\{r, j, k, l\}| = 4$ then $a_{lj} = 0 = c_{lr}$ and $a_{lr} = 1 = c_{lj}$. So, we have to six possibilities:

(1) If $j < k < l < r$ then $A[\{j, k, l, r\}]$ has the form of B_{S_2}.

(2) If $j < r < k < l$ then $A[\{j, r, k, l\}]$ has the form of B_{S_1}.

(3) If $j < k < r < l$ then $A[\{j, k, r, l\}]$ has the form of B_{S_3}.

(4) If $k < j < r < l$ then $A[\{k, j, r, l\}]$ has the form of B_{S_2}.

(5) If $k < l < j < r$ then $A[\{k, l, j, r\}]$ has the form of B_{S_1}.

(6) If $k < j < l < r$ then $A[\{k, j, l, r\}]$ has the form of B_{S_3}.

All have $\sigma_{pc}(A) > \sigma_{pc}(C)$ if $(p, c) \in (\{k, \ldots, l-1\} \times \{j, \ldots, r-1\}) \cup (\{j, \ldots, r-1\} \times \{k, \ldots, l-1\})$. In (1), (3), (4), and (6), we actually have $\sigma_{pc}(A) - 1 > \sigma_{pc}(C)$.

Let D be the matrix obtained from A by the backward switch on rows and columns k, j, l, r as described. Then, $A \prec_{BG} D$, implying $A \prec_B D$. Since $A \prec_B C$ and using the observations described we get $A \prec_B D \preceq_B C$. Therefore, $D = C$ and $A \prec_{\mathrm{BG}} C$. ∎

Lemma 3.2 *Let G be a labeled graph $T \cup Q \cup N_t$, where T is a star, Q is a union of complete graphs K_2, and N_t is the null graph with t vertices. Let R be the degree sequence of G. Then the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{sym}(R)$ coincide.*

Proof. Let A and C be matrices of $\mathcal{A}^0_{sym}(R)$. We know that $A \prec_{BG} C$ implies that $A \prec_B C$. Now we show the converse.

Suppose that $A \prec_B C$ and there is no matrix D in $\mathcal{A}_{sym}(R)$ with $A \prec_B D \prec_B C$. Since T is a star, let k be the row of A that correspond to the central vertex of T. So, if h is an integer such that $1 \leq h \leq n$ and h is different from k then the h-row sum and the h-column sum of A (respectively C) is less than or equal to 1.

Since $A \prec_B C$, let (i, j) be the first position for the lexicographic order such that $i < j$, $a_{ij} = 1$ and $\sigma_{ij}(A) > \sigma_{ij}(C)$. Then $c_{ij} = 0$. Let s and t be integers with (s, t) lexicographically maximal such that

$$(r, c) \in \{i, \ldots, s-1\} \times \{j, \ldots, t-1\} \Longrightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

By Lemma 4.1 in [3], there exists $(i_0, j_0) \in \{i+1, \ldots, s\} \times \{j+1, \ldots, t\}$ with $a_{i_0 j_0} = 1$.

Case 1 If $i_0 = j$ then we have two 1's in column j because $a_{ij} = 1 = a_{i_0 j_0} = a_{j_0 i_0} = a_{j_0 j}$. So, $j = k$ and column j and row j correspond to the central vertex of T.

If there is $(u,v) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$ such that $v \neq j$, $u < v$, $a_{uv} = 1$ and $\sigma_{uv}(A) > \sigma_{uv}(C)$, let s_1 and t_1 be integers with (s_1, t_1) lexicographically maximal such that

$$(r,c) \in \{u,\ldots,s_1-1\} \times \{v,\ldots,t_1-1\} \Longrightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

By Lemma 4.1 in [3], there exists $(i_1, j_1) \in \{u+1,\ldots,s_1\} \times \{v+1,\ldots,t_1\}$ with $a_{i_1j_1} = 1$. In this case, $|\{u, v, i_1j_1\}| = 4$ and the prove is similar to case 2 or 3.

Suppose that if $(u,v) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$ such that $v \neq j$, $u < v$, $a_{uv} = 1$ then $\sigma_{uv}(A) = \sigma_{uv}(C)$. Then $a_{rc} = 0$ if

$$(r,c) \in \{i,\ldots,s-1\} \times \{j,\ldots,t-1\}.$$

Since if $(g,f) \in (\{1,\ldots,j\} \times \{1,\ldots,i\}) \setminus (\{j\} \times \{i,\ldots,j\})$ then $\sigma_{gf}(A) \geq \sigma_{gf}(C)$ we get $a_{gf} = c_{gf}$ if $(g,f) \in (\{1,\ldots,j\} \times \{1,\ldots,i\}) \setminus (\{j\} \times \{i,\ldots,j\})$.

Let s be the maximal such that $i \leq s < j$, $a_{sj} = 1$ and $c_{sj} = 0$.

If there is p_1 such that $s < p_1 < j$, $a_{p_1,j} = 0$ and $c_{p_1,j} = 1$, by maximality of s, $\sigma_{p_1j}(A) = \sigma_{p_1j}(C)$, which is impossible.

Let y be the minimal such that $y > j$, $a_{j,y} = 1$ (recall that j_0 verifies this condition). Since $\sigma_{jy}(A) = \sigma_{jy}(C)$ then there are, at least, y_1, y_2 such that $j < y_1 < y_2 < y$, $a_{j,y_1} = a_{j,y_2} = 0$ and $c_{f_1,y_1} = c_{f_2,y_2} = 1$, with $f_1, f_2 \in \{s+1,\ldots,j\}$.

Since $\sigma_{y_1y_2}(A) \geq \sigma_{y_1y_2}(C)$ then $a_{y_1,y_2} = 1$. This implies that $A[\{s,j,y_1,y_2\}$ has the form B_{S_2}. Moreover, $\sigma_{x_1x_2}(A) > \sigma_{x_1x_2}(C)$ if $(x_1,x_2) \in \{i,\ldots,j-1\} \times \{j,\ldots,y_1-1\}$ and $\sigma_{x_1x_2}(A) - 1 > \sigma_{x_1x_2}(C)$ if $(x_1,x_2) \in \{j,\ldots,y_1-1\} \times \{j,\ldots,y_1-1\}$. By a prove similar to case 3, we conclude the result.

Case 2 If $i_0 < j$, then $a_{ij_0} = 0$ (if not, rows i and j_0 will be two ones) and $a_{i_0j} = 0$. So, $A[\{i,i_0,j,j_0\}]$ has the form B_{S_1}. Let D be the matrix obtained from A by a backward (i,i_0,j,j_0)-switch. Then, $A \prec_{BG} D$, implying $A \prec_B D$. Since, for $(r,c) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$,

$$\sigma_{rc}(D) = \begin{cases} \sigma_{rc}(A) - 1 & \text{if } (r,c) \in (\{i,\ldots,i_0-1\} \times \{j,\ldots,j_0-1\}) \cup \\ & \cup (\{j,\ldots,j_0-1\} \times \{i,\ldots,i_0-1\}) \\ \sigma_{rc}(A) & \text{otherwise} \end{cases}$$

and $\sigma_{rc}(A) > \sigma_{rc}(C)$ for any $(r,c) \in (\{i,\ldots,i_0-1\} \times \{j,\ldots,j_0-1\}) \cup (\{j,\ldots,j_0-1\} \times \{i,\ldots,i_0-1\})$, then $A \prec_B C$ implies $A \prec_B D \preceq_B C$. Therefore, $D = C$ and $A \prec_{\widehat{B}} C$.

Case 3 If $j < i_0 < j_0$ (or $j < j_0 < i_0$,) then $A[\{i, j, i_0, j_0\}]$ (or $A[\{i, j, j_0, i_0\}]$) has the form B_{S_2}. Let D be the matrix obtained from A by a backward (i, j, i_0, j_0)-switch (or a backward (i, j, j_0, i_0)-switch). Then, $A \prec_{BG} D$, implying $A \prec_B D$. Since, for $(r, c) \in \{1, \dots, n\} \times \{1, \dots, n\}$,

$$\sigma_{rc}(D) = \begin{cases} \sigma_{rc}(A) - 1 & \text{if } (r,c) \in (\{i, \dots, j_0 - 1\} \times \{j, \dots, i_0 - 1\}) \cup \\ & \cup(\{j, \dots, i_0 - 1\} \times \{i, \dots, j_0 - 1\})) \setminus \\ & \setminus \{j, \dots, i_0 - 1\} \times \{j, \dots, i_0 - 1\} \\ \sigma_{rc}(A) - 2 & \text{if } (r,c) \in \{j, \dots, i_0 - 1\} \times \{j, \dots, i_0 - 1\} \\ \sigma_{rc}(A) & \text{otherwise} \end{cases}$$

and, $\sigma_{rc}(A) > \sigma_{rc}(C)$ for any $(r, c) \in ((\{i, \dots, j_0 - 1\} \times \{j, \dots, i_0 - 1\}) \cup (\{j, \dots, i_0 - 1\} \times \{i, \dots, j_0 - 1\})) \setminus \{j, \dots, i_0 - 1\} \times \{j, \dots, i_0 - 1\}$ and $\sigma_{rc}(A) > \sigma_{rc}(C)$ for any $(r, c) \in \{j, \dots, i_0 - 1\} \times \{j, \dots, i_0 - 1\}$, then $A \prec_B C$ implies $A \prec_B D \preceq_B C$ (if $j < j_0 < i_0$, with similar arguments we conclude the same). Therefore, $D = C$ and $A \prec_{BG} C$. ■

The next theorem is a consequence of Theorem 2.7 and Lemmas 3.1 and 3.2.

Theorem 3.3 *Let T be a switch invariant labeled forest with degree sequence R. Then the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{\text{sym}}(R)$ coincide.*

Since the adjacency matrix of the union of $\frac{n}{2}$ complete graphs K_2 is a matrix in $\mathcal{A}^0_{\text{sym}}(n, 1)$, then the following result is a consequence of Theorem 3.3 (recall that the $\cup K_2$ is a switch invariant forest).

Corollary 3.4 *For $n \geq 1$, the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{\text{sym}}(n, 1)$ coincide.*

From Corollary 3.4 we conclude that for all labelings of the vertices of the union $\cup K_2$ of $\frac{n}{2}$ complete graphs K_2, the relation between the adjacency matrices of any two of these graphs in the Bruhat-graph order is the same as in the Bruhat order.

4 The class $\mathcal{A}^0_{\text{sym}}(2)$

By Corollary 3.4, the Bruhat order on $\mathcal{A}^0_{\text{sym}}(n, 1)$ coincides with the Bruhat-graph order. In this section we extend this result to $\mathcal{A}^0_{\text{sym}}(2)$.

Theorem 4.1 *For $n \geq 3$, the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{sym}(n,2)$ coincide.*

Proof. Let A and C be matrices in $\mathcal{A}_{sym}(n,2)$. We know that $A \prec_{BG} C$ implies $A \prec_B C$. So we need to prove that if $A \prec_B C$ implies $A \prec_{BG} C$. It suffices to show this property when C covers A. So we assume that C covers A for the Bruhat order.

Let (i,j) be the position of A such that $\sigma_{ij}(A) > \sigma_{ij}(C)$, $a_{ij} = 1$, $i < j$ and $i + j$ is as large as possible.

Applying Lemma 4.1 of [3], we choose $(i_0, j_0) \in \{i+1, \ldots, n\} \times \{j+1, \ldots, n\}$ such that $a_{i_0 j_0} = 1$ and for $(r,c) \in \{i, \ldots, i_0 - 1\} \times \{j, \ldots, j_0 - 1\}$, $\sigma_{rc}(A) > \sigma_{rc}(C)$. We consider four cases.

Case 1: $a_{i_0 j} = a_{i j_0} = 0$ and $i_0 \neq j$.

In this case, a backward switch that replaces $A[\{i, i_0, j, j_0\}] = B_{S_1}$ with F_{S_1}, if $i_0 < j$, or replaces $A[\{i, j, i_0, j_0\}] = B_{S_3}$ with F_{S_3}, if $j_0 > i_0 > j$, or replaces $A[\{i, j, j_0, i_0\}] = B_{S_2}$ with F_{S_2}, if $i_0 > j_0$ results in a matrix D with $A \prec_{BG} D$. Since for any $(r,c) \in \{i, \ldots, i_0 - 1\} \times \{j, \ldots, j_0 - 1\}$, $\sigma_{rc}(A) > \sigma_{rc}(C)$ and $A \prec_B C$ then $D \preceq_B C$. Using the fact that C covers A for the Bruhat order, we conclude that $D = C$ and $A \prec_{BG} C$.

Case 2: $a_{i_0 j} = a_{i j_0} = 0$ and $i_0 = j$.

In this case $i_0 < j_0$.

If there is $b \in \{i+1, \ldots, i_0 - 1\}$ such that $a_{b,j_0} = 1$, using the maximality of $i + j$ we conclude that $a_{b,j} = 0$ and we can argue as in Case 1 with rows and columns i, b, j, j_0. Thus, we assume that if $b \in \{i+1, \ldots, i_0 - 1\}$ then $a_{b,j_0} = 0$. We consider two subcases.

Subcase 2.1: Suppose that $\sigma_{jj}(A) = \sigma_{jj}(C)$.

We know that $\sigma_{j-1,j}(A) > \sigma_{j-1,j}(C)$.

We claim that $\sigma_{j-1,j-1}(A) > \sigma_{j-1,j-1}(C)$. Suppose that $\sigma_{j-1,j-1}(A) = \sigma_{j-1,j-1}(C)$. Then

$$0 + \sigma_{j-1,j}(A) + \sigma_{j,j-1}(A) - \sigma_{j-1,j-1}(A) = \sigma_{j,j}(A) =$$

$$\sigma_{j,j}(C) = 0 + \sigma_{j-1,j}(C) + \sigma_{j,j-1}(C) - \sigma_{j-1,j-1}(C).$$

Consequently,

$$\sigma_{j-1,j}(A) + \sigma_{j,j-1}(A) = \sigma_{j-1,j}(C) + \sigma_{j,j-1}(C)$$

and

$$\sigma_{j-1,j}(A) = \sigma_{j-1,j}(C),$$

a contradiction. Therefore, $\sigma_{j-1,j-1}(A) > \sigma_{j-1,j-1}(C)$. Applying Lemma 4.3 of [3], there is $(i_1, j_1) \in \{1, \ldots, j-1\} \times \{1, \ldots, j-1\}$ such that $a_{i_1,j_1} = 1$ and for any $(r, c) \in \{i_1, \ldots, j-1\} \times \{j_1, \ldots, j-1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$. Note that $i_1 \neq j_1$. We assume that $i_1 + j_1$ is as large as possible. We now consider three cases.

Subcase 2.1.1: $i_1 = i$ (or $j_1 = i$).

In this case we have already identified two positions in row i (or in column i) of A that are occupied by 1's, (i, j) and (i_1, j_1) (or (j, i) and (i_1, j_1)). Moreover, we know that $a_{ij_0} = 0 = a_{j_0i}$. Since $a_{i_0i} = a_{ii_0} = a_{ij} = 1$ and we know that $a_{i_0j_0} = a_{j_0i_0} = 1$ then $a_{i_0j_1} = 0$ (or $a_{i_1i_0} = 0$). In addition we have, for any $(r, c) \in \{i_1, \ldots, i_0 - 1\} \times \{j_1, \ldots, j_0 - 1\}$ (or $(r, c) \in \{i_1, \ldots, j_0 - 1\} \times \{j_1, \ldots, i_0 - 1\}$), $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i_1, i_0, j_1, j_0, B_{S_p} with F_{S_p}, for some $p \in \{1, 2, 3\}$, results in a matrix D with $A \prec_B D \preceq_B C$.

Subcase 2.1.2: $j_1 > i_1 > i$ (or $i_1 > j_1 > i$).

In this case we have already identified two positions in row i_0 of A that are occupied by 1's, (i_0, i) and (i_0, j_0). Then $a_{i_0,j_1} = 0$. Moreover, we know that $a_{i_1j_0} = 0$. In addition, $a_{i_1j_1} = 1$ and for any $(r, c) \in \{i_1, \ldots, i_0 - 1\} \times \{j_1, \ldots, j_0 - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i_1, i_0, j_1, j_0, B_{S_p} with F_{S_p}, for some $p \in \{1, 2, 3\}$, results in a matrix D with $A \prec_B D \preceq_B C$.

Subcase 2.1.3: $i_1 < i$ and $i \neq j_1$ (or $j_1 < i$ and $i \neq i_1$).

Using the maximality on $i_1 + j_1$ then $a_{i,j_1} = 0$ (or $a_{i_1,i} = 0$). In this case we have already identified two positions in column i_0 of A that are occupied by 1's, (i, i_0) and (j_0, i_0). Then $a_{i_1,i_0} = 0$ (or $a_{i_0,j_1} = 0$). Moreover, $a_{i_1j_1} = 1$ and for any $(r, c) \in \{i_1, \ldots, i-1\} \times \{j_1, \ldots, j-1\}$ (or $(r, c) \in \{i_1, \ldots, j-1\} \times \{j_1, \ldots, i-1\}$), $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i_1, i, j_1, j, B_{S_p} with F_{S_p}, for some $p \in \{1, 2, 3\}$, results in a matrix D with $A \prec_B D \preceq_B C$.

Since C covers A for the Bruhat order, in each subcase of 2.1, we conclude that $D = C$ and $A \prec_{BG} C$.

Subcase 2.2: Suppose that $\sigma_{jj}(A) > \sigma_{jj}(C)$.

Applying Lemma 4.1 of [3], there is $(i_2, j_2) \in \{j+1, \ldots, n\} \times \{j+1, \ldots, n\}$ such that $a_{i_2,j_2} = 1$, and for any $(r, c) \in \{j, \ldots, i_2 - 1\} \times \{j, \ldots, j_2 - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

If $j_2 > j_0$ then $\sigma_{j,j_0}(A) > \sigma_{j,j_0}(C)$ or if $i_2 > j_0$ then $\sigma_{j_0,j}(A) > \sigma_{j_0,j}(C)$. Using the maximality on $i + j$ and the fact that $a_{j,j_0} = 1 = a_{j_0,j}$ we obtain a contradiction. So, $j_2 \leq j_0$ and $i_2 \leq j_0$.

Again, using the maximality on $i + j$ we have $a_{i,j_2} = 0$. We have already identified two positions in column j of A that are occupied by 1's, (i, j) and (j_0, j). Then $a_{i_2,j} = 0$. In addition we have, for any $(r, c) \in \{i, \ldots, i_2 - 1\} \times \{j, \ldots, j_2 - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i, i_2, j, j_2, B_{S_p} with F_{S_p}, for some $p \in \{1, 2, 3\}$, results in a matrix D with $A \prec_B D \preceq_B C$. Since C covers A for the Bruhat order, we conclude that $D = C$ and $A \prec_{BG} C$.

Case 3: $a_{i_0 j} = 1$.

Because $i_0 > i$, by maximality condition on $i+j$, we know that $\sigma_{i_0,j}(A) = \sigma_{i_0,j}(C)$. We also know that $\sigma_{i_0-1,j}(A) > \sigma_{i_0-1,j}(C)$.

We claim that $\sigma_{i_0-1,j-1}(A) > \sigma_{i_0-1,j-1}(C)$. Suppose that $\sigma_{i_0-1,j-1}(A) = \sigma_{i_0-1,j-1}(C)$. Then

$$1 + \sigma_{i_0-1,j}(A) + \sigma_{i_0,j-1}(A) - \sigma_{i_0-1,j-1}(A) = \sigma_{i_0,j}(A) =$$

$$\sigma_{i_0,j}(C) \leq 1 + \sigma_{i_0-1,j}(C) + \sigma_{i_0,j-1}(C) - \sigma_{i_0-1,j-1}(C).$$

Consequently,

$$\sigma_{i_0-1,j}(A) + \sigma_{i_0,j-1}(A) \leq \sigma_{i_0-1,j}(C) + \sigma_{i_0,j-1}(C)$$

and

$$0 < \sigma_{i_0-1,j}(A) - \sigma_{i_0-1,j}(C) \leq \sigma_{i_0,j-1}(C) - \sigma_{i_0,j-1}(A).$$

Thus $\sigma_{i_0,j-1}(C) > \sigma_{i_0,j-1}(A)$, a contradiction. Therefore, $\sigma_{i_0-1,j}(A) > \sigma_{i_0-1,j}(C)$. Applying Lemma 4.3 of [3], there is $(i_1, j_1) \in \{1, \ldots, i_0 - 1\} \times \{1, \ldots, j - 1\}$ such that $a_{i_1,j_1} = 1$ and for any $(r, c) \in \{i_1, \ldots, i_0 - 1\} \times \{j_1, \ldots, j - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$. Note that $i_1 \neq j_1$. We now consider two subcases.

Subcase 3.1: $i_1 = i$.

In this case we have already identified two positions in row i of A that are occupied by 1's, (i, j) and (i_1, j_1). Moreover, we know that $a_{ij_0} = 0$.

Since $a_{i_0j} = a_{i_0j_0} = 1$ then $a_{i_0j_1} = 0$. In addition we have, for any $(r,c) \in \{i_1, \dots, i_0 - 1\} \times \{j_1, \dots, j_0 - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i_1, i_0, j_1, j_0, B_{S_p} with F_{S_p}, for some $p \in \{1,2,3\}$, results in a matrix D with $A \prec_B D \preceq_B C$.

Subcase 3.2: $i_1 \neq i$.

In this case we have already identified two positions in column j of A that are occupied by 1's, (i,j) and (i_0,j). Then $a_{i_1,j} = 0$. We also have already identified two positions in row i_0 of A that are occupied by 1's, (i_0,j) and (i_0,j_0). Then $a_{i_0,j_1} = 0$.

In addition we have, for any $(r,c) \in \{i_1, \dots, i_0 - 1\} \times \{j_1, \dots, j - 1\}$, $\sigma_{r,c}(A) > \sigma_{r,c}(C)$.

As in Case 1, a backward switch that replaces the submatrix of A, in rows and columns i_1, i_0, j_1, j_0, B_{S_p} with F_{S_p}, for some $p \in \{1,2,3\}$, results in a matrix D with $A \prec_B D \preceq_B C$.

Since C covers A for the Bruhat order, in each subcase of 3 we conclude that $D = C$ and $A \prec_{BG} C$.

Case 4: $a_{ij_0} = 1$.

The proof is similar to the proof in Case 3. ■

As shown in Proposition 4, the graph $K_3 \cup K_3$, whose degree sequence is $R = (2,2,2,2,2,2)$, is not switch invariant. On the other hand, by Theorem 4.1, the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{sym}(6,2)$ coincide. Thus, we can conclude that, even if the Bruhat order and Bruhat-graph order on a class $\mathcal{A}^0_{sym}(R)$ coincide, there may exist switch invariant graphs with degree sequence R.

In the next example we show that the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{\mathrm{sym}}(R)$ do not coincide in general.

Example 4.2 First let B_1 and B_2 be $(0,1)$-matrices defined as follows:

$$B_1 = \left[\begin{array}{c|c|c|c} 1 & & & \\ \hline 1 & & 1 & 1 \\ \hline 1 & 1 & & 1 \\ \hline & & & 1 \end{array}\right] \text{ and } B_2 = \left[\begin{array}{c|c|c|c} & & & 1 \\ \hline 1 & 1 & & 1 \\ \hline 1 & & 1 & 1 \\ \hline 1 & & & \end{array}\right].$$

Thus B_1 and B_2 have row sum vector $R = (1,3,3,1)$ and column sum vector

$S = (3, 1, 1, 3)$. Then

$$\Sigma_{B_1} = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 7 \\ \hline 3 & 4 & 5 & 8 \end{array}\right] \text{ and } \Sigma_{B_2} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 1 \\ \hline 1 & 2 & 2 & 4 \\ \hline 2 & 3 & 4 & 7 \\ \hline 3 & 4 & 5 & 8 \end{array}\right],$$

where

$$\Sigma_{B_1} \geq \Sigma_{B_2}$$

and hence, in the usual Bruhat order on $(0,1)$-matrices, we have

$$B_1 \preceq_B B_2.$$

Now let A_1 and A_2 be the 8-by-8 symmetric $(0,1)$-matrices with zero trace defined by

$$A_1 = \left[\begin{array}{cc} O & B_1 \\ B_1^t & O \end{array}\right] \text{ and } A_2 = \left[\begin{array}{cc} O & B_2 \\ B_2^t & O \end{array}\right],$$

with row sum vector $(1, 3, 3, 1, 3, 1, 1, 3)$. We claim that $A_1 \preceq_B A_2$ holds but $A_1 \not\preceq_{BG} A_2$.

Since $\Sigma_{B_1} \geq \Sigma_{B_2}$, it follows easily that $\Sigma_{A_1} \geq \Sigma_{A_2}$ and hence $A_1 \preceq_B A_2$. Thus if A_1 and A_2 are comparable in the Bruhat-graph order, it must be that $A_1 \preceq_{BG} A_2$.

So suppose that $A_1 \preceq_{BG} A_2$. Then in the Bruhat-graph order there is a chain from A_2 down to A_1, and thus a sequence of forward switches that sends A_2 into A_1. But we cannot do a forward switch that puts a 1 in the initial 4×4 principal zero submatrix of A_1 and A_2 for then eventually we would have to do a backward switch to eliminate it to get to A_1. But then there is only one possible forward switch as shown below and it does not give A_1.

$$\left[\begin{array}{cccc||cccc} & & & & & & & 1 \\ \hline & & & & 1 & 1 & & 1 \\ \hline & & & & 1 & & 1 & 1 \\ \hline & & & & 1 & & & \\ \hline\hline & 1 & 1 & 1 & & & & \\ \hline & 1 & & & & & & \\ \hline & & 1 & & & & & \\ \hline 1 & 1 & 1 & & & & & \end{array}\right] \rightarrow \left[\begin{array}{cccc||cccc} & & & & 1 & & & \\ \hline & & & & 1 & 1 & & 1 \\ \hline & & & & 1 & & 1 & 1 \\ \hline & & & & & & & 1 \\ \hline\hline 1 & 1 & 1 & & & & & \\ \hline & 1 & & & & & & \\ \hline & & 1 & & & & & \\ \hline & 1 & 1 & 1 & & & & \end{array}\right].$$

Notice that, in Example 4.2, if $G(A_1)$ is an unlabeled graph that has A_1 as its adjacency matrix and $G(A_2)$ is an unlabeled graph that has A_2 as its adjacency matrix then $G(A_1)$ and $G(A_2)$ are isomorphic. Also notice that these graphs are not trees since there is a 4-cycle.

For $k = 3$ the Bruhat order and Bruhat-graph order on $\mathcal{A}^0_{sym}(n, 3)$ do not coincide, as the following example shows.

Example 4.3 *Consider the three matrices*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$T_A = \left[\begin{array}{c|c} 0 & A \\ \hline A^t & 0 \end{array}\right], \qquad T_C = \left[\begin{array}{c|c} 0 & C \\ \hline C^t & 0 \end{array}\right], \qquad T_D = \left[\begin{array}{c|c} 0 & D \\ \hline D^t & 0 \end{array}\right].$$

A calculation shows that

$$\Sigma_{T_A} > \Sigma_{T_D} > \Sigma_{T_C}.$$

Thus,

$$T_A \prec_B T_D \prec_B T_C.$$

However, with arguments similar to those used in the previous example, we can see that T_D and T_A are incomparable in the Bruhat-graph order.

We conclude this section by showing that the Bruhat order and Bruhat-graph order do not coincide for different labelings of a path.

Example 4.4 Consider the adjacency matrices of two different labelings of a path P_7:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Computation shows that

$$\Sigma_{A_1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 4 & 5 & 5 & 5 \\ 1 & 2 & 4 & 6 & 7 & 7 & 7 \\ 1 & 3 & 5 & 7 & 8 & 9 & 9 \\ 1 & 3 & 5 & 7 & 9 & 0 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 & 12 \end{bmatrix} \text{ and } \Sigma_{A_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 5 & 5 & 5 \\ 0 & 2 & 3 & 4 & 6 & 7 & 7 \\ 1 & 3 & 5 & 6 & 8 & 9 & 9 \\ 1 & 3 & 5 & 7 & 9 & 10 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 & 12 \end{bmatrix}$$

so that $A_1 \preceq_B A_2$. Moreover, A_1 cannot be obtained from A_2 by one switch but it can be obtained by a type (2) forward switch followed by a type (1) forward switch; however, the matrix obtained after the first switch has a cycle. In fact, there does not exist a matrix C such that $C \neq A_1, A_2$, $G(C)$ is isomorphic to P_n and $A_1 \preceq_{BG} C \preceq_{BG} A_2$.

5 Coda

Using the notion of switching, we have modified the classical Bruhat order on the class of adjacency matrices $\mathcal{A}^0_{\text{sym}}$ of undirected graphs, without loops and with n labeled vertices with a given degree sequence, to the Bruhat-graph order. We have introduced the idea of a switch invariant graph and have described some graphs that are switch invariant. We described certain classes $\mathcal{A}^0_{\text{sym}}$ for which the Bruhat order and Bruhat-graph order coincide, in particular for the classes corresponding to regular graphs of degree 1 and 2. We have shown that this does not hold for degree 3.

In a subsequent paper, we plan to characterize the minimal and maximal matrices in the Bruhat order for regular graphs of degree 1 and 2.

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