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The maximum multiplicity and the two largest multiplicities of eigenvalues in a Hermitian matrix whose graph is a tree

Abstract: The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, $M_1$, was understood fully (from a combinatorial perspective) by C.R. Johnson, A. Leal-Duarte (Linear Algebra and Multilinear Algebra 46 (1999) 139-144). Among the possible multiplicity lists for the eigenvalues of Hermitian matrices whose graph is a tree, we focus upon $M_2$, the maximum value of the sum of the two largest multiplicities when the largest multiplicity is $M_1$. Upper and lower bounds are given for $M_2$. Using a combinatorial algorithm, cases of equality are computed for $M_2$.

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1 Introduction

Let $T$ be a tree on $n \geq 2$ vertices. We denote by $S(T)$ the collection of all $n$-by-$n$ complex Hermitian matrices whose graph is $T$. No restriction is placed upon the diagonal entries of matrices in $S(T)$.

For convenience, when $A \in S(T)$, we place in non-increasing order the multiplicities of the eigenvalues of $A$. We refer to such a list of multiplicities as the unordered multiplicity list and we denote it by $(m_1(A), m_2(A), \ldots, m_k(A))$, where $k(A)$ is the number of distinct eigenvalues of $A$. So, $m_j(A)$ is the $j$th largest multiplicity of an eigenvalue in the multiplicity list of $A$.

Definition 1.1. Let $L(T)$ be the set of all positive integer lists (unordered multiplicity lists) $(p_1, p_2, \ldots, p_s)$ satisfying:

1. $p_1 \geq p_2 \geq \ldots \geq p_s \geq 1$;
2. $\sum_{i=1}^s p_i = n$;
3. There is an $A \in S(T)$ with $(m_1(A), m_2(A), \ldots, m_{k(A)}(A)) = (p_1, p_2, \ldots, p_s)$.

For $j \geq 1$, we denote by

$$M_j(T) = \max_{(p_1, p_2, \ldots, p_s) \in L(T)} (p_1 + \ldots + p_j).$$

It is well known that $M_1(T)$ is equal to the path cover number $P(T)$, the smallest number of nonintersecting induced paths of $T$ that cover all the vertices of $T$; this is the same as $\max(p - q)$, where $p$ is the number of paths remaining when $q$ vertices have been removed from $T$ in such a way as to leave only induced paths [3].

Remark 1.2. In [7] a combinatorial algorithm was given to compute $M_2(T)$. It is easy to see that if $(p_1, p_2, \ldots, p_s) \in L(T)$ then

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(1) \( p_1 \leq M_1(T) \).
(2) \( p_1 + p_2 \leq M_2(T) \).
(3) \( p_1 + p_2 \geq 2, p_2 \neq 0 \) (because if \( T \) is a tree and \( A \in S(T) \) then the largest and the smallest eigenvalues of \( A \) have multiplicities one. So, each list in \( \mathcal{L}(T) \) has at least two \( 1 \)'s, [4]).
(4) Using the definition of \( M_1(T) \), there exists \((p_1, p_2, \ldots, p_s) \in \mathcal{L}(T) \) such that \( p_1 = M_1(T) \).

Given \( M_1(T) \) and \( M_2(T) \), we cannot say there exists a list \((p_1, p_2, \ldots, p_s) \in \mathcal{L}(T) \) such that \( p_1 = M_1(T) \) and \( p_2 = M_2(T) - M_1(T) \). For example, [7], the double star \( D_{3,3} \) has \( M_1(D_{3,3}) = 4 \), \( M_2(D_{3,3}) = 6 \) but \((4, 2, 1, 1) \notin \mathcal{L}(D_{3,3}) \) (we can prove this using the Parter-Wiener theorem [5]). \( M_1(D_{3,3}) = 4 \) because \((4, 1, 1, 1, 1) \in \mathcal{L}(D_{3,3}) \), for example, consider the matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

\( M_2(D_{3,3}) = 6 \) because \((3, 3, 1, 1) \in \mathcal{L}(D_{3,3}) \), for example, consider the matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}.
\]

So, it is important to know when given \( M_2(T) \), we can say that there is a list \((p_1, p_2, \ldots, p_s) \in \mathcal{L}(T) \) such that \( p_1 = M_1(T) \) and \( p_2 = M_2(T) - M_1(T) \).

Let \( \overline{M}_2(T) \) (or simply \( \overline{M}_2 \)) denote the maximum value of the sum of the two largest integers among the lists \((p_1, p_2, \ldots, p_s) \in \mathcal{L}(T) \), when \( p_1 = M_1(T) \), i.e.,

\[
\overline{M}_2(T) = \max_{(M_1(T), p_2, \ldots, p_s) \in \mathcal{L}(T)} (M_1(T) + p_2).
\]

Using the definition of \( M_2(T) \), we have \( \overline{M}_2(T) \leq M_2(T) \). In this paper we give upper and lower bounds for \( \overline{M}_2 \) and in some cases, a method for calculating \( \overline{M}_2 \).

## 2 Assignments

Let \( T \) be a tree on \( n \geq 2 \) vertices. If \( A \in S(T) \) and \( v \) is a vertex of \( T \) then \( A(v) \) denotes the principal submatrix of \( A \) resulting from deleting row and column associated with \( v \), and \( m_A(\lambda) \) denotes the multiplicity of eigenvalue \( \lambda \) of matrix \( A \). The Parter theorem, [8], indicates that if \( A \in S(T) \) and \( m_A(\lambda) \geq 2 \), then there is at least one vertex \( v \) of \( T \), of degree at least 3, such that \( m_{A(v)}(\lambda) = m_A(\lambda) + 1 \). Moreover, \( v \) may be chosen so that \( \lambda \) is an eigenvalue of at least three principal submatrices of \( A \) associated with branches of \( T \) at \( v \). So, we refer to any vertex \( v \) of degree greater or equal to 3 as a high-degree vertex, or HDV. The Parter theorem was refined by Wiener [9] and more fully in [5]. A vertex \( v \) of \( T \) is a Parter vertex for \( A \in S(T) \) and \( \lambda \) when \( m_A(\lambda) \geq 1 \) and \( m_{A(v)}(\lambda) = m_A(\lambda) + 1 \). The Parter theorem guarantees the existence of at least one Parter HDV
for any multiple eigenvalue. If a principal submatrix of $A$ associated with some branch at $v$ again has $\lambda$ as a multiple eigenvalue, then this theorem may again be applied to that branch. Parter vertices for $\lambda$ may be removed in this fashion until (fully) fragmenting $T$ into many subtrees when $\lambda$ occurs as an eigenvalue in such a submatrix associated with the subtree at most once. Such a set of Parter vertices is called a fully fragmented Parter set for $\lambda$, and it is known that each successive Parter vertex is also a Parter vertex for $A$ and $\lambda$ in the original tree.

If $X$ is a set or collection (or graph), then $|X|$ denotes the cardinality of (number of vertices in) $X$. If $V$ is a set of vertices and $X$ is a graph then $V \cap X$ denotes the set of vertices in both $V$ and $X$. If $X$ is a tree then $\mathcal{P}(X)$ denotes the collection of all subtrees of $X$, including $X$.

**Definition 2.1.** [7] (Assignment) Let $T$ be a tree on $n \geq 2$ vertices and let
\[
(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i})
\]
be a non-increasing list of positive integers, with $\sum_{i=1}^k p_i \leq n$. The notation $1^l$ denotes that the last $l$ entries of the list are 1. Note that some of the $p_i$'s may be 1. An assignment $A$ of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to $T$ is a collection $A = ((A_1, V_1), \ldots, (A_k, V_k))$ of $k$ collections $A_i$ of subtrees of $T$ and $k$ collections $V_i$ of vertices of $T$, with the following properties.

1. (Specification of Parter vertices) For each integer $i$ between 1 and $k$,
   1a. Each subtree in $A_i$ is a connected component of $T - V_i$.
   1b. $|A_i| = p_i + |V_i|$.
   1c. For each vertex $v \in V_i$, there exists a vertex $x$ adjacent to $v$ such that $x$ is in one of the subtrees in $A_i$.

2. (No overloading) We require that no subtree $S$ of $T$ is assigned more than $|S|$ integers; define
   \[
c_i(S) = |A_i \cap \mathcal{P}(S)| - |V_i \cap S|,
   \]
   the difference between the number of subtrees contained in $S$ and the number of Parter vertices in $S$ for the $i$th integer. So, we require that
   \[
   \sum_{i=1}^k \max(0, c_i(S)) \leq |S|, \text{ for each } S \in \mathcal{P}(T).
   \]
   If this condition is violated at any subtree, then that subtree is said to be overloaded.

**Definition 2.2.** [7] A collection $A = ((A_1, V_1), \ldots, (A_k, V_k))$ of $k$ collections $A_i$ of subtrees of $T$ and $k$ collections $V_i$ of vertices of $T$ is:

1. an assignment candidate of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to $T$ when $A$ satisfies condition 1, but not necessarily 2 of Definition 2.1.
2. a near-assignment of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to $T$ when $A$ satisfies conditions 1a, 1b, 2, but not necessarily 1c of Definition 2.1.
3. a near-assignment candidate of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to $T$ when $A$ satisfies conditions 1a, 1b, but not necessarily 1c or 2 of Definition 2.1.

In [7] a simplification of assignments of the list $(p_1, p_2, 1^l)$ is considered.

**Lemma 2.3.** (Overloading Lemma) If $T$ is a tree and $A$ is an assignment candidate (or a near-assignment candidate) of the list $(p_1, p_2, 1^l)$ to $T$, but $A$ is not an assignment (or a near-assignment, respectively), then there must exist a single vertex in $T$ that is overloaded by $A$.

**Example 2.4.** Let $T$ be the following tree
and let \((3, 2, 1^3)\) be a list.
If we consider \(\mathcal{A} = ((A_1, V_1), (A_2, V_2))\) where
\[
A_1 = T - \{4, 5\}, \quad A_2 = T - \{5\}, \quad V_1 = \{4, 5\} \quad \text{and} \quad V_2 = \{5\},
\]
then \(A_1\) has 5 connected components and \(A_2\) has 3 connected components. So, \(|A_1| = 5\) and \(|A_2| = 3\).
\(A\) is an assignment candidate of \((3, 2, 1^3)\) to \(T\) but not an assignment because the subtree \(\{6\}\) of \(T\) satisfies
\[
\max(0, c_1(\{6\})) + \max(0, c_2(\{6\})) = 1 + 1 = 2 > 1 = |\{6\}|.
\]
If we consider \(A' = ((A'_1, V'_1), (A'_2, V'_2))\), where
\[
A'_1 = T - \{4\}, \quad A'_2 = T - \{5\}, \quad V'_1 = \{4\} \quad \text{and} \quad V'_2 = \{5\},
\]
then \(A'_1\) has 4 connected components and \(A'_2\) has 3 connected components. So, \(|A'_1| = 4\) and \(|A'_2| = 3\).
\(A'\) satisfies condition 1 of Definition 2.1.
If \(S = \{1\} \quad \text{or} \quad S = \{2\} \quad \text{or} \quad S = \{3\}, \) then
\[
\max(0, c_1(S)) + \max(0, c_2(S)) = 1 + 0 = |S|.
\]
If \(S = \{4\} \quad \text{or} \quad S = \{5\} \quad \text{or} \quad S = \{7\} \quad \text{or} \quad S = \{8\}, \) then
\[
\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 0 < |S| = 1.
\]
If \(S = \{6\}\) then
\[
\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 1 = |S|.
\]
Using Lemma 2.3, \(A'\) is an assignment of \((3, 2, 1^3)\) to \(T\).

Example 2.5. Let \(T\) be the following tree

and let \((2, 2, 1^4)\) be a list.
If we consider \(\mathcal{A} = ((A_1, V_1), (A_2, V_2))\), where
\[
A_1 = T - \{5, 6, 7, 8\}, \quad A_2 = T - \{6\}, \quad V_1 = \{5, 6\} \quad \text{and} \quad V_2 = \{6\},
\]
then \(A_1\) has 4 connected components and \(A_2\) has 3 connected components. So, \(|A_1| = 4\) and \(|A_2| = 3\).
\(A\) is a near-assignment of \((2, 2, 1^4)\) to \(T\) (to prove condition 2 of Definition 2.1 use Lemma 2.3) but not an assignment because \(6 \in V_1\) and there is not a vertex of \(T\) adjacent to 6 in a subtree of \(A_1\).
Using the Overloading Lemma (Lemma 2.3), another important result appears.

**Lemma 2.6.** Let $T$ be a tree. Then
there exists a near-assignment of the list $(p_1, p_2, 1^1)$ to $T$ if and only if there exists an assignment of the list $(p_1, p_2, 1^1)$ to $T$.

**Proof** Suppose there exists a near-assignment $A = ((A_1, V_1), (A_2, V_2))$ of the list $(p_1, p_2, 1^1)$ to $T$. If $A$ satisfies 1c of Definition 2.1, then $A$ is an assignment of $(p_1, p_2, 1^1)$ to $T$.

Suppose that $A$ does not satisfy 1c. Then $V_1$ or $V_2$ does not satisfy 1c. Suppose, without loss of generalization, that $V_1$ does not satisfy 1c. So, there exists a vertex $v_1 \in V_1$ such that there is not a vertex $x$ adjacent to $v_1$ in a subtree of $A_1$.

Since $|A_1| = p_1 + |V_1|$, remove $v_1$ from $V_1$ and remove a subtree $R_1$ from $A_1$. We obtain $A_1' = A_1 \setminus R_1$ and $V_1' = V_1 \setminus \{v_1\}$. Since $|A_1'| = p_1 + |V_1'|$, we conclude that $A' = ((A_1', V_1'), (A_2, V_2))$ is a near-assignment candidate of the list $(p_1, p_2, 1^1)$ to $T$.

If $A'$ is not a near-assignment, by Lemma 2.3, there must exist a single vertex $y$ in $T$ that is overloaded by $A'$. Using the fact that $A$ is a near-assignment, $y = v_1$. But $v_1$ does not belong to $A_1$. Consequently, $S = \{v_1\}$ satisfies condition 2 of Definition 2.1. Contradiction. Therefore, $A'$ is a near-assignment.

If $A'$ satisfies 1c of Definition 2.1, then $A'$ is an assignment of $(p_1, p_2, 1^1)$ to $T$. If $A'$ does not satisfy 1c of Definition 2.1, repeat the process.

Repeating this process we obtain an assignment because $p_1, p_2 \geq 1$ and in each process we have a collection of subtrees of $T$ satisfying condition 1a of Definition 2.1.

Conversely, the proof is trivial. \qed

**Definition 2.7.** If $A \in \mathcal{S}(T)$ and $S$ is a subgraph of $T$ then

1. $A[S]$ denotes the principal submatrix of $A$ lying on rows and columns associated with the vertices of $S$.
2. $A(S)$ denotes the principal submatrix of $A$ resulting from deleting rows and columns associated with the vertices of $S$.

Using the interlacing theorem for Hermitian matrices [2], if $x$ is a vertex of $T$ (tree) and $\lambda$ is an eigenvalue of $A \in \mathcal{S}(T)$, then there is a simple relation between $m_{A[l]}(\lambda)$ and $m_{A}(\lambda)$:

$$m_{A[l]}(\lambda) = m_{A}(\lambda) - 1 \quad \text{or} \quad m_{A[l]}(\lambda) = m_{A}(\lambda) \quad \text{or} \quad m_{A[l]}(\lambda) = m_{A}(\lambda) + 1.$$ 

**Definition 2.8.** [7] Let $T$ be a tree on $n \geq 2$ vertices. We call an assignment $A = ((A_1, V_1), \ldots, (A_k, V_k))$ of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_i p_i})$ to $T$ realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $(p_1, p_2, \ldots, p_k, 1^{n-\sum_i p_i})$, such that, for each $i$ between 1 and $k$, if $s_i$ is the eigenvalue of $B$ associated with $p_i$, i.e, $m_{B}(s_i) = p_i$, then:

1. For each subtree $R$ of $T$ in $A_i$, $m_{B[R]}(s_i) = 1$.
2. For each connected component $Q$ of $T - V_i$ that is not in $A_i$, $m_{B[Q]}(s_i) = 0$.
3. For each $x \in V_i$, $x$ is a Parter vertex for $B$ and $s_i$.

**Remark 2.9.** Note that if $C \in \mathcal{S}(T)$ is a matrix that satisfies conditions 1 and 2 of Definition 2.8, then for each $i$ between 1 and $k$, $m_{C}(s_i) = p_i \geq 1$.

Using the interlacing theorem for Hermitian matrices, if $x \in V_i$, then $m_{C[l]}(s_i)$ is equal to

$$m_{C}(s_i) - 1 \quad \text{or} \quad m_{C}(s_i) \quad \text{or} \quad m_{C}(s_i) + 1.$$ 

By conditions 1 and 2 of Definition 2.8, $m_{C[V_i]}(s_i) = |A_i|$. But $A$ is an assignment, so, $|A_i| = p_1 + |V_i|$. Thus,

$$m_{C[l]}(s_i) = m_{C}(s_i) + 1.$$ 

Therefore, $C$ satisfies Definition 2.8. \qed
Using the last remark, we can rewrite Definition 2.8.

**Definition 2.8** Let $T$ be a tree on $n \geq 2$ vertices. We call an assignment $\mathcal{A} = ((A_1, V_1), \ldots, (A_k, V_k))$ of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i})$ to $T$ realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i})$, such that, for each $i$ between 1 and $k$, if $s_i$ is the eigenvalue of $B$ associated with $p_i$, i.e., $m_B(s_i) = p_i$, then:

1. For each subtree $R$ of $T$ in $\mathcal{A}_i$, $m_{B[R]}(s_i) = 1$.
2. For each connected component $Q$ of $T - V_1$ that is not in $\mathcal{A}_i$, $m_{B[Q]}(s_i) = 0$.

**Definition 2.10.** If $T$ is a tree on $n \geq 2$ vertices, $\mathcal{A}$ is a realizable assignment of $(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i})$ to $T$ and $B \in \mathcal{S}(T)$ is a matrix that satisfies Definition 2.8, then we say that $B$ realizes the assignment $\mathcal{A}$.

There are assignments that are not realizable. For instance see Example 2.3 in [7]. However when we study the list $(p_1, p_2, 1^l)$ we have the following result.

**Theorem 2.11.** [7] Given a tree $T$ on $n = p_1 + p_2 + 1$ vertices, a near-assignment of the list $(p_1, p_2, 1^l)$ to $T$, $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$, and any distinct real numbers $\alpha$ and $\beta$, then there exists $A \in \mathcal{S}(T)$ satisfying the following conditions:

If $R$ is a connected component of $T - V_1$, then

\[
\alpha \text{ is an eigenvalue of } A[R] \text{ if and only if } R \in \mathcal{A}_1.
\]

Similarly, if $S$ is a connected component of $T - V_2$, then

\[
\beta \text{ is an eigenvalue of } A[S] \text{ if and only if } S \in \mathcal{A}_2.
\]

Using Lemma 2.6, Theorem 2.11 and the new version of Definition 2.8 we obtain the following result.

**Theorem 2.12.** Given a tree $T$ on $n = p_1 + p_2 + 1$ vertices, a near-assignment $\mathcal{A}$ of the list $(p_1, p_2, 1^l)$ to $T$, and any distinct real numbers $\alpha$ and $\beta$, then

1. there exists a realizable assignment $B$ of $(p_1, p_2, 1^l)$ to $T$.
2. there exists $A \in \mathcal{S}(T)$ that realizes the assignment $B$ with $m_A(\alpha) = p_1$ and $m_A(\beta) = p_2$.

Therefore, we immediately have as a consequence:

**Corollary 2.13.** For any tree $T$, if there exists a near-assignment of the list $(M_1(T), p_2, 1^l)$ to $T$, then

\[
\overline{M}_2(T) \geq M_1(T) + p_2.
\]

### 3 Upper and lower bounds for $\overline{M}_2$

In this section, using the reduction theorem for $M_2$, [7], we directly compute $\overline{M}_2$ for particular trees. For other kinds of trees, we give bounds on $\overline{M}_2$.

In [7], the authors directly computed $M_2$ for generalized stars (for the notion of generalized star see [6]).

**Definition 3.1.** [6] Let $T$ be a tree and $x_0$ be a vertex of $T$. A generalized star $T$ with central vertex $x_0$ is a tree such that $T - \{x_0\}$ is a union of paths (arms), each one of them is adjacent to $x_0$ by an endpoint.

**Proposition 3.2.** [7] Let $T$ be a generalized star on $n \geq 2$ vertices, with $f$ arms of length 1 and $g$ arms of length at least 2. Then:

(A) If $g \geq 2$, then $M_2(T) = f + 2g - 2$. 


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(B) If \( g \leq 1 \) and \( T \) is not a path, then \( M_2(T) = f + g \).

(C) If \( T \) is a path, then \( M_2(T) = 2 \).

**Definition 3.3.** [ Peripheral HDV, peripheral arm] Given a tree \( T \) and a high-degree vertex \( v \), \( v \) is a peripheral HDV of \( T \) if and only if there is a branch of \( T \) at \( v \) that contains all the other high-degree vertices in \( T \). A peripheral arm of a tree \( T \) is a branch of \( T \) at a peripheral HDV such that the branch does not itself contain any HDV.

**Definition 3.4.** Throughout this section, we will consider a peripheral HDV \( v \) in a tree \( T \).

The subtree of \( T \) consisting of \( v \) and its peripheral arms will be called \( S \) however, if \( v \) is the only HDV in \( T \), we will let \( S \) be \( v \) and all but one of its peripheral arms (chosen arbitrarily). The point is that \( S \) should be a generalized star containing everything except a single branch of \( T \) at \( v \).

\[ (T - S) + w K_1 \] the tree obtained from \( T - S \) by putting a vertex adjacent to \( v \) that is not in \( S \). We denote by \( (T - S) + w K_1 \) the tree obtained from \( T - S \) by putting a vertex adjacent to \( v \) that is not in \( S \).

**Theorem 3.5.** [ M_2 Reduction Theorem] Let \( T \) be a tree and \( v \) a peripheral HDV, with \( S \) as defined earlier in this section. Suppose that \( S \) has \( f \) arms of length \( 1 \) and \( g \) arms of length at least \( 2 \). Then:

(A) If \( g \geq 2 \), then \( M_2(T - S) = M_2(T) - f - 2g + 2 \).

(B) If \( g \leq 1 \), then \( M_2((T - S) + w K_1) = M_2(T) - f - g + 1 \).

In [1] a class of trees was introduced that contains the generalized stars, the superstars.

**Definition 3.6.** [ Peripheral HDV, peripheral arm] Let \( T \) be a tree and \( x_0 \) be a vertex of \( T \). A superstar \( T \) with central vertex \( x_0 \) is a tree such that \( T - \{x_0\} \) is a union of paths.

The focus of this section is to directly compute \( M_2 \) for a subclass of superstars.

**Definition 3.7.** Let \( T \) be a superstar with central vertex \( x_0 \). A small pincer of \( T \) is a path, \( P \), of \( T - \{x_0\} \) such that:

1. \( P \) is adjacent to \( x_0 \) by a vertex \( u \) of degree two in \( P \).
2. At least one path of \( P - u \) is a vertex.

**Definition 3.8.** Let \( T \) be a superstar with central vertex \( x_0 \). \( T \) is a small superstar if all paths of \( T - \{x_0\} \) are small pincers or are adjacent to \( x_0 \) by an endpoint (arms).

**Example 3.9.** The superstar \( T \) of Example 2.4 is a small superstar with central vertex \( 4 \). The superstar \( T \) of Example 2.5 is a small superstar with central vertex \( 5 \). All stars and generalized stars are small superstars.

The following superstar is not a small superstar

**Definition 3.10.** Let \( T \) be a tree and \( A \) an assignment of \((M_1(T), p_2, 1^1)\) to \( T \).

1. We refer to \( A \) as an \( M_2 \) assignment to \( T \).
2. If \( M_1(T) + p_2 = M_2(T) \), we refer to \( A \) as an \( M_2 \)-maximal assignment to \( T \).
Remark 3.11. Let $T$ be a tree and $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^1)$ to $T$. Because $M_1(T) = |A_1| - |V_1|$, 

1. All components of $T - V_1$ are in $A_1$.
2. We can assume that if $v \in V_1$ then $v$ is a HDV.
3. Since all components of $T - V_1$ are paths, if $v$ if a peripheral HDV of degree greater or equal to 4 in $T$ then $v \in V_1$.
4. If $v$ is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then they are in $A_1$ and no one is in $A_2$ (see Lemma 2.3).

Remark 3.12. Let $T$ be a tree and $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^1)$ to $T$. Because $\overline{M_2}(T) = |A_1| - |V_1| + |A_2| - |V_2|$, 

1. All components of $T - V_2$ with more than one vertex are in $A_2$.
2. We can assume that if $v \in V_2$ then $v$ is a HDV.
3. All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in $A_2$.
4. If $v$ is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then using Remark 3.11, 4, we conclude that $v \notin V_2$.
5. If $v$ is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1, except one, then there is an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^1)$ to $T$ such that $v \notin V_2$.

Remark 3.13. In some proofs we construct an $\overline{M_2}$-maximal (or simply an $\overline{M_2}$) assignment of $(M_1(T), p_2, 1^1)$ to $T$, for some integer $p_2$. In these cases, first we construct an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^1)$ to $T$, $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ by putting the elements in $A_1$ and in $V_1$, next we put the elements in $A_2$ and in $V_2$, using Remarks 3.11 and 3.12. This construction is in such a way that $M_1(T) = |A_1| - |V_1|$ and $M_1(T) + p_2 = |A_1| - |V_1| + |A_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13, we say that $\overline{M_2}(T) = M_1(T) + p_2$. 

Proposition 3.14. Let $T$ be a small superstar on $n \geq 2$ vertices, with $f$ arms of length 1, $g$ arms of length at least 2 and $h$ small pincers, with $f + g \geq 2$ or $h \geq 2$. Then:

A. If $g \geq 2$, then $\overline{M_2}(T) = 2h + f + 2g - 2$.
B. If $g \leq 1$ and $T$ is not a path, then $\overline{M_2}(T) = 2h + f + g$.
C. If $T$ is a path, then $\overline{M_2}(T) = 2$.

Proof Let $x$ be the central vertex of $T$. If $S$ is a small pincer of $T$, by Theorem 3.5,

$$M_2((T - S) + x, K_1) = M_2(T) - 1.$$ 

Since $T$ has $h$ small pincers,

$$M_2(T') = M_2(T) - h,$$

where $T'$ is obtained from $T$ by removing all small pincers and by putting $h$ vertices adjacent to $x$. Consequently, $T'$ is a generalized star with $f + h$ arms of length 1 and $g$ arms of length at least 2. Using Proposition 3.2

$$M_2(T') = \begin{cases} 
  f + h + 2g - 2 & \text{if } g \geq 2 \\
  f + h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\
  2 & \text{if } T' \text{ is a path}.
\end{cases}$$

Therefore,

$$M_2(T) = \begin{cases} 
  f + 2h + 2g - 2 & \text{if } g \geq 2 \\
  f + 2h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\
  2 + h & \text{if } T' \text{ is a path}.
\end{cases}$$

Note that if $T'$ is a path with $h = 2$ and $f = g = 0$ then $T$ is not a path and $M_2(T) = M_2(T') + h = 2 + 2 = 4 = 2h + g$. By hypothesis, if $h < 2$ then $f + g \geq 2$. In this case, if $T'$ is a path then $h = 0$ and $f + g = 2$. Consequently, $T$ is a path.
So, we conclude that

\[
M_2(T) = \begin{cases} 
  f + 2h + 2g - 2 & \text{if } g \geq 2 \\
  f + 2h + g & \text{if } g \leq 1 \text{ and } T \text{ is not a path} \\
  2 & \text{if } T \text{ is a path.}
\end{cases}
\]

Since \(\overline{M}_2(T) \leq M_2(T)\), we have

(A) If \(g \geq 2\), then \(\overline{M}_2(T) \leq 2h + f + 2g - 2\).
(B) If \(g \leq 1\) and \(T\) is not a path, then \(\overline{M}_2(T) \leq 2h + f + g\).
(C) If \(T\) is a path, then \(\overline{M}_2(T) \leq 2\).

Conversely, since \(T\) is a tree,

\[
M_1(T) = \begin{cases} 
  f + h + g - 1 & \text{if } f + g \geq 2 \\
  h + 1 & \text{if } f + g \leq 1.
\end{cases}
\]

We are going to construct an \(\overline{M}_2\) assignment of \((M_1(T), p_2, 1')\), for some integer \(p_2\), to \(T\) (see Remark 3.13).

**Case 1** If \(f + g \geq 2\), we put the central vertex of \(T\) in \(V_1\) and we put the \(f + h + g\) paths obtained by removing the central vertex of \(T\) in \(A_1\).

If \(g \geq 2\), we put the central vertex of \(T\) in \(V_2\) and we put the \(h + g\) paths of length at least 2, obtained by removing the central vertex of \(T\) in \(A_2\). So, \(|A_1| - |V_1| = f + g + h - 1 = M_1(T)\). Using Remark 3.13, \(\overline{M}_2(T) \geq f + h + g - 1 + h + g - 1 = f + 2h + 2g - 2\).

If \(g \leq 1\), we put the central vertex of each small pincer of \(T\) in \(V_2\), we put the \(2h + 1\) subtrees obtained by removing the central vertex of all small pincers of \(T\) in \(A_2\). Since \(|A_2| - |V_2| = f + g + h - 1 = M_1(T)\), using Remark 3.13, \(\overline{M}_2(T) \geq f + h + g - 1 + 2h + 1 - h = f + 2h + g\).

Note that if \(T\) is a path and \(f + g \geq 2\) then \(f + g = 2\) and \(h = 0\). Thus, if \(g = 2\), then \(\overline{M}_2(T) \geq f + 2h + 2g - 2 = 2\) and if \(g \leq 1\), then \(\overline{M}_2(T) \geq f + 2h + g = 2\).

**Case 2** If \(f + g \leq 1\) then \(h \geq 2\) and \(T\) is not a path. We put the central vertex of each small pincer of \(T\) in \(V_1\) and we put the \(2h + 1\) subtrees obtained by removing the central vertex of all small pincers of \(T\) in \(A_1\). We put the central vertex of \(T\) in \(V_2\) and we put the \(f + h + g\) paths obtained by removing the central vertex of \(T\) in \(A_2\). Since \(|A_1| - |V_1| = h + 1 = M_1(T)\), by Remark 3.13, \(\overline{M}_2(T) \geq h + 1 + f + g + h - 1 = f + g + 2h\).

Consequently,

(A) If \(g \geq 2\), then \(\overline{M}_2(T) \geq 2h + f + 2g - 2\).
(B) If \(g \leq 1\) and \(T\) is not a path, then \(\overline{M}_2(T) \geq 2h + f + g\).
(C) If \(T\) is a path, then \(\overline{M}_2(T) \geq 2\).

Therefore,

(A) If \(g \geq 2\), then \(\overline{M}_2(T) = 2h + f + 2g - 2\).
(B) If \(g \leq 1\) and \(T\) is not a path, then \(\overline{M}_2(T) = 2h + f + g\).
(C) If \(T\) is a path, then \(\overline{M}_2(T) = 2\).

**Proposition 3.15.** Let \(T\) be a tree and \(v\) a peripheral HDV, with \(S\) as defined earlier in this section. Suppose that \(S\) has 3 arms of length 1 and 0 arms of length at least 2 and \(T \neq S\). Then

\[
\overline{M}_2(T - S) + 2 \leq \overline{M}_2(T) \leq \overline{M}_2(T - S) + 3.
\]

**Proof** Let \(A = ((A_1, V_1), (A_2, V_2))\) be an \(\overline{M}_2\)-maximal assignment to \(T\). We are going to construct an \(\overline{M}_2\) assignment to \(T - S\), \(A' = ((A_1', V_1'), (A_2', V_2'))\) (see Remark 3.13). Note that \(M_1(T - S) = M_1(T) - 2\). Because \(v\) has degree 4, by Remark 3.11, 3, \(v \in V_1\), the peripheral arms of \(S\) are in \(A_1\) and no one is in \(A_2\). Using Remark 3.12, 4, \(v \notin V_2\). So, let \(F\) be the component of \(T - V_2\) containing \(S\). By Remark 3.12, 1, \(F\) is in \(A_2\).

Let \(A_1' = A_1 \setminus \{\text{the peripheral arms of } S\}\), \(V_1' = V_1 \setminus \{v\}\), \(V_2' = V_2\) and

\[
A_2' = \begin{cases} 
  A_2 \setminus \{F\} & \text{if } A_2 \neq \{F\} \\
  T - S & \text{if } A_2 = \{F\}
\end{cases}
\]
By Remark 3.13, \( A' = ((A_1', V_1'), (A_2', V_2')) \) is an \( M_2 \) assignment to \( T - S \) and \( M_2(T - S) \geq M_2(T) - 2 - 1 = M_2(T) - 3 \).

Let \( A = ((A_1, V_1), (A_2, V_2)) \) be an \( M_2 \)-maximal assignment to \( T - S \). We are going to construct an \( M_2 \) assignment to \( T - S \), \( A' = ((A_1', V_1'), (A_2', V_2')) \) (see Remark 3.13). Note that \( M_1(T) = M_1(T - S) + 2 \). Let \( w \) be the vertex of \( T - S \) adjacent to \( v \) in \( T \). If \( w \notin V_2 \), then let \( R \) be the component of \( (T - S) - V_2 \) containing \( w \) and let \( P \) be the component of \( T - V_2 \) containing \( S \).

Let \( A'_1 = A_1 \cup \{ \text{the peripheral arms of } S \}, V'_1 = V_1 \cup \{ v \}, V'_2 = V_2 \) and

\[
A'_2 = \begin{cases}
A_2 \setminus \{ R \} \cup \{ P \} & \text{if } R \in A_2 \text{ and } w \notin V_2 \\
A_2 & \text{otherwise}
\end{cases}
\]

By Remark 3.13, \( A' = ((A_1', V_1'), (A_2', V_2')) \) is an \( M_2 \) assignment to \( T \) and \( M_2(T) \geq M_2(T) + 2 \).

**Proposition 3.16.** Let \( T \) be a tree and \( v \) a peripheral HDV, with \( S \) as defined earlier in this section. Suppose that \( S \) has 1 arm of length 1 and 1 arm of length at least 2 (or \( T \) has 2 arms of length 1 and 0 arms of length at least 2) and \( T \neq S \). Then

\[
M_2(T - S) + 1 \leq M_2(T) \leq M_2(T - S) + 2.
\]

**Proof** Let \( A = ((A_1, V_1), (A_2, V_2)) \) be an \( M_2 \)-maximal assignment to \( T \). We are going to construct an \( M_2 \) assignment to \( T - S \), \( A' = ((A_1', V_1'), (A_2', V_2')) \) (see Remark 3.13). Note that \( M_1(T - S) = M_1(T) - 1 \).

Using Remark 3.11, 1, if \( v \) is in \( V_1 \), then the peripheral arms of \( S - v \) are in \( A_1 \). Using Remark 3.12, 5, without loss of generality, we can assume that \( v \notin V_2 \). Let \( F \) be the component of \( T - V_2 \) containing \( S \). By Remark 3.12, 1 and 3, \( F \) is in \( A_2 \).

Let \( A'_1 = A_1 \setminus \{ \text{the peripheral arms of } S \}, V'_1 = V_1 \setminus \{ v \}, V'_2 = V_2 \) and

\[
A'_2 = \begin{cases}
A_2 \setminus \{ F \} & \text{if } A_2 \neq \{ F \} \\
T - S & \text{if } A_2 = \{ F \}
\end{cases}
\]

By Remark 3.13, \( A' = ((A_1', V_1'), (A_2', V_2')) \) is an \( M_2 \) assignment to \( T - S \) and \( M_2(T - S) \geq M_2(T) - 1 - 1 = M_2(T) - 2 \).

If \( v \) is not in \( V_1 \), since \( v \) has degree 3 in \( T \), then \( w \in V_1 \). By Remark 3.11, 1, \( S \) is in \( A_1 \). By Remark 3.12, 3, we can assume, without loss of generality, that \( v \in V_2 \) and the peripheral arms of \( S \) are in \( A_2 \).

Let \( A'_1 = A_1 \setminus \{ S \}, V'_1 = V_1 \setminus \{ v \}, V'_2 = V_2 \), \( A'_2 = A_2 \setminus \{ \text{the peripheral arms of } S \} \).

By Remark 3.13, \( A' = ((A_1', V_1'), (A_2', V_2')) \) is an \( M_2 \) assignment to \( T - S \) and \( M_2(T - S) \geq M_2(T) - 1 - 1 = M_2(T) - 2 \).

Let \( A = ((A_1, V_1), (A_2, V_2)) \) be an \( M_2 \)-maximal assignment to \( T - S \). We are going to construct an \( M_2 \) assignment to \( T \), \( A' = ((A_1', V_1'), (A_2', V_2')) \) (see Remark 3.13). Note that \( M_1(T) = M_1(T - S) + 1 \). Let \( w \) be the vertex of \( T - S \) adjacent to \( v \) in \( T \). If \( w \notin V_2 \), then let \( F \) be the component of \( (T - S) - V_2 \) containing \( w \) and let \( P \) be the component of \( T - V_2 \) containing \( S \).

Let \( A'_1 = A_1 \cup \{ \text{the peripheral arms of } S \}, V'_1 = V_1 \cup \{ v \}, V'_2 = V_2 \) and

\[
A'_2 = \begin{cases}
A_2 \setminus \{ F \} \cup \{ P \} & \text{if } F \in A_2 \text{ and } w \notin V_2 \\
A_2 & \text{otherwise}
\end{cases}
\]

By Remark 3.13, \( A' = ((A_1', V_1'), (A_2', V_2')) \) is an \( M_2 \) assignment to \( T \) and \( M_2(T) \geq M_2(T - S) + 1 \).

**Example 3.17.** Let \( T \) be the following tree.
Let $H$ be the subtree obtained from $T$ by removing vertices 11, 12, 13. By Proposition 3.16,
\[ \overline{M_2}(H) + 1 \leq \overline{M_2}(T) \leq \overline{M_2}(H) + 2. \]

Since $H$ is a small superstar with central vertex 4, by Proposition 3.14, $\overline{M_2}(H) = 4 + 0 + 4 - 2 = 2$. So,
\[ 5 \leq \overline{M_2}(T) \leq 6. \]

### 4 An algorithm for $\overline{M_2}$

The purpose of this section is to find simple reductions of the initial tree in such a way that we know the effect of each reduction on $\overline{M_2}$. The process may be continued until a small superstar, for which $\overline{M_2}$ is known (Proposition 3.14), or until a subtree for which $\overline{M_2}$ has bounds (Section 3).

**Definition 4.1.** (Peripheral SHDV, peripheral super path) Let $T$ be a tree that is not a small superstar. A peripheral superstar high degree vertex (SHDV) $v$ of $T$ is an HDV vertex of $T$ such that

1. there is a unique subtree of $T - v$, $R$, that contains high-degree vertices;
2. $T - R$ is a small superstar;
3. if $w \in R$ and $w$ is adjacent to $v$, then $w$ does not satisfy 1, 2.

A peripheral super path of $T$ at $v$ ($v$ is a SHDV) is a path of $(T - R) - v$. There are two kinds of peripheral super paths of $T$ at $v$ (SHDV): peripheral arms and small pincers.

**Example 4.2.** Consider the tree $T$ of Example 3.17.

The vertices 4 and 8 are peripheral superstar high degree vertices.

The vertex 2 is not a peripheral superstar high degree vertex because it is adjacent to vertex 4 and this vertex satisfies conditions 1 and 2 of Definition 4.1.

The subtree of $T$ generated by vertices 1, 2, 3 is a peripheral super path of $T$ at 4, but it is not a peripheral arm of $T$ at 4 (it is a small pincer).

**Definition 4.3.** Throughout this section, we will consider a peripheral SHDV $v$ in a tree $T$ that is not a small superstar. The subtree of $T$ consisting of $v$ and its peripheral super paths will be called $Q$. Let $w$ be the one vertex adjacent to $v$ that is not in $Q$.

**Remark 4.4.** Let $T$ be a tree and $A = (A_1, V_1), (A_2, V_2)$ an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^1)$ to $T$. Because $M_1(T) = |A_1| - |V_1|$, we have

1. All components of $T - V_1$ are in $A_1$.
2. We can assume that if $v \in V_1$ then $v$ has degree greater than two in $T$.
3. Since all components of $T - V_1$ are paths, if $v$ is a peripheral SHDV of degree greater or equal to 4 in $T$ then $v \in V_1$ or there is at most one peripheral arm adjacent to $v$ and the central vertex of each small pincer adjacent to $v$ is in $V_1$.
4. If $v$ is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to $v$ have length 1 then they are in $A_1$ and no one is in $A_2$ (see Lemma 2.3).
Remark 4.5. Let $T$ be a tree and $A = ((A_1, V_1), (A_2, V_2))$ an $\overline{M}_2$-maximal assignment of $(M_1(T), p_2, 1^1)$ to $T$. Because $\overline{M}_2(T) = |A_1| - |V_1| + |A_2| - |V_2|$, we can assume that if $v \in V_2$ then $v$ has degree greater than two in $T$.

(1) All components of $T - V_2$ with more than one vertex are in $A_2$.

(2) We can assume that if $v \in V_2$ then $v$ has degree greater than two in $T$.

(3) All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in $A_2$.

(4) If $v$ is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to $v$ have length 1, then using Remark 4.4, we conclude that $v \notin V_2$.

(5) If $v$ is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to $v$ have length 1, except one, then there is an $\overline{M}_2$-maximal assignment of $(M_1(T), p_2, 1^1)$ to $T$ such that $v \notin V_2$.

Remark 4.6. In some proofs we construct an $\overline{M}_2$-maximal (or simply an $\overline{M}_2$) assignment of $(M_1(T), p_2, 1^1)$ to $T$, for some integer $p_2$. In these cases, first we construct an $\overline{M}_2$ assignment of $(M_1(T), p_2, 1^1)$ to $T$, $A = ((A_1, V_1), (A_2, V_2))$ by putting the elements in $A_1$ and in $V_1$, next we put the elements in $A_2$ and in $V_2$, using Remarks 4.4 and 4.5. This construction is in such a way that $M_1(T) = |A_1| - |V_1|$ and $M_1(T) + p_2 = |A_1| - |V_1| + |A_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13, we say that $\overline{M}_2(T) \geq M_1(T) + p_2$.

Proposition 4.7. Let $T$ be a tree that is not a small superstar and $v$ a peripheral SHDV, with $Q$ as defined earlier in this section. Suppose that $Q$ has $h \geq 1$ small pincers and the degree of $v$ in $T$ is greater than 4. Let $H$ be the graph obtained from $T$ by removing one small pincer of $Q$. Then

$$\overline{M}_2(H) = \overline{M}_2(T) - 2.$$ 

Proof By Proposition 3.16, $\overline{M}_2(H) \geq \overline{M}_2(T) - 2$.

Let $A = ((A_1, V_1), (A_2, V_2))$ be an $\overline{M}_2$-maximal assignment to $H$. We are going to construct an $\overline{M}_2$ assignment to $T$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6). Note that $M_1(T) = M_1(H) + 1$. Since the degree of $v$ in $T$ is greater than 4, we conclude that $v \in V_1 \cup V_2$.

Suppose that $v \in V_1 \cap V_2$. Let $P$ be the small pincer $T - H$.

Let $A'_1 = A_1 \cup \{P\}$, $V'_1 = V_1$, $V'_2 = V_2$ and $A'_2 = A_2 \cup \{P\}$.

By Remark 4.6, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ is an $\overline{M}_2$ assignment to $T$ and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

Suppose that $v \notin V_1 \cap V_2$. Let $x$ be the central vertex of the small pincer, $P$, of $T - H$.

Let $A'_1 = A_1 \cup \{P\}$, $V'_1 = V_1$, $A'_2 = A_2 \cup \{P\}$ (the peripheral arms of $P$ at $x$) and $V'_2 = V_2 \cup \{x\}$.

By Remark 4.6, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ is an $\overline{M}_2$ assignment to $T$ and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

Suppose that $v \notin V_1 \cup V_2$. Let $x$ be the central vertex of the small pincer, $P$, of $T - H$.

Let $V'_1 = V_1 \cup \{x\}$, $A'_1 = A_1 \cup \{P\}$ (the peripheral arms of $P$ at $x$), $A'_2 = A_2 \cup \{P\}$ and $V'_2 = V_2$.

By Remark 4.6, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ is an $\overline{M}_2$ assignment to $T$ and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

Consequently, $\overline{M}_2(T) = \overline{M}_2(H) + 2$. 

Lemma 4.8. Let $T$ be a tree that is not a small superstar. Suppose that $v$ is a peripheral SHDV in $T$ with $Q$, $w$ as defined earlier in this section. Then, there exists an $\overline{M}_2$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$, in which $v \in V_1 \cup V_2$.

Moreover,

(1) If $v$ has at least two peripheral arms of length at least 2, then there exists an $\overline{M}_2$-maximal assignment, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ in which $v \notin V'_1 \cap V'_2$.

(2) If $v$ has at most one peripheral arm of length at least 2 and $w$ has degree two in $T$, then there exists an $\overline{M}_2$-maximal assignment, $A'' = ((A''_1, V''_1), (A''_2, V''_2))$ such that $v$ is in exactly one $V''_1$ or $V''_2$.

(3) If $Q$ has $f$ peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2, $f + g > 2$ and $A''' = ((A'''_1, V'''_1), (A'''_2, V'''_2))$ is an $\overline{M}_2$-maximal assignment to $T$, then $v \in V'''_1$. 

(4) If $Q$ has $f$ peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2, $f + g > 2$ and $A''' = ((A'''_1, V'''_1), (A'''_2, V'''_2))$ is an $\overline{M}_2$-maximal assignment to $T$, then $v \in V'''_1$. 


Proof Let $A = ((A_1, V_1), (A_2, V_2))$ be an $\overline{M}_T$-maximal assignment to $T$ in which $v \not\in V_1 \cup V_2$. Suppose that $Q$ has $f$ peripheral arms of length 1 and $g$ peripheral arms of length at least 2. We are going to construct an $\overline{M}_T$-maximal assignment to $T$, $B = ((B_1, U_1), (B_2, U_2))$ (see Remark 4.6).

If $f + g \geq 2$, then by Remark 4.4, 1, the component, $R$, of $T - V_1$ containing $v$ is in $A_1$. Note that the peripheral arms of $Q$ might be in $R$.

Let $B_1 = (A_1 \setminus \{R\}) \cup \{\text{two peripheral arms of } Q\}$, $U_1 = V_1 \cup \{v\}$, $B_2 = A_2$ and $U_2 = V_2$.

By Remark 4.6 and the cardinality of $B$, $B = ((B_1, U_1), (A_2, V_2))$ is an $\overline{M}_T$-maximal assignment to $T$ in which $v \in U_1$.

If $f + g \leq 1$, by Remark 4.4, 3 and Remark 4.5, 1 and 3, the central vertex of each small pincer of $Q$ is in $V_1 \cup V_2$. By Remark 4.5, 1, the component, $R$, of $T - V_2$ containing $v$, is in $A_2$.

Let $B_1 = A_1$, $U_1 = V_1$, $B_2 = (A_1 \setminus \{R\}) \cup \{\text{two peripheral super paths of } Q\}$ and $U_2 = V_2 \cup \{v\}$.

By Remark 4.6 and the cardinality of $B$, $B = ((B_1, U_1), (B_2, U_2))$ is an $\overline{M}_T$-maximal assignment to $T$ in which $v \in U_2$. So, there exists an $\overline{M}_T$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$ in which $v \in V_1 \cup V_2$.

1) By what we just proved, there exists an $\overline{M}_T$-maximal assignment to $T$,

$$A = ((A_1, V_1), (A_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose without loss of generality that $v \in V_1 \setminus V_2$. We are going to construct an $\overline{M}_T$-maximal assignment to $T$, $A' = ((A_1', V_1'), (A_2', V_2'))$, in which $v \in V_1' \cap V_2'$. (see Remark 4.6). By Remark 4.5, 1 and 3, the component, $R$, of $T - V_2$ containing $v$, is in $A_2$. Note that the peripheral arms of $Q$ might be in $R$.

Let $A_1' = A_1$, $V_1' = V_1$, $V_2' = V_2 \cup \{v\}$ and $A_2' = (A_2 \setminus \{R\}) \cup \{\text{two peripheral arms of length at least two of } Q\}$.

Since $|A_2'| - |V_2'| = |A_2'| - |V_2'|$, by Remark 4.6 and the cardinality of $A'$, $A' = ((A_1', V_1'), (A_2', V_2'))$ is an $\overline{M}_T$-maximal assignment to $T$, in which $v \in V_1 \cap V_2$.

2) By what we just proved, there exists an $\overline{M}_T$-maximal assignment,

$$A = ((A_1, V_1), (A_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose $v \in V_1 \cap V_2$. We are going to construct an $\overline{M}_T$-maximal assignment to $T$, $A'' = ((A_1'', V_1''), (A_2'', V_2''))$, in which $v \in V_1'' \cap V_2''$. (see Remark 4.6) Using Remark 4.4, 1, each peripheral super path of $Q$ is in $A_1$. By Remark 4.5, 1, the longer arm of $Q$ and the small pincers of $Q$ are in $A_2$ and there is not a peripheral arm of length 1 of $Q$ in $A_2$. By Remark 4.5, 2, $w \not\in V_2$. Let $R$ be the component of $T - V_2$ containing $w$ and let $F$ be the component of $T - ((V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$ containing $v$ and $w$. By Remark 4.5, 1, $F \in A_2$.

Let $A_1'' = A_1$, $V_1'' = V_1$, $V_2'' = V_2 \cup \{v\}$ and $A_2'' = (A_2 \setminus \{R\}) \cup \{\text{the peripheral super paths of length at least two of } Q, R\} \cup \{v\}$.

If $Q$ does not have a longer arm or $R \not\in A_2$ then $|A_2''| - |V_2''| > |A_2| - |F_2|$. This is impossible because $A$ is an $\overline{M}_T$-maximal assignment to $T$. So, $v \not\in V_1 \cap V_2$.

If $Q$ has a longer arm and $R \in A_2$ then $|A_2| - |V_2| = |A_2''| - |V_2''|$. By Remark 4.6 and using the cardinality of $A'$, $A'' = ((A_1', V_1'), (A_2'', V_2''))$ is an $\overline{M}_T$-maximal assignment to $T$, in which $v \in V_1'' \cap V_2''$.

3) By what we just proved, there exists an $\overline{M}_T$-maximal assignment to $T$, $A'' = ((A_1'', V_1''), (A_2'', V_2''))$.

Lemma 4.9. Let $T$ be a tree that is not a small superstar. Suppose that $v$ is a peripheral SHDV in $T$ with $Q$, $w$ as defined earlier in this section. Suppose that $Q$ has $f$ peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2 and the degree of $w$ in $T$ is 2. Then, there exists an $\overline{M}_T$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$, in which:
(1) If \( f + g \geq 1 \), then \( v \in V_1 \) and the central vertex of each small pincer of \( Q \) belongs to \( V_2 \).

(2) If \( f + g = 0 \), then \( v \in V_2 \) and the central vertex of each small pincer of \( Q \) belongs to \( V_1 \).

**Proof**

(1) By 2 of Lemma 4.8, let \( A = ((A_1, V_1), (A_2, V_2)) \) be an \( \overline{M}_2 \)-maximal assignment to \( T \) such that \( v \) is exactly one \( V_1 \) or \( V_2 \).

If \( f + g > 1 \), since \( v \) is a peripheral SHDV and \( w \notin V_1 \) (the degree of \( w \) in \( T \) is 2), by Remark 4.4, 1, each peripheral super path of \( Q \) belongs to \( A_1 \) and \( v \in V_1 \). In this case, because \( v \notin V_2 \) and \( A \) is an \( \overline{M}_2 \)-maximal assignment to \( T \), we conclude that the central vertex of each small pincer of \( Q \) is in \( V_2 \) and the peripheral arms of each small pincer of \( Q \) are in \( A_2 \).

Suppose that \( f + g = 1 \) and \( v \in V_2 \), then by Remark 4.4, 1, the central vertex of each small pincer of \( Q \) is in \( V_1 \) and the peripheral arms of each small pincer of \( Q \) are in \( A_1 \). By Remark 4.5, 1 and 3, the peripheral super paths of \( Q \) are in \( A_2 \). Since \( w \) has degree two in \( T \), we can assume that \( w \notin V_1 \cup V_2 \). We are going to construct an \( \overline{M}_2 \)-maximal assignment to \( T \), \( A' = ((A'_1, V'_1), (A'_2, V'_2)) \), in which \( v \in V_1 \) and the central vertex of each small pincer of \( Q \) is in \( V_2 \) (see Remark 4.6). Let \( R \) be the component of \( T - V_1 \) containing \( v, w \). By Remark 4.4, 1, \( R \in A_1 \). Let \( P \) be the component of \( T - V_2 \), containing \( w \). Since \( P \neq R \), by Remark 4.5, 1 and 3, \( P \in A_2 \). Let \( B \) be the component of \( T - (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\} \), containing \( v \) and \( w \). Let \( C \) be the component of \( T - (V_1 \cup \{v\}) \), containing \( w \). Note that \( B \neq C \).

Let

\[
A'_1 = (A_1 \setminus \{\text{the peripheral arms of each small pincer of } Q, R\}) \cup \{C, \text{ the peripheral super paths of } Q\},
\]

\[
V'_1 = (V_1 \setminus \{\text{the central vertex of each small pincer of } Q\}) \cup \{v\},
\]

\[
A'_2 = (A_2 \setminus \{\text{the peripheral super paths of } Q, P\}) \cup \{\text{the peripheral arms of each small pincer of } Q, B\}
\]

and \( V'_2 = (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\} \).

Since \( |A'_1| - |V'_1| = |A_1| - |V_1| \) and \( |A'_2| - |V'_2| = |A_2| - |V_2| \) and by Remark 4.6, we get an \( \overline{M}_2 \)-maximal assignment to \( T \), \( A' = ((A'_1, V'_1), (A'_2, V'_2)) \), where \( v \in V'_1 \) and the central vertex of each small pincer of \( Q \) belongs to \( V'_2 \).

(2) By 2 of Lemma 4.8, let \( A = ((A_1, V_1), (A_2, V_2)) \) be an \( \overline{M}_2 \)-maximal assignment to \( T \) such that \( v \) is exactly one \( V_1 \) or \( V_2 \). Since \( f + g = 0 \), \( v \) is a peripheral SHDV and \( w \notin V_1 \), if \( v \in V_1 \) then by Remark 4.4, 1, the peripheral super paths of \( Q \) are in \( A_1 \). Let \( F \) be the component of \( T - V_1 \) containing \( w \). By Remark 4.4, 1, \( F \in A_1 \). Let \( H \) be the component of \( T - (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\} \) containing \( w \) and \( v \). Let

\[
A'_1 = (A_1 \setminus \{\text{the peripheral super paths of } Q, F\}) \cup \{\text{the peripheral arms of each small pincer of } Q, H\},
\]

\[
V'_1 = (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}.
\]

Since \( |A'_1| - |V'_1| = |A_1| - |V_1| + 1 \) we conclude that \( A \) is not an \( \overline{M}_2 \)-maximal assignment to \( T \). Impossible. Consequently, \( v \notin V_1 \) and \( v \in V_2 \).

Therefore, the central vertex of each small pincer of \( Q \) belongs to \( V_1 \). \( \square \)

**Theorem 4.10.** (\( \overline{M}_2 \) Reduction Theorem) Let \( T \) be a tree that is not a small superstar and \( v \) a peripheral SHDV, with \( Q, w \) as defined earlier in this section. Suppose that \( Q \) has \( f \) peripheral arms of length 1, \( g \) peripheral arms of length at least 2 and \( h \) small pincers. Then:

(A) If \( g \geq 2 \), then \( \overline{M}_2(T - Q) = \overline{M}_2(T) - f - 2g - 2h + 2 \).

(B) If \( g \leq 1 \) and the degree of \( w \) in \( T \) is 2, then \( \overline{M}_2((T - Q) + w K_1) = \overline{M}_2(T) - f - g - 2h + 1 \), where \((+w K_1)\) means that we put a vertex adjacent to \( w \).

(C) If \( g \leq 1 \), the degree of \( w \) in \( T \) is greater than 2 and \( f + g > 2 \) then

\[
\overline{M}_2((T - Q) + w S_4) = \overline{M}_2(T) - f - g - 2h + 3,
\]
where $S_4$ is the star with 3 arms of length 1 and $(+_w S_4)$ means that $S_4$ is adjacent to $w$ by the central vertex.

**Proof Part A:** Let $A = ((A_1, V_1), (A_2, V_2))$ be an $\overline{M}_2$-maximal assignment to $T - Q$. We are going to construct an $\overline{M}_2$ assignment to $T$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6).

Let $A'_1 = A_1 \cup \{\text{the peripheral super paths of } Q\}$, $V'_1 = V_1 \setminus \{v\}$, $A'_2 = A_2 \cup \{\text{the peripheral super paths of length at least two of } Q\}$ and $V'_2 = V_2 \setminus \{v\}$.

Since $M_1(T) = M_1(T - Q) + f + g + h - 1$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $T$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2(T - Q) + f + g + h - 1 = \overline{M}_2(T) + f + 2g + 2h - 2$.

Conversely, by Lemma 4.8, 1, there exists an $\overline{M}_2$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$, in which $v$ is in $V_1 \cap V_2$. We are going to construct an $\overline{M}_2$ assignment to $T - Q$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$. By 10 Remarks 4.4, 1 and 45.1, each of the $f + g + h$ peripheral super paths of $Q$ might be in $A_1$ and each of the $g + h$ peripheral super paths of length at least 2 of $Q$ might be in $A_2$.

Let $A'_1 = A_1 \setminus \{R\} \cup \{\text{the peripheral super paths of } Q, P\}$, $V'_1 = V_1 \setminus \{v\}$, $A'_2 = A_2 \setminus \{U\} \cup \{\text{the peripheral arms of each small pincer of } Q\}$ and $V'_2 = V_2 \setminus \{\text{the central vertex of each small pincer of } Q\}$.

Since $M_1(T) = M_1((T - Q) + w K_1) + f + g + h - 1$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $T$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) + w K_1) + f + g + h - 1 + 2h = \overline{M}_2(T) + f + g + 2h - 1$.

Suppose that $f + g \geq 0$. Let $B$ be the component of $T - (V_2 \cup \{v\})$ containing $w$ and $C$ be the component of $T - (V_1 \cup \{v\})$ containing $w$ and $v$.

Let $A'_1 = A_1 \setminus \{R\} \cup \{\text{the peripheral arms of each small pincer of } Q, C\}$, $V'_1 = V_1 \setminus \{v\}$ and $A'_2 = (A_2 \setminus \{U\}) \cup \{\text{the peripheral super paths of } P\}$.

Since $M_1(T) = M_1(T - Q + w K_1) + f + g + h - 1$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $T$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) + w K_1) + f + g + h + 2h - 1 = \overline{M}_2(T - Q) + f + g + 2h - 1$.

Conversely, suppose that $f + g \geq 1$. By Lemma 4.9, 1, there exists an $\overline{M}_2$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$, in which $v$ is in $V_1$ and the central vertex of each small pincer of $Q$ is in $V_2$. We are going to construct an $\overline{M}_2$ assignment to $(T - Q) + w K_1$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6). By Remarks 4.4, 1 and 2, and 45.1, 3, each of the $f + g + h$ peripheral super paths of $Q$ might be in $A_1$, the peripheral arms of each small pincer of $Q$ might be in $A_2$ and $w \not\in V_1 \cup V_2$. Let $R$ be the component of $T - V_1$ containing $w$ and $v$. By Remarks 4.4, 1 and 45.1, $R \in A_1$ and $P \in A_2$.

Let $A'_1 = A_1 \setminus \{R\} \cup \{\text{the peripheral super paths of } Q\}$, $V'_1 = V_1 \setminus \{v\}$, $A'_2 = (A_2 \setminus \{P\}) \cup \{\text{the peripheral arms of each small pincer of } Q\}$ and $V'_2 = V_2 \setminus \{\text{the central vertex of each small pincer of } Q\}$.

Since $M_1((T - Q) + w K_1) = M_1(T) - f - g - h + 1$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $(T - Q) + w K_1$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) + w K_1) + f - g - h + 1 + 2h = \overline{M}_2(T) - f - g - 2h + 1$.

Suppose that $f + g = 0$. By Lemma 4.9, 2, there exists an $\overline{M}_2$-maximal assignment to $T$, $A = ((A_1, V_1), (A_2, V_2))$, in which $v$ is in $V_2$, the central vertex of each small pincer of $Q$ is in $V_1$ and $w \not\in V_1 \cup V_2$. We are going to construct an $\overline{M}_2$ assignment to $(T - Q) + w K_1$, $A' = ((A'_1, V'_1), (A'_2, V'_2))$. By Remarks 4.4, 1 and 50...
4.5, 1 and 3, each of the $h$ small pinners of $Q$ might be in $A_2$ and the peripheral arms of each small pinner of $Q$ might be in $A_1$. Let $R$ be the component of $T - V_1$ containing $v$, $w$ and let $P$ be the component of $T - V_2$ containing $w$. By Remarks 4.4, 1 and 4.5, 1 and 3, $R \in A_1$ and $P \in A_2$. Let $P'$ be the component of $((T - Q) + w \ K_1) - (V_2 \setminus \{v\})$ containing $wand K_1$, and let $R'$ be the component of $((T - Q) + w \ K_1) - (V_1 \setminus \{the central vertex of each small pinner of $Q\})$ containing $w$ and $K_1$.

Let $A_1' = (A_1 \setminus \{R, the peripheral arms of each small pinner of $Q\}) \cup \{R'\}$, $V_1' = V_1 \setminus \{the central vertex of each small pinner of $Q\}$, $V_2' = V_2 \setminus \{v\}$ and $A_2' = (A_2 \setminus \{P, the peripheral super paths of $Q\}) \cup \{P'\}$.

Since $M_1((T - Q) + w \ K_1) = M_1(T) - h$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $(T - Q) + w \ K_1$, $A' = ((A_1', V_1'), (A_2', V_2'))$ and $\overline{M}_2((T - Q) + w \ K_1) \geq \overline{M}_2(T) - 2h + h + h + 1 = \overline{M}_2(T) - f + g - 2h + 1$.

So, we have $\overline{M}_2((T - Q) + w \ K_1) = \overline{M}_2(T) - f + g - 2h + 1$.

**Part C:** Let $A = ((A_1, V_1), (A_2, V_2))$ be an $\overline{M}_2$-maximal assignment to $(T - Q) + w \ S_4$. We are going to construct an $\overline{M}_2$ assignment to $T, A' = ((A_1', V_1'), (A_2', V_2'))$. Let $x$ be the central vertex of $S_4$.

By Lemma 4.8, 3 and by Remark 4.4, 1, $x \in V_1$ and the peripheral arms of $S_4$ are in $A_1$. By Remark 4.5, 4, $x \notin V_2$. Let $R$ be the component of $((T - Q) + w \ S_4) - V_2$ containing $S_4$. By Remark 4.5, 1, $R$ is in $A_2$. Let $R'$ be the component of $T - (V_2 \cup \{the central vertex of each small pinner of $Q\})$ containing $v$.

Let $A_1' = (A_1 \setminus \{the peripheral arms of $S_4$\}) \cup \{the peripheral super paths of $Q\}, V_1' = (V_1 \setminus \{x\}) \cup \{v\}, V_2' = V_2 \cup \{the central vertex of each small pinner of $Q\}$ and $A_2' = (A_2 \setminus \{R\}) \cup \{the peripheral arms of each small pinner of $Q, R'\}$.

Since $M_1(T) = M_1((T - Q) + w \ S_4) + f + g + h + 3$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $T, A' = ((A_1', V_1'), (A_2', V_2'))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) + w \ S_4) + f + g + h + 3$.

Conversely, let $A = ((A_1, V_1), (A_2, V_2))$ be an $\overline{M}_2$-maximal assignment to $T$. By Lemma 4.8, 3, $v$ is in $V_1$.

If $v \notin V_2$ then by Remark 4.5, 1, the longer arm and the small pinners of $Q$ are in $A_2$. By Remark 4.4, 1, each of the $f + g + h$ peripheral super paths of $Q$ might be in $A_1$. If $w \notin V_2$, then let $F$ be the component of $T - V_2$ containing $w$. Let $H$ be the component of $T - ((V_2 \setminus \{v\}) \cup \{the central vertex of each small pinner of $Q\})$ containing $v$.

Let $B_1 = A_1, U_1 = V_1, B_2 = (A_2 \setminus \{F, the longer arm and the small pinners of $Q\}) \cup \{the peripheral arms of each small pinner of $Q, H\}$ and $U_2 = (V_2 \setminus \{v\}) \cup \{the central vertex of each small pinner of $Q\}$.

By Remark 4.6, this creates an $\overline{M}_2$ assignment to $T, B = ((B_1, U_1), (B_2, U_2))$. Using the cardinality of $B$ we conclude that $g = 1, w \notin V_2$ and $F \in A_2$.

We are going to construct, $A' = ((A_1', V_1'), (A_2', V_2'))$, an $\overline{M}_2$ assignment to $(T - Q) + w \ S_4$. Let $x$ be the central vertex of $S_4$. Let $R'$ be the component of $((T - Q) + w \ S_4) - (V_2 \setminus \{v\})$ containing $x$.

Let $A_1' = (A_1 \setminus \{the peripheral super paths of $Q\}) \cup \{the peripheral arms of $S_4\}, V_1' = (V_1 \setminus \{v\}) \cup \{x\}, A_2' = (A_2 \setminus \{F, the longer arm and the small pinners of $Q\}) \cup \{R'\}$ and $V_2' = V_2 \setminus \{v\}$.

Since $M_1((T - Q) + w \ S_4) = M_1(T) - f - g - h + 3$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $(T - Q) + w \ S_4, A' = ((A_1', V_1'), (A_2', V_2'))$ and $\overline{M}_2((T - Q) + w \ S_4) \geq \overline{M}_2(T) - f - g - h + 3 + 1 - 1 - h + 1 + 1 = \overline{M}_2(T) - f - g - 2h + 3$.

If $v \notin V_2$, using the maximality of $\{A_1 \setminus \{V_2\}, then the central vertex of each small pinner of $Q$ is in $V_2$.

We are going to construct an $\overline{M}_2$ assignment to $(T - Q) + w \ S_4, A' = ((A_1, V_1'), (A_2', V_2'))$. By Remarks 4.4, 1 and 4.5, 1 and 3, each of the $f + g + h$ peripheral super paths of $Q$ might be in $A_1$ and the peripheral arms of each small pinner of $Q$ might be in $A_2$. Let $R$ be the component of $T - V_2$ containing $v$. Let $R'$ be the component of $((T - Q) + w \ S_4) - V_2$ containing $x$ (x is the central vertex of $S_4$).

Let $A_1' = (A_1 \setminus \{the peripheral super paths of $Q\}) \cup \{the peripheral arms of $S_4\}, V_1' = (V_1 \setminus \{v\}) \cup \{x\}, A_2' = (A_2 \setminus \{R, the peripheral arms of each small pinner of $Q \cup \{R'\} and V_2' = V_2 \setminus \{the central vertex of each small pinner of $Q\}.

Since $M_1((T - Q) + w \ S_4) = M_1(T) - f - g - h + 3$, by Remark 4.6, this creates an $\overline{M}_2$ assignment to $(T - Q) + w \ S_4, A' = ((A_1, V_1), (A_2', V_2))$ and $\overline{M}_2((T - Q) + w \ S_4) \geq \overline{M}_2(T) - f - g - h + 3 + 2h + h - \overline{M}_2(T) - f - g - 2h + 3$.

Consequently, we have $\overline{M}_2((T - Q) + w \ S_4) = \overline{M}_2(T) - f - g - 2h + 3$.

**Example 4.11.** Let $T$ be the tree of Example 3.17. Let $Q$ be the subtree of $T generated by vertices 1, 2, 3, 4, 5, 6. Since $Q$ is a small superstar (star is not a small superstar) with 1 arm of length 1, 1 small pinner, and 7 is a vertex
of $T$ with degree 2, by Theorem 4.10,
\[ M_2(T) = M_2((T - Q) +_w K_1) + 2, \]
where $w$ is the vertex 7. So, $(T - Q) +_w K_1$ (that is a small superstar with central vertex 8) is the tree

\begin{verbatim}
\node (1) at (0,0) [circle,fill,inner sep=2pt] {10}
\node (2) at (-1,-1) [circle,fill,inner sep=2pt] {9}
\node (3) at (1,-1) [circle,fill,inner sep=2pt] {11}
\node (4) at (-2,-2) [circle,fill,inner sep=2pt] {4}
\node (5) at (-1,-2) [circle,fill,inner sep=2pt] {7}
\node (6) at (1,-2) [circle,fill,inner sep=2pt] {8}
\node (7) at (-2,-3) [circle,fill,inner sep=2pt] {12}
\node (8) at (-1,-3) [circle,fill,inner sep=2pt] {13}
\node (9) at (1,-3) [circle,fill,inner sep=2pt] {10}
\end{verbatim}

By Proposition 3.14,
\[ M_2((T - Q) +_w K_1) - J) = 2 + 4 - 2 = 2. \]
Therefore,
\[ M_2(T) = 6. \]

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